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# EXISTENCE OF SOLUTIONS FOR A CLASS OF SECOND-ORDER $p$-LAPLACIAN SYSTEMS WITH IMPULSIVE EFFECTS

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*Abstract.* The purpose of this paper is to study the existence and multiplicity of a periodic solution for the non-autonomous second-order system

$$\frac{\mathrm{d}}{\mathrm{d}t}(|\dot{u}(t)|^{p-2}\dot{u}(t)) = \nabla F(t,u(t)), \quad \text{a.e. } t \in [0,T],$$
$$u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0,$$
$$\Delta\dot{u}^i(t_j) = \dot{u}^i(t_j^+) - \dot{u}^i(t_j^-) = I_{ij}(u^i(t_j)),\ i = 1,2,\ldots,N;\ j = 1,2,\ldots,m.$$

By using the least action principle and the saddle point theorem, some new existence theorems are obtained for second-order $p$-Laplacian systems with or without impulse under weak sublinear growth conditions, improving some existing results in the literature.

*Keywords*: second-order $p$-Laplacian Hamiltonian systems; impulsive effect; critical point theory

*MSC 2010*: 34C25, 58E50

## 1. Introduction

Consider the second-order $p$-Laplacian system with impulsive effects

$$(1.1)\quad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(|\dot{u}(t)|^{p-2}\dot{u}(t)) = \nabla F(t,u(t)), \quad \text{a.e. } t \in [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \\ \Delta\dot{u}^i(t_j) = \dot{u}^i(t_j^+) - \dot{u}^i(t_j^-) = I_{ij}(u^i(t_j)),\ i = 1,2,\ldots,N;\ j = 1,2,\ldots,m, \end{cases}$$

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where $p > 1$, $T > 0$, $t_0 = 0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = T$, $u(t) = (u^1(t), u^2(t), \ldots, u^N(t))$, $I_{ij}\colon \mathbb{R} \to \mathbb{R}$ $(i = 1, 2, \ldots, N;\ j = 1, 2, \ldots, m)$ are continuous and $F\colon [0, T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption:

(A) $F(t, x)$ is measurable in $t$ for every $x \in \mathbb{R}^N$ and continuously differentiable in $x$ for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b \in L^1([0, T], \mathbb{R}^+)$ such that

$$|F(t, x)| \leqslant a(|x|)b(t), \quad |\nabla F(t, x)| \leqslant a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$.

For the sake of convenience, in the sequel we define $A = \{1, 2, \ldots, N\}$, $B = \{1, 2, \ldots, m\}$.

When $I_{ij} \equiv 0, p = 2$, (1.1) degenerates to the second order Hamiltonian system

$$\begin{cases} \ddot{u}(t) = \nabla F(t, u(t)), & \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases} \tag{1.2}$$

It has been proved that problem (1.2) has at least one solution by the least action principle and the minimax methods (see [2], [7]–[9], [11], [12], [15]–[18], [20]–[22], [25], [26]). Many solvability conditions are given, such as the coercive condition (see [2]), the periodicity condition (see [20]), the convexity condition (see [7]), the subadditive condition (see [15]), the bounded condition (see [8]).

When the nonlinearity $\nabla F(t, x)$ is bounded sublinearly, that is, there exist $f, g \in L^1([0, T], \mathbb{R}^+)$ and $\alpha \in [0, 1)$ such that

$$|\nabla F(t, x)| \leqslant f(t)|x|^{\alpha} + g(t) \tag{1.3}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$, Tang [17] also proved the existence of solutions for problem (1.2) when $I_{ij} \equiv 0$ under the condition

$$\lim_{|x| \to \infty} |x|^{-2\alpha} \int_0^T F(t, x)\,\mathrm{d}t \to \infty, \tag{1.4}$$

or

$$\lim_{|x| \to \infty} |x|^{-2\alpha} \int_0^T F(t, x)\,\mathrm{d}t \to -\infty, \tag{1.5}$$

which generalizes Mawhin-Willem's results under the boundedness condition (see [8]).

When $\alpha = 1$, condition (1.2) reduces to the linearly bounded gradient condition, in this case, Zhao and Wu [21], [22] also proved the existence of solutions for problem

(1.1) under the condition

$$\int_0^T f(t)\,\mathrm{d}t < \frac{12}{T} \tag{1.6}$$

and (1.4) or (1.5) with $\alpha = 1$.

However, there exists $F$ that satisfies neither (1.4) nor (1.5).

