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# Boundedness of Solutions of Certain System of Second-order Ordinary Differential Equations

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**Abstract**

We extend, in this paper, some known results on the boundedness of solutions of certain second order nonlinear scalar differential equations to system of second order nonlinear differential equations.

**Key words:** boundedness, Lyapunov function, differential equations of second-order

**2010 Mathematics Subject Classification:** 34C11

## 1 Introduction

Let $\mathbb{R}$ denote the real line $-\infty < t < \infty$, and let $\mathbb{R}^n$ denote the real $n-$dimensional Euclidean space equipped with the usual Euclidean norm which will be represented throughout the sequel by $\|.\|$. We shall consider here differential equations of the form

$$\ddot{X} + F(X, \dot{X})\dot{X} + H(X) = P(t, X, \dot{X}) \tag{1.1}$$

in which $X\colon \mathbb{R} \to \mathbb{R}^n$, $H\colon \mathbb{R}^n \to \mathbb{R}^n$, $P\colon \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $F$ is an $n \times n$ continuous symmetric positive definite matrix function for the argument displayed explicitly, and the dots as usual indicate differentiation with respect to $t$. The equation is the vector version for the system of real second-order differential equations

$$\begin{aligned}&\ddot{x}_i + \sum_{k=1}^{n} f_{ik}(x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n)\dot{x}_k + h_i(x_1, \ldots, x_n)\\ &= p_i(t, x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n), \quad (i = 1, 2, \ldots, n).\end{aligned}$$

It is also assumed that $H$ and $P$ are continuous for the argument displayed explicitly. Moreover, the existence and uniqueness of the solutions of (1.1) will be assumed (see Picard–Lindelof theorem in [10]).

Our study of (1.1) here is concerned primarily with the problems of the boundedness of solutions of (1.1). For over four decades, many authors have dealt with extension of results obtained for scalar differential equations ([11]) to vector differential equations. See, for example, [1–4, 6–9, 12, 15, 16] and the references cited therein for third and higher order cases.

However, only few results on second order scalar differential equations have been extended to second order vector equations. The special case in which $F(X, \dot{X})$ is an $n \times n$ matrix $A$ in (1.1), and (1.1) is an $n$ vector and matrix equations have been studied [5, 14]. In [5], for example, the author dealt with the boundedness problem in connection with his study of convergence properties of solutions, while in [14] the asymptotic stability, boundedness and existence of solutions was discussed. The conditions obtained in each of these previous investigations are generalization in some form or the other, of the conditions: $a > 0$ and $b > 0$ for the scalar equation

$$\ddot{x} + a\dot{x} + bx = p(t),$$

with $a, b$ constants, which conditions ensure the ultimate boundedness of all solutions if $p$ is bounded and the existence of periodic solutions if $p$ is periodic in $t$. Our motivation comes from [13]. With respect to our observations in the literature, no work based on [13] was found. Consequently, our present investigations are related to [13], and we shall provide extensions of some of the results obtained to $n$-dimensional equations of the form (1.1).

## 2 Notations

We shall use the notation as given in [1]. Throughout this paper, $\delta$'s, $\Delta$'s and $D$'s with or without suffices will denote positive constants whose magnitudes depend on an $n \times n$ matrix function $F(X, \dot{X})$ and vector functions $H(X)$, $P(t, X, \dot{X})$. The $\delta$'s, $\Delta$'s and $D$'s with numerical or alphabetical suffixes shall retain fixed magnitudes, while those without suffixes are not necessarily the same at each occurrence. It should also be noted that $\alpha_0, \alpha_1, \xi_0, \beta, \gamma$ are positive constants as defined later.

Also, we shall denote the scalar product $\langle X, Y \rangle$ of any vectors $X, Y$ in $\mathbb{R}^n$, with respective components $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ by $\sum_{i=1}^{n} x_i y_i$. In particular, $\langle X, X \rangle = \|X\|^2$. Finally, by $\operatorname{sgn} X$, we mean $(\operatorname{sgn} x_1, \operatorname{sgn} x_2, \ldots, \operatorname{sgn} x_n)$ and $\|\operatorname{sgn} X\| = \sqrt{n} > 0$.

## 3 Main results

Before stating our main results, we give a well-known algebraic result which will be required in the proofs.