Let

$$F(t,x) = \sin\Bigl(\frac{2\pi t}{T}\Bigr)|x|^{7/4} + (0.6T - t)|x|^{3/2}.$$

It is easy to see that

$$|\nabla F(t,x)| \leqslant \frac{7}{4}\Bigl|\sin\Bigl(\frac{2\pi t}{T}\Bigr)\Bigr||x|^{3/4} + \frac{3}{2}|0.6T - t||x|^{1/2} \leqslant \frac{7}{4}\Bigl(\Bigl|\sin\Bigl(\frac{2\pi t}{T}\Bigr)\Bigr| + \varepsilon\Bigr)|x|^{3/4} + \frac{T^3}{\varepsilon^2}$$

for all $x \in \mathbb{R}^N$ and $t \in [0,T]$, where $\varepsilon > 0$. The above shows (1.2) holds with $\alpha = 3/4$ and

$$f(t) = \frac{7}{4}\Bigl(\Bigl|\sin\Bigl(\frac{2\pi t}{T}\Bigr)\Bigr| + \varepsilon\Bigr), \quad g(t) = \frac{T^3}{\varepsilon^2}.$$

However, $F(t,x)$ satisfies neither (1.4) nor (1.5). In fact,

$$|x|^{-2\alpha}\int_0^T F(t,x)\,\mathrm{d}t = |x|^{-3/2}\int_0^T \Bigl[\sin\Bigl(\frac{2\pi t}{T}\Bigr)|x|^{7/4} + (0.6T - t)|x|^{3/2}\Bigr]\,\mathrm{d}t = 0.1T^2.$$

The above example shows that it is valuable to further improve (1.4) and (1.5).

For $I_{ij} \not\equiv 0, i \in A, j \in B$, there are a few papers which discussed the existence of solution for (1.1) by variational method (see [28]). Hence, it is necessary to improve (1.4) or (1.5) for problem (1.1).

Impulsive differential equations arising from the real world describe the dynamics of processes in which sudden, discontinuous jumps occur. For the background, theory and applications of impulsive differential equations, we refer the readers to the monographs and some recent contributions as [1], [3], [4], [6], [13], [20], [24].

Some classical tools such as fixed point theorems in cones [1], [5], [19] or the method of lower and upper solutions [3], [23] have been widely used to study impulsive differential equations.

Recently, the Dirichlet and periodic boundary conditions problems with impulses in the derivative have been studied by variational method. For general and recent works on the critical point theory and variational methods we refer the readers to [10], [14], [19], [27], [28]. It is a new approach to apply variational methods to the impulsive boundary value problem (IBVP for short). All results of [10], [14], [19], [27], [28] can be seen as generalizations of the corresponding ones for second order ordinary differential equations. The results of this paper show that under appropriate conditions, system (1.1) possesses at least one periodic solution, which generalizes some existing results in the literature. In particular, some results show that they have relationship both with the nonlinear term $F$ and the impulsive terms $I$; to the best of the authors' knowledge, there is still no result in the literature.

Inspired by the above results [15], [19], [21], [22], [28], we study the existence of solutions for problem (1.1) under weak sublinear growth conditions. Our results generalize the previous work, which seems not to have been considered in the literature.

Throughout this paper, we let $q \in (1, \infty)$ such that $1/p + 1/q = 1$.

## 2. Preliminaries and the variational setting

In this section, we recall some basic facts which will be used in the proofs of our main results. In order to apply the critical point theory, we construct a variational structure. With this variational structure, we can reduce the problem of finding solutions of (1.1) to that of seeking the critical points of a corresponding functional.

Let $W_T^{1,p}$ be the Sobolev space

$$W_T^{1,p} = \{u\colon [0,T] \to \mathbb{R}^N;\ u \,\text{absolutely continuous}, u(0) = u(T),\ \dot{u} \in L^p([0,T], \mathbb{R}^N)\},$$

which is a reflexive Banach space with the norm defined by

$$\|u\| = \|u\|_{W_T^{1,p}} = \left(\int_0^T [|\dot{u}(t)|^p + |u(t)|^p] \,\mathrm{d}t\right)^{1/p}$$

for $u \in W_T^{1,p}$.

Let us recall that

$$\|u\|_p = \left(\int_0^T |u(t)|^p \,\mathrm{d}t\right)^{1/p} \quad \text{and} \quad \|u\|_\infty = \max_{t\in[0,T]} |u(t)|.$$

We have the following fact.

Take $v \in W_T^{1,p}$ and multiply both sides of the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}(|\dot{u}(t)|^{p-2}\dot{u}(t)) = \nabla F(t, u(t)), \tag{2.1}$$

by $v$ and integrate from 0 to $T$:

$$\int_0^T ((|\dot{u}(t)|^{p-2}\dot{u}(t))', v(t)) \,\mathrm{d}t = \int_0^T (\nabla F(t, u(t)), v(t)) \,\mathrm{d}t.$$

The first term is now

$$\int_0^T ((|\dot{u}(t)|^{p-2}\dot{u}(t))', v(t))\,\mathrm{d}t = \sum_{j=0}^{m} \int_{t_j}^{t_{j+1}} ((|\dot{u}(t)|^{p-2}\dot{u}(t))', v(t))\,\mathrm{d}t$$

and

$$\begin{aligned}
&\int_{t_j}^{t_{j+1}} ((|\dot{u}(t)|^{p-2}\dot{u}(t))', v(t))\,\mathrm{d}t \\
&\quad = (|\dot{u}(t_{j+1}^-)|^{p-2}\dot{u}(t_{j+1}^-), v(t_{j+1}^-)) - (|\dot{u}(t_j^+)|^{p-2}\dot{u}(t_j^+), v(t_j^+)) \\
&\qquad - \int_{t_j}^{t_{j+1}} (|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{v}(t))\,\mathrm{d}t \\
&\quad = \sum_{i=1}^{N} (|\dot{u}^i(t_{j+1}^-)|^{p-2}\dot{u}^i(t_{j+1}^-)v^i(t_{j+1}^-) - |\dot{u}^i(t_j^+)|^{p-2}\dot{u}^i(t_j^+)v^i(t_j^+)) \\
&\qquad - \int_{t_j}^{t_{j+1}} (|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{v}(t))\,\mathrm{d}t.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_0^T ((|\dot{u}(t)|^{p-2}\dot{u}(t))', v(t))\,\mathrm{d}t = {} & \sum_{j=1}^{m}\sum_{i=1}^{N} \Delta\dot{u}^i(t_j)v^i(t_j) + |\dot{u}(T)|^{p-2}\dot{u}(T)v(T) \\
& - |\dot{u}(0)|^{p-2}\dot{u}(0)v(0) - \int_0^T (|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{v}(t))\,\mathrm{d}t \\
= {} & -\sum_{j=1}^{m}\sum_{i=1}^{N} I_{ij}(u^i(t_j))v^i(t_j) - \int_0^T (|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{v}(t))\,\mathrm{d}t.
\end{aligned}$$

Combining it with (2.1), we get

$$\sum_{j=1}^{m}\sum_{i=1}^{N} I_{ij}(u^i(t_j))v^i(t_j) + \int_0^T (|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{v}(t))\,\mathrm{d}t + \int_0^T (\nabla F(t, u(t), v(t)))\,\mathrm{d}t = 0.$$

Now, we introduce a weak formulation of the problem (1.1).

**Definition 2.1.** We say that a function $u \in W_T^{1,p}$ is a weak solution of problem (1.1) if the identity

$$\int_0^T (|\dot{u}(t)|^{p-2}\dot{u}(t), \dot{v}(t))\,\mathrm{d}t + \sum_{j=1}^{m}\sum_{i=1}^{N} I_{ij}(u^i(t_j))v^i(t_j)) = -\int_0^T (\nabla F(t, u(t)), v(t))\,\mathrm{d}t$$

holds for any $v \in W_T^{1,p}$.

The corresponding functional $\varphi$ on $W_T^{1,p}$ is given by

$$\begin{aligned}\varphi(u) &= \frac{1}{p}\int_0^T |\dot u(t)|^p \,\mathrm{d}t + \int_0^T F(t,u(t))\,\mathrm{d}t + \sum_{j=1}^m\sum_{i=1}^N \int_0^{u^i(t_j)} I_{ij}(t)\,\mathrm{d}t \\ &= \psi(u) + \varphi(u),\end{aligned} \tag{2.2}$$

where

$$\psi(u) = \frac{1}{p}\int_0^T |\dot u(t)|^p \,\mathrm{d}t + \int_0^T F(t,u(t))\,\mathrm{d}t \quad \text{and}\ \varphi(u) = \sum_{j=1}^m\sum_{i=1}^N \int_0^{u^i(t_j)} I_{ij}(t)\,\mathrm{d}t.$$

It follows from assumption (A) that $\psi \in C^1(W_T^{1,p},\mathbb{R})$. By the continuity of $I_{ij}$, $i\in A$, $j\in B$, one has that $\varphi \in C^1(W_T^{1,p},\mathbb{R})$. Thus, $\varphi \in C^1(W_T^{1,p},\mathbb{R})$. For any $v\in W_T^{1,p}$, we have

$$\begin{aligned}\langle \varphi'(u), v\rangle &= \int_0^T (|\dot u(t)|^{p-2}\dot u(t), \dot v(t))\,\mathrm{d}t + \sum_{j=1}^m\sum_{i=1}^N I_{ij}(u^i(t_j)v^i(t_j)) \\ &\quad + \int_0^T (\nabla F(t,u(t)), v(t))\,\mathrm{d}t.\end{aligned} \tag{2.3}$$

By Definition 2.1, the weak solutions of problem (1.1) correspond to the critical points of $\varphi$.

**Definition 2.2** ([9]). Let $X$ be a Banach space and $\varphi\colon X\to\mathbb{R}$ a $C^1$-functional. We say that $\varphi$ satisfies the Palais-Smale condition, denoted (PS), if any sequence $(u_n)$ in $X$ such that $\varphi(u_n)$ is bounded and $\varphi'(u_n)\to 0$, admits a convergent subsequence.