**Lemma 1** *Let $A$ be a real symmetric positive definite $n \times n$-matrix. Then for any $X \in \mathbb{R}^n$*

$$\delta_a \|X\|^2 \leq \langle AX, X \rangle \leq \Delta_a \|X\|^2,$$

*where $\delta_a$ and $\Delta_a$ are, respectively, the least and greatest eigenvalues of the matrix $A$.*

**Proof** See [7], [12], [16]. □

Our main theorems are the following.

**Theorem 1** *Let all the basic assumptions imposed on $F, H$ and $P$ hold, and that $F(0,0) = 0, H(0) = 0$. Suppose further that for any arbitrary $X \in \mathbb{R}^n$,*

*(i)* *$JH(X)$ is symmetric and positive definite;*

*(ii)* *the eigenvalues $\lambda_i(F(X,\dot{X}))$ of $F(X,\dot{X})$ satisfy*

$$0 < \delta_f \leq \lambda_i(F(X,\dot{X})) \leq \Delta_f, \tag{3.1}$$

*where $\delta_f, \Delta_f$ are respectively, the least and greatest eigenvalues of matrix $F(X,\dot{X})$;*

*(iii)*

$$\lim_{\|X\| \to \infty} \{\alpha \langle H(X), \operatorname{sgn} X \rangle - 2\gamma \Delta_f\} > 2\gamma\beta, \tag{3.2}$$

*where $\alpha = \operatorname{sgn}(\langle H(X), \operatorname{sgn} X \rangle)$, and $\gamma = \sqrt{n}$, $\beta$ are positive constants.*

*(iv)* *the finite constants $\delta_f$ and $\beta$ are such that*

$$\delta_f - \beta > 0; \tag{3.3}$$

*(v)* *the function $H(X)$ satisfies either*

$$\langle H(X), \operatorname{sgn} X \rangle \to +\infty \text{ as } \|X\| \to \infty \tag{3.4}$$

*or*

$$\langle H(X), \operatorname{sgn} X \rangle \to -\infty \text{ as } \|X\| \to \infty \tag{3.5}$$

*and*

*(vi)* *for all $t, X$ and $\dot{X}$,*

$$\|P(t, X, \dot{X})\| \leq \beta. \tag{3.6}$$

*Then, there exists a constant $D, 0 < D < \infty$, whose magnitude depends only on the constant $\beta, \delta_f, \Delta_f$ as well as on $F(X,\dot{X}), JH(X)$ and $P(t, X, \dot{X})$ such that every solution $X(t)$ of (1.1) ultimately satisfies*

$$\|X(t)\| \leq D, \quad \|\dot{X}(t)\| \leq D. \tag{3.7}$$

Suppose we relax the restriction on $P(t, X, \dot{X})$ in (3.6) so that we have

**Theorem 2** *In addition to the conditions (i), (ii), (iv) and (v) of Theorem 1, suppose*

*(i)*

$$\lim_{\|X\|\to\infty} \{\alpha\langle H(X), \operatorname{sgn} X\rangle - 2\gamma\Delta_f\} > 2\gamma\beta_*, \tag{3.8}$$

*where*

$$\beta_* = \max\left\{\frac{\gamma}{8}(\Delta_f + \beta)^2(\delta_f - \beta)^{-1}, \beta\right\} \tag{3.9}$$

*(ii) for all $t, X$ and $\dot{X}$*

$$\|P(t, X, \dot{X})\| \le \beta\|\dot{X}\|. \tag{3.10}$$

*Then, there exists a constant $D, 0 < D < \infty$, whose magnitude depends only on the constants $\beta$, $\delta_f$, $\Delta_f$ as well as on $F(X, \dot{X})$, $JH(X)$ and $P(t, X, \dot{X})$ such that every solution $X(t)$ of (1.1) ultimately satisfies (3.7).*

## 4 Some preliminaries

The following result will be basic to the proofs of Theorems 1 and 2.

**Lemma 2** *Let $H(0) = 0$ and assume that the matrices $A$ and $JH(X)$ are symmetric and commute for all $X \in \mathbb{R}^n$. Then*

$$\langle H(X), AX\rangle = \int_0^1 X^T AJH(\sigma X)X\, d\sigma. \tag{4.1}$$

**Proof** See [7], [12], [16]. □

**Lemma 3** *Let $H(0) = 0$ and assume that $JH(X)$ is symmetric for arbitrary $X \in \mathbb{R}^n$. Then*

$$\frac{d}{dt}\int_0^1 \langle H(\sigma X), X\rangle\, d\sigma = \langle H(X), \dot{X}\rangle \quad \textit{for all } X \in \mathbb{R}^n \tag{4.2}$$