**Lemma 2.1.** *If $u\in W_T^{1,p}$, letting $\bar u = \frac{1}{T}\int_0^T u(t)\,\mathrm{d}t$ and $\tilde u(t) = u(t)-\bar u$, we have*

$$\|\tilde u\|_\infty \leqslant T^{1/q}\|\dot u\|_{L^p}, \tag{2.4}$$

*and*

$$\|\tilde u\|_{L^p} \leqslant T\|\dot u\|_{L^p}. \tag{2.5}$$

Proof. Since $\tilde u(t) = u(t)-\bar u$, it is easy to verify that $\int_0^T \tilde u(t)\,\mathrm{d}t = 0$.

Let $\tilde u(\tau) = \frac{1}{T}\int_0^T \tilde u(t)\,\mathrm{d}t = 0$. Using the Hölder inequality, we have

$$|\tilde u(t)| = |\tilde u(\tau) + \int_\tau^t \dot{\tilde u}(s)\,\mathrm{d}s| \leqslant \int_0^T |\dot u(s)|\,\mathrm{d}s \leqslant T^{1/q}\Big(\int_0^T |\dot u(s)|^p\,\mathrm{d}s\Big)^{1/p} = T^{1/q}\|\dot u\|_{L^p}.$$

Thus, (2.4) holds.

It follows from (2.4) that

$$|\tilde{u}(t)|^p \leqslant T^{p/q}\|\dot{\tilde{u}}\|_{L^p}^p.$$

Then

$$\|\tilde{u}(t)\|_{L^p}^p = \int_0^T |\tilde{u}(t)|^p \,\mathrm{d}t \leqslant \int_0^T T^{p/q}\|\dot{\tilde{u}}\|_{L^p}^p \,\mathrm{d}t = T^{1+p/q}\|\dot{\tilde{u}}\|_{L^p}^p = T^p\|\dot{\tilde{u}}\|_{L^p}^p.$$

Thus, (2.5) holds. The proofs are completed. □

## 3. Main results and their proofs

**Theorem 3.1.** *Suppose that* (A) *holds and* $F$, $I_{ij}$ *satisfy the following conditions:*
(I1) *For any* $i \in A$, $j \in B$,

$$I_{ij}(t) \geqslant 0 \quad \forall\, t \in \mathbb{R}; \tag{3.1}$$

(F1) *there exist* $f, g \in L^1([0,T], \mathbb{R}^+)$ *and* $\alpha \in [0, p-1)$ *such that*

$$|\nabla F(t,x)| \leqslant f(t)|x|^\alpha + g(t) \tag{3.2}$$

*for all* $x \in \mathbb{R}^N$ *and a.e.* $t \in [0,T]$;
(F2)

$$\liminf_{|x|\to\infty} |x|^{-q\alpha} \int_0^T F(t,x)\,\mathrm{d}t > \frac{2^{q\alpha}T}{q}\Big(\int_0^T f(t)\,\mathrm{d}t\Big)^q. \tag{3.3}$$

*Then problem* (1.1) *has at least one solution in the sense of Definition* 2.1 *which minimizes the functional* $\varphi$ *on* $W_T^{1,p}$ .

Remark 3.1. When $I_{ij} \equiv 0$, problem (1.1) degenerates to the second-order Hamiltonian system. The conclusion of Theorem 3.1 still holds.

**Theorem 3.2.** *Suppose that* (A) *and* (F1) *hold, and the following conditions are satisfied:*
(I2) *There exist* $a_{ij} > 0$ *and* $\beta_{ij} \in (0,1), \alpha \in [0, p-1)$ *such that*

$$|I_{ij}(t)| \leqslant a_{ij} + b_{ij}|t|^{\alpha\beta_{ij}} \quad \text{for every } t \in \mathbb{R},\ i \in A,\ j \in B; \tag{3.4}$$

(I3) *for any* $i \in A$, $j \in B$,

$$I_{ij}(t)t \leqslant 0 \quad \forall\, t \in \mathbb{R}; \tag{3.5}$$

(F3)
$$\limsup_{|x|\to\infty} |x|^{-q\alpha} \int_0^T F(t,x)\,\mathrm{d}t < -2^{q\alpha+1}T\Big(\int_0^T f(t)\,\mathrm{d}t\Big)^q - \frac{2^{\alpha+1}bmN}{p}, \tag{3.6}$$

*where $b$ is defined in* (3.14).

*Then problem* (1.1) *has at least one solution in $W_T^{1,p}$ in the sense of Definition* 2.1.

Remark 3.2. When $I_{ij} \equiv 0$, problem (1.1) degenerates to the second-order Hamiltonian system. The conclusion of Theorem 3.2 still holds if we replace Hypothesis (F3) by

$$\limsup_{|x|\to\infty} |x|^{-q\alpha} \int_0^T F(t,x)\,\mathrm{d}t < -2^{q\alpha+1}T\Big(\int_0^T f(t)\,\mathrm{d}t\Big)^q. \tag{F3$'$}$$

Introduce the condition

(F1′) there exist $f, g \in L^1([0,T], \mathbb{R}^+)$ such that

$$|\nabla F(t,x)| \leqslant f(t)|x| + g(t) \tag{3.7}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

Zhao and Wu [21], [25] proved the existence of solutions for problem (1.1) when $p = 2$ with no impulse, that is, condition (F1) reduces to linearly bounded gradient condition (F1′). Inspired by this case, we generalize the results.