**Proof** See [7], [12], [16]. □

## 5 Proof of Theorem 1

Let us replace system of differential equations of form (1.1) in the equivalent system form

$$\dot{X} = Y, \quad \dot{Y} = -F(X, Y)Y - H(X) + P(t, X, Y) \tag{5.1}$$

for which a typical solution will be $(X(t), Y(t))$. Define the continuous function $V = V(X, Y)$ adapted from [13], with suitable modification by

$$V = V_1 + V_2, \tag{5.2}$$

where

$$2V_1 = \langle Y, Y\rangle + \int_0^1 \langle H(\sigma X), X\rangle \, d\sigma \tag{5.3}$$

$$V_2 = \begin{cases} \alpha\langle Y, \operatorname{sgn} X\rangle & \text{if } \|Y\| \leq \|X\|; \\ \langle X, \operatorname{sgn} Y\rangle & \text{if } \|X\| \leq \|Y\|. \end{cases} \tag{5.4}$$

We shall show that $V(X, Y)$ satisfies

$$V(X, Y) \to +\infty \text{ as } \|X\|^2 + \|Y\|^2 \to +\infty \tag{5.5}$$

and that, for any solution $(X(t), Y(t))$ of (5.1) the derivative of $V = V(X, Y)$ exists and satisfies

$$\dot{V} \leq -D_0 \text{ if } \|X\|^2 + \|Y\|^2 \geq D_1 \tag{5.6}$$

for some finite constants $D_0 > 0$, $D_1 > 0$. As will be clear from Yoshizawa-type technique employed in [9, 17], the two results (5.5) and (5.6) together imply ultimately that

$$\|X\|^2 + \|Y\|^2 \leq D$$

and hence (3.7).

To verify (5.5), note from (5.4) that $|V_2| \leq \delta\|Y\|$ where $\delta$ is a positive constant.

Also, by Lemma 1 with $A \equiv I_n$ and Lemma 2, we have that

$$\int_0^1 \langle H(\sigma X), X\rangle \, d\sigma = \int_0^1 \int_0^1 \sigma X^T J H(\tau\sigma X) X d\tau \, d\sigma \geq \delta_h \|X\|^2$$

since $JH(X)$ is positive definite.

Therefore $V$ satisfies

$$2V \geq 2\delta_h\|X\|^2 + \|Y\|^2 - \delta\|Y\| \geq \delta_1(\|X\|^2 + \|Y\|^2) - \delta\|Y\|,$$

where $\delta_1 = \min\{1, 2\delta_h\}$. The right hand side here tends to $+\infty$ as

$$\|X\|^2 + \|Y\|^2 \to +\infty.$$

We are now left to show that $\dot{V}$ exists and satisfies (5.6) thus,

$$\dot{V} = \dot{V}_1 + \dot{V}_2.$$

Using Lemma 3 and (5.1), we have that

$$\dot{V}_1 = -\langle F(X, Y)Y, Y\rangle + \langle Y, P(t, X, Y)\rangle \tag{5.7}$$

and

$$\dot{V}_2 = \begin{cases} -\alpha\{\langle F(X, Y)Y, \operatorname{sgn} X\rangle + \langle H(X), \operatorname{sgn} X\rangle - \langle P(t, X, Y), \operatorname{sgn} X\rangle\} \\ \quad \text{if } \|Y\| \leq \|X\|; \\ \langle Y, \operatorname{sgn} Y\rangle \quad \text{if } \|X\| \leq \|Y\|. \end{cases} \tag{5.8}$$

Thus,

$$\begin{aligned}\dot V = -\alpha\langle H(X), \operatorname{sgn} X\rangle - \langle F(X,Y)Y, Y\rangle - \alpha\langle F(X,Y)Y, \operatorname{sgn} X\rangle \\ + \langle Y + \alpha \operatorname{sgn} X, P(t,X,Y)\rangle, \quad \text{if } \|Y\| \le \|X\| \end{aligned} \tag{5.9}$$

or

$$\dot V = -\langle F(X,Y)Y, Y\rangle + \langle Y, \operatorname{sgn} Y\rangle + \langle Y, P(t,X,Y)\rangle, \quad \text{if } \|X\| \le \|Y\|. \tag{5.10}$$

$$\begin{aligned}\dot V \le -\alpha\langle H(X), \operatorname{sgn} X\rangle - \langle F(X,Y)Y, Y\rangle + |\langle F(X,Y)Y, \operatorname{sgn} X\rangle| \\ + (\|Y\| + \gamma)\|P(t,X,Y)\|, \quad \text{if } \|Y\| \le \|X\| \end{aligned} \tag{5.11}$$

or

$$\dot V = -\langle F(X,Y)Y, Y\rangle + (\|P(t,X,Y)\| + \gamma)\|Y\|, \quad \text{if } \|X\| \le \|Y\|. \tag{5.12}$$