**Theorem 3.3.** *Suppose that* (A), (I1) *hold, and the following conditions are satisfied:*

(f)
$$\int_0^T f(t)\,\mathrm{d}t < \frac{2^{1-p}T^{-p/q}}{p}; \tag{3.8}$$

(F4) *there exist $f, g \in L^1([0,T], \mathbb{R}^+)$ such that*

$$|\nabla F(t,x)| \leqslant f(t)|x|^{p-1} + g(t) \tag{3.9}$$

*for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$;*

(F5)
$$\liminf_{|x|\to\infty} |x|^{-p} \int_0^T F(t,x)\,\mathrm{d}t > \frac{2^pT^q}{(1 - 2^{p-1}pT^{p/q}\int_0^T f(t)\,\mathrm{d}t)^{q/p}}\Big(\int_0^T f(t)\,\mathrm{d}t\Big)^q. \tag{3.10}$$

*Then problem* (1.1) *has at least one solution in the sense of Definition* 2.1 *which minimizes the functional $\varphi$ on $W_T^{1,p}$.*

Remark 3.3. When $I_{ij} \equiv 0$, problem (1.1) degenerates to the second-order Hamiltonian system. The conclusion of Theorem 3.3 still holds.

**Theorem 3.4.** *Suppose that* (A), (f), (I3), (F4) *hold and the following conditions are satisfied:*

(I4) *There exist* $a_{ij} > 0$ *and* $\beta_{ij} \in (0,1)$, $\gamma \in (0, p-1)$ *such that*

$$|I_{ij}(t)| \leqslant a_{ij} + b_{ij}|t|^{\gamma\beta_{ij}} \quad \text{for every } t \in \mathbb{R},\ i \in A,\ j \in B; \tag{3.11}$$

(F6)

$$\begin{aligned}\limsup_{|x|\to\infty} |x|^{-p} \int_0^T F(t,x)\,\mathrm{d}t < -\Big(\int_0^T f(t)\,\mathrm{d}t\Big)^q \\ \times \Big[\frac{2^pT(1+2^{p-1}T^{p/q}\int_0^T f(t)\,\mathrm{d}t)}{(1-2^{p-1}T^{p/q}\int_0^T f(t)\,\mathrm{d}t)(1-2^{p-1}pT^{p/q}\int_0^T f(t)\,\mathrm{d}t)} + \frac{2^pT(\frac{q}{p})^{q/p}}{q}\Big] \\ - \frac{2^{p-1}(1+2^{p-1}T^{p/q}\int_0^T f(t)\,\mathrm{d}t)qbmN}{1-2^{p-1}T^{p/q}\int_0^T f(t)\,\mathrm{d}t},\end{aligned} \tag{3.12}$$

*where* $b$ *is defined in* (3.14).

*Then problem* (1.1) *has at least one solution in* $W_T^{1,p}$ *in the sense of Definition* 2.1.

Remark 3.4. When $I_{ij} \equiv 0$, problem (1.1) degenerates to the second-order Hamiltonian system. The conclusion of Theorem 3.4 still holds if we replace Hypothesis (F6) by

$$\begin{aligned}\limsup_{|x|\to\infty} |x|^{-p} \int_0^T F(t,x)\,\mathrm{d}t < -\Big(\int_0^T f(t)\,\mathrm{d}t\Big)^q \\ \times \Big[\frac{2^pT(1+2^{p-1}T^{p/q}\int_0^T f(t)\,\mathrm{d}t)}{(1-2^{p-1}T^{p/q}\int_0^T f(t)\,\mathrm{d}t)(1-2^{p-1}pT^{p/q}\int_0^T f(t)\,\mathrm{d}t)} + \frac{2^pT(\frac{q}{p})^{q/p}}{q}\Big].\end{aligned} \tag{F6$'$}$$

For the sake of convenience, we denote

$$M_1 = \int_0^T f(t)\,\mathrm{d}t, \quad M_2 = \int_0^T g(t)\,\mathrm{d}t, \tag{3.13}$$

$$a = \max_{i\in A, j\in B} a_{ij}, \quad b = \max_{i\in A, j\in B} b_{ij}. \tag{3.14}$$

Proof of Theorem 3.1. By (F2), we can choose an $a_1 > T^{1/q}$ such that

$$\liminf_{|x|\to\infty} |x|^{-q\alpha} \int_0^T F(t,x)\,\mathrm{d}t > \frac{2^{q\alpha}a_1^q}{q} M_1^q. \tag{3.15}$$

It follows from (2.4), (2.5) and the Young inequality that

$$
\begin{aligned}
(3.16)\qquad &\Big|\int_0^T (F(t,u(t)) - F(t,\bar u))\,\mathrm{d}t\Big| \\
&= \Big|\int_0^T\int_0^1 (\nabla F(t,\bar u + s\tilde u(t)), \tilde u(t))\,\mathrm{d}s\,\mathrm{d}t\Big| \\
&\leqslant \int_0^T\int_0^1 f(t)|\bar u + s\tilde u(t)|^\alpha|\tilde u(t)|\,\mathrm{d}s\,\mathrm{d}t + \int_0^T\int_0^1 g(t)|\tilde u(t)|\,\mathrm{d}s\,\mathrm{d}t \\
&\leqslant 2^\alpha\int_0^T f(t)(|\bar u|^\alpha + |\tilde u(t)|^\alpha)|\tilde u(t)|\,\mathrm{d}t + \int_0^T g(t)|\tilde u|\,\mathrm{d}t \\
&\leqslant 2^\alpha(|\bar u|^\alpha\|\tilde u\|_\infty + \|\tilde u\|_\infty^{\alpha+1})\int_0^T f(t)\,\mathrm{d}t + \|\tilde u\|_\infty\int_0^T g(t)\,\mathrm{d}t \\
&= 2^\alpha M_1|\bar u|^\alpha\|\tilde u\|_\infty + 2^\alpha M_1\|\tilde u\|_\infty^{\alpha+1} + M_2\|\tilde u\|_\infty \\
&\leqslant \frac{1}{pa_1^p}\|\tilde u\|_\infty^p + \frac{2^{q\alpha}a_1^q}{q}M_1^q|\bar u|^{q\alpha} + 2^\alpha M_1\|\tilde u\|_\infty^{\alpha+1} + M_2\|\tilde u\|_\infty \\
&\leqslant \frac{T^{p/q}}{pa_1^p}\|\dot u\|_{L^p}^p + \frac{2^{q\alpha}a_1^q}{q}M_1^q|\bar u|^{q\alpha} + 2^\alpha T^{(\alpha+1)/q}M_1\|\dot u\|_{L^p}^{\alpha+1} + T^{1/q}M_2\|\dot u\|_{L^p}.
\end{aligned}
$$

Hence, we have by (I1) and (3.16)

$$
\begin{aligned}
(3.17)\qquad \varphi(u) &= \frac{1}{p}\|\dot u\|_{L^p}^p + \int_0^T [F(t,u(t)) - F(t,\bar u)]\,\mathrm{d}t + \int_0^T F(t,\bar u)\,\mathrm{d}t + \varphi(u) \\
&\geqslant \Big(\frac{1}{p} - \frac{T^{p/q}}{pa_1^p}\Big)\|\dot u\|_{L^p}^p - 2^\alpha T^{(\alpha+1)/q}M_1\|\dot u\|_{L^p}^{\alpha+1} - T^{1/q}M_2\|\dot u\|_{L^p} \\
&\quad + (|\bar u|^p)^{q\alpha/p}\Big(|\bar u|^{-q\alpha}\int_0^T F(t,\bar u)\,\mathrm{d}t - \frac{2^{q\alpha}a_1^q}{q}M_1^q\Big).
\end{aligned}
$$

In the Sobolev space $W_T^{1,p}$, for $u \in W_T^{1,p}$, we have $\|u\| \to \infty$ if and only if $(|\bar u|^p + \|\dot u\|_{L^p}^p)^{1/p} \to \infty$; (F2) and (3.17) show that $\varphi(u) \to \infty$ as $\|u\| \to \infty$. Similarly to the proof of Lemma 3.1 in [28], $\varphi$ is weakly lower semi-continuous on $W_T^{1,p}$, and by Theorem 1.1 and Corollary 1.1 in [8], $\varphi$ has a minimum point on $W_T^{1,p}$, which is a critical point of $\varphi$. Thus we complete the proof of Theorem 3.1. □

P r o o f of Theorem 3.2. Suppose that $\{u_n\} \subset W_T^{1,p}$ is a (PS) sequence of $\varphi$, that is $\varphi'(u_n) \to 0$ as $n \to \infty$ and $\{\varphi(u_n)\}$ is bounded. By (F3), we can choose an $a_2 > T^{1/q}$ such that

$$
(3.18)\qquad \limsup_{|x|\to\infty} |x|^{-q\alpha}\int_0^T F(t,x)\,\mathrm{d}t < -2^{q\alpha}a_2^qM_1^q - \frac{2^\alpha bmN}{p}.
$$

In a way similar to the proof of Theorem 3.1, we have

$$\left|\int_0^T (\nabla F(t,u_n(t)),\tilde u_n(t))\,\mathrm{d}t\right| \leqslant \frac{T^{p/q}}{pa_2^p}\|\dot u_n\|_{L^p}^p + \frac{2^{q\alpha}a_2^q}{q}M_1^q|\bar u_n|^{q\alpha} + 2^\alpha T^{(\alpha+1)/q}M_1\|\dot u_n\|_{L^p}^{\alpha+1} + T^{1/q}M_2\|\dot u_n\|_{L^p}.$$