The condition (3.2) implies the existence of finite constants $\alpha_0 > 0$, $D_2 > 0$ such that

$$\|X\| \ge \alpha_0 \Longrightarrow \alpha\langle H(X), \operatorname{sgn} X\rangle - 2\gamma\Delta_f - 2\gamma\beta \ge D_2. \tag{5.13}$$

Let

$$\alpha_1 = \max\{1, \alpha_0, \mu\}, \tag{5.14}$$

where

$$\mu = \delta_f^{-1}(\beta + \gamma).$$

We assert that, for some finite constants $D_3 > 0$,

$$\dot V \le -D_3 \quad \text{if } \|X\| \ge \alpha_1. \tag{5.15}$$

Indeed, if $\|Y\| \le \|X\|$ so that $\dot V$ satisfies (5.11) and, if $\|Y\| \ge 1$, then by (3.1) and (3.6),

$$\begin{aligned}\dot V &\le -\alpha\langle H(X), \operatorname{sgn} X\rangle - \|Y\|(\delta_f\|Y\| - \gamma\Delta_f) + \beta(\|Y\| + \gamma) \\ &\le -\alpha\langle H(X), \operatorname{sgn} X\rangle - (\delta_f\|Y\| - \gamma\Delta_f) + \beta(\|Y\| + \gamma) \\ &\le -\alpha\langle H(X), \operatorname{sgn} X\rangle - (\delta_f - \beta)\|Y\| + \gamma(\beta + \Delta_f) \\ &\le -\alpha\langle H(X), \operatorname{sgn} X\rangle + 2\gamma(\beta + \Delta_f).\end{aligned}$$

Thus,

$$\dot V \le -D_2 \quad \text{if } \|X\| \ge \alpha_1 \tag{5.16}$$

by (5.13) and (5.14).

Suppose however, that $\|Y\| \le 1$, then

$$\begin{aligned}\dot V &\le -\alpha\langle H(X), \operatorname{sgn} X\rangle - \delta_f\|Y\|^2 + \gamma\Delta_f\|Y\| + \beta(\|Y\| + \gamma) \\ &\le -\alpha\langle H(X), \operatorname{sgn} X\rangle + \gamma\Delta_f + \beta(1 + \gamma) \\ &\le -\alpha\langle H(X), \operatorname{sgn} X\rangle + 2\gamma(\beta + \Delta_f),\end{aligned}$$

so that (5.13), (5.16) still hold in this case. We are now left with the case: $\|X\| \leq \|Y\|$ for which $\dot{V}$ satisfies (5.12). If we note that $\|X\| \geq \alpha_1$ implies that $\|Y\| \geq \alpha_1$, with $\alpha_1$ fixed by (5.14), we have that

$$\begin{aligned} \dot{V} \leq & -\delta_f\|Y\|^2 + \gamma\|Y\| + \beta\|Y\| = -\{\delta_f\|Y\| - (\gamma+\beta)\}\|Y\| \leq -1 \\ & \text{for } \|Y\| \geq \mu = (\gamma+\beta)\delta_f^{-1}. \end{aligned}$$

That is, $\|X\| \geq \alpha_1 \Rightarrow \dot{V} \leq -1$. This together with (5.16) show that (5.15) holds with $D_3 = \max\{1, D_2\}$.

To complete our discussion, suppose, on the contrary, that $\|X\| \leq \alpha_1$ and assume for a start that $\|Y\| \geq \alpha_1$. Then $\|Y\| \geq \|X\|$ and so $\dot{V}$ satisfies

$$\dot{V} \leq -\delta_f\|Y\|^2 + \gamma\|Y\| + \beta\|Y\| \leq -1 \quad \text{for } \|Y\| \geq \mu. \tag{5.17}$$

The results (5.15) and (5.17) show clearly that

$$\dot{V} \leq -D_3 \quad \text{if } \|X\|^2 + \|Y\|^2 \geq 2\alpha_1,$$

$D_3 = \max\{1, D_2\}$. This concludes the proof of (5.6) and, as earlier remarked, the theorem now follows.