Hence, we get

$$\begin{aligned}
(3.19)\quad \|\tilde u_n\| &\geqslant \langle \varphi'(u_n),\tilde u_n\rangle \\
&= \|\dot u_n\|_{L^p}^p + \int_0^T (\nabla F(t,u_n(t)),\tilde u_n(t))\,\mathrm{d}t + \sum_{j=1}^m\sum_{i=1}^N I_{ij}(u_n^i(t))\tilde u_n^i(t) \\
&\geqslant \Big(1-\frac{T^{p/q}}{pa_2^p}\Big)\|\dot u_n\|_{L^p}^p - \frac{2^{q\alpha}a_2^q}{q}M_1^q|\bar u_n|^{q\alpha} - 2^\alpha T^{(\alpha+1)/q}M_1\|\dot u_n\|_{L^p}^{\alpha+1} \\
&\quad - T^{1/q}M_2\|\dot u_n\|_{L^p} - \sum_{j=1}^m\sum_{i=1}^N (a_{ij}+b_{ij}|u_n^i(t)|^{\alpha\beta_{ij}})|\tilde u_n^i(t)| \\
&= \Big(1-\frac{T^{p/q}}{pa_2^p}\Big)\|\dot u_n\|_{L^p}^p - \frac{2^{q\alpha}a_2^q}{q}M_1^q|\bar u_n|^{q\alpha} - 2^\alpha T^{(\alpha+1)/q}M_1\|\dot u_n\|_{L^p}^{\alpha+1} \\
&\quad - T^{1/q}M_2\|\dot u_n\|_{L^p} - \sum_{j=1}^m\sum_{i=1}^N (a_{ij}+b_{ij}|\bar u_n^i(t)+\tilde u_n^i(t)|^{\alpha\beta_{ij}})|\tilde u_n^i(t)| \\
&\geqslant \Big(1-\frac{T^{p/q}}{pa_2^p}\Big)\|\dot u_n\|_{L^p}^p - \frac{2^{q\alpha}a_2^q}{q}M_1^q|\bar u_n|^{q\alpha} - 2^\alpha T^{(\alpha+1)/q}M_1\|\dot u_n\|_{L^p}^{\alpha+1} \\
&\quad - T^{1/q}M_2\|\dot u_n\|_{L^p} - amN\|\tilde u_n\|_\infty - b\sum_{j=1}^m\sum_{i=1}^N 2^\alpha(|\bar u_n|^{\alpha\beta_{ij}} + \|\tilde u_n\|_\infty^{\alpha\beta_{ij}})\|\tilde u_n\|_\infty \\
&\geqslant \Big(1-\frac{T^{p/q}}{pa_2^p}\Big)\|\dot u_n\|_{L^p}^p - \frac{2^{q\alpha}a_2^q}{q}M_1^q|\bar u_n|^{q\alpha} - 2^\alpha T^{(\alpha+1)/q}M_1\|\dot u_n\|_{L^p}^{\alpha+1} \\
&\quad - T^{1/q}M_2\|\dot u_n\|_{L^p} - amNT^{1/q}\|\dot u_n\|_{L^p} - 2^\alpha b\sum_{j=1}^m\sum_{i=1}^N \frac{\beta_{ij}}{q}|\bar u_n|^{q\alpha} \\
&\quad - 2^\alpha b\sum_{j=1}^m\sum_{i=1}^N \frac{q-\beta_{ij}}{q}\|\tilde u_n\|_\infty^{q/(q-\beta_{ij})} - 2^\alpha b\sum_{j=1}^m\sum_{i=1}^N \|\tilde u_n\|_\infty^{\alpha\beta_{ij}+1} \\
&\geqslant \Big(1-\frac{T^{p/q}}{pa_2^p}\Big)\|\dot u_n\|_{L^p}^p - \frac{2^{q\alpha}a_2^q}{q}M_1^q|\bar u_n|^{q\alpha} - 2^\alpha T^{(\alpha+1)/q}M_1\|\dot u_n\|_{L^p}^{\alpha+1} \\
&\quad - T^{1/q}M_2\|\dot u_n\|_{L^p} - amNT^{1/q}\|\dot u_n\|_{L^p} - \frac{2^\alpha}{q}bmN|\bar u_n|^{q\alpha} \\
&\quad - b\sum_{j=1}^m\sum_{i=1}^N \frac{2^\alpha(q-\beta_{ij})}{q}T^{1/(q-\beta_{ij})}\|\dot u_n\|_{L^p}^{q/(q-\beta_{ij})} \\
&\quad - 2^\alpha b\sum_{j=1}^m\sum_{i=1}^N T^{(\alpha\beta_{ij}+1)/q}\|\dot u_n\|_{L^p}^{\alpha\beta_{ij}+1}.
\end{aligned}$$