# 6 Proof of Theorem 2

The procedure here is the same as that used for Theorem 1. The proof of Theorem 2 is immediate as soon as we show (5.5) and (5.6). The verification of (5.5) given in §5 carries over with obvious modifications.

To verify (5.6), our starting point will be the estimate (5.11) and (5.12) which are still valid in this case. In view of (3.8) there are constants $\alpha_0 > 0, D_4 > 0$ such that

$$\|X\| \geq \alpha_0 \Longrightarrow \{\alpha\langle H(X), \operatorname{sgn} X\rangle - 2\gamma\Delta_f - 2\gamma\beta_*\} \geq D_4. \tag{5.18}$$

Suppose also that $\xi_0 > 0$ is a constant such that

$$\|Y\| \geq \xi_0 \Longrightarrow -(\delta_f - \beta)\|Y\|^2 + \|Y\| \leq -1 \tag{5.19}$$

and set

$$\alpha_1 = \max\{1, \alpha_0, \xi_0\}. \tag{5.20}$$

First, we show that for some constant $D_5 > 0$,

$$\|X\| \geq \alpha_1 \Longrightarrow \dot{V} \leq -D_5. \tag{5.21}$$

As before we consider the two cases $\|Y\| \leq \|X\|$ and $\|X\| \leq \|Y\|$ separately.

Let $\|Y\| \leq \|X\|$ and suppose that $\|Y\| \geq 1$. Then on using (5.11), we have

$$\begin{aligned}
\dot{V} &\leq -\alpha\langle H(X), \operatorname{sgn} X\rangle - \delta_f\|Y\|^2 + \gamma\Delta_f\|Y\| + \beta\|Y\|(\|Y\| + \gamma) \\
&= -\alpha\langle H(X), \operatorname{sgn} X\rangle - (\delta_f - \beta)\left\{\|Y\| - \frac{\gamma(\Delta_f + \beta)}{2(\delta_f - \beta)}\right\}^2 + \frac{\gamma^2(\Delta_f + \beta)^2}{4(\delta_f - \beta)} \\
&\leq -\alpha\langle H(X), \operatorname{sgn} X\rangle + \frac{\gamma^2(\Delta_f + \beta)^2}{4(\delta_f - \beta)} \\
&\leq -\alpha\langle H(X), \operatorname{sgn} X\rangle + 2\gamma\Delta_f + \frac{\gamma^2(\Delta_f + \beta)^2}{4(\delta_f - \beta)}.
\end{aligned}$$

If, however, $\|Y\| \leq 1$,

$$\begin{aligned}
\dot{V} &\leq -\alpha\langle H(X), \operatorname{sgn} X\rangle - \delta_f\|Y\|^2 + \gamma\Delta_f\|Y\| + \beta\|Y\|(\|Y\| + \gamma) \\
&= -\alpha\langle H(X), \operatorname{sgn} X\rangle - (\delta_f - \beta)\|Y\|^2 + \gamma(\Delta_f + \beta)\|Y\| \\
&\leq -\alpha\langle H(X), \operatorname{sgn} X\rangle + 2\gamma(\Delta_f + \beta).
\end{aligned}$$

By (5.18), (3.8) and (5.20) it is clear that in either case (5.21) holds.

Suppose now that $\|X\| \leq \|Y\|$. Then $\|X\| \geq \alpha_1$ implies that $\|Y\| \geq \alpha_1 \geq \xi_0$ by (5.20). Thus,

$$\dot{V} \leq -(\delta_f - \beta)\|Y\|^2 + \gamma\|Y\| \leq -1 \text{ for } \|Y\| \geq \gamma(\delta_f - \beta)^{-1}.$$

Suppose on the contrary that $\|X\| \leq \alpha_1$ and assume $\|Y\| \geq \alpha_1$. Then $\|Y\| \geq \|X\|$ and so we have

$$\dot{V} \leq -\{(\delta_f - \beta)\|Y\| - \gamma\}\|Y\| \leq -1 \quad \text{for } \|Y\| \geq \gamma(\delta_f - \beta)^{-1}$$

since $\|Y\| \geq \alpha_1$. This together with (5.21) show that

$$\dot{V} \leq -D_5 \quad \text{if } \|X\|^2 + \|Y\|^2 \geq 2\alpha_1$$

which verifies (5.6).

**Remark 1** For the case $n = 1$, (that is in $\mathbb{R}$) Theorems 1 and 2 reduce to Theorems 1 and 2 in [13], with obvious modifications.

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