On the other hand, by (2.5) we have

$$\|\tilde{u}_n\| \leqslant (1+T^p)^{1/p}\|\dot{u}_n\|_{L^p}. \tag{3.20}$$

We have by (I2), (3.19), and (3.20)

$$\begin{aligned}\Big(\frac{2^{q\alpha}a_2^q}{q}M_1^q + \frac{2^\alpha}{q}bmN\Big)|\bar{u}_n|^{q\alpha} &\geqslant \Big(1-\frac{T^{p/q}}{pa_2^p}\Big)\|\dot{u}_n\|_{L^p}^p - T^{(\alpha+1)/q}M_1\|\dot{u}_n\|_{L^p}^{\alpha+1}\\ &- [(1+T^p)^{1/p}+T^{1/q}M_2]\|\dot{u}_n\|_{L^p}\\ &- b\sum_{j=1}^m\sum_{i=1}^N \frac{2(q-\beta_{ij})}{q}T^{1/(q-\beta_{ij})}\|\dot{u}_n\|_{L^p}^{q/(q-\beta_{ij})}\\ &- 2b\sum_{j=1}^m\sum_{i=1}^N T^{(\alpha\beta_{ij}+1)/q}\|\dot{u}_n\|_{L^p}^{\alpha\beta_{ij}+1} \geqslant \frac{1}{q}\|\dot{u}_n\|_{L^p}^p + C_1,\end{aligned} \tag{3.21}$$

where

$$\begin{aligned}C_1 &= \min_{s\in[0,\infty)}\{G(s)\},\\ G(s) &= \frac{a_2^p - T^{p/q}}{pa_2^p}s^p - T^{(\alpha+1)/q}M_1 s^{\alpha+1} - [(1+T^p)^{1/p}+T^{1/q}M_2]s\\ &- b\sum_{j=1}^m\sum_{i=1}^N \frac{2(q-\beta_{ij})}{q}T^{1/(q-\beta_{ij})}s^{q/(q-\beta_{ij})} - 2b\sum_{j=1}^m\sum_{i=1}^N T^{(\alpha\beta_{ij}+1)/q}s^{\alpha\beta_{ij}+1}.\end{aligned}$$

The fact that $a_2 > T^{1/q}$ implies that $-\infty < C_1 < 0$. So it follows from (3.21) that

$$\|\dot{u}_n\|_{L^p}^p \leqslant (2^{q\alpha}a_2^qM_1^q + 2^\alpha bmN)|\bar{u}_n|^{q\alpha} - qC_1, \tag{3.22}$$

and so

$$\|\dot{u}_n\|_{L^p} \leqslant (2^{q\alpha}a_2^qM_1^q + 2^\alpha bmN)^{1/p}|\bar{u}_n|^{q\alpha/p} + C_2, \tag{3.23}$$

where $C_2 > 0$. By the proof of Theorem 3.1, we have

$$\begin{aligned}\Big|\int_0^T (F(t,u_n(t)) - F(t,\bar{u}_n))\,\mathrm{d}t\Big| &= M_1|\bar{u}_n|^\alpha\|\tilde{u}_n\|_{L^p} + M_1\|\tilde{u}_n\|_\infty^{\alpha+1} + M_2\|\tilde{u}_n\|_\infty\\ &\leqslant \frac{1}{pa_2^p}\|\tilde{u}_n\|_\infty^p + \frac{2^{q\alpha}a_2^q}{q}M_1^q|\bar{u}_n|^{q\alpha} + M_1\|\tilde{u}_n\|_\infty^{\alpha+1} + M_2\|\tilde{u}_n\|_\infty\\ &\leqslant \frac{T^{p/q}}{pa_2^p}\|\dot{u}_n\|_{L^p}^p + \frac{2^{q\alpha}a_2^q}{q}M_1^q|\bar{u}_n|^{q\alpha} + T^{(\alpha+1)/q}M_1\|\dot{u}_n\|_{L^p}^{\alpha+1}\\ &+ T^{1/q}M_2\|\dot{u}_n\|_{L^p}.\end{aligned} \tag{3.24}$$

It follows from the boundedness of $\varphi(u_n)$, (3.22), (3.23), (3.24), and (I3) that

$$
\begin{aligned}
C_3 &\leqslant \varphi(u_n)\\
&= \frac{1}{p}\|\dot{u}_n\|_{L^p}^p + \int_0^T [F(t,u_n(t)) - F(t,\bar{u}_n)]\,\mathrm{d}t + \int_0^T F(t,\bar{u}_n)\,\mathrm{d}t + \varphi(u_n(t))\\
&\leqslant \Big(\frac{1}{p} + \frac{T^{p/q}}{pa_2^p}\Big)\|\dot{u}_n\|_{L^p}^p + \frac{2^{q\alpha}a_2^q}{q}M_1^q|\bar{u}|^{q\alpha} + T^{(\alpha+1)/q}M_1\|\dot{u}_n\|_{L^p}^{\alpha+1}\\
&\quad + T^{1/q}M_2\|\dot{u}_n\|_{L^p} + \int_0^T F(t,\bar{u}_n)\,\mathrm{d}t + \sum_{j=1}^{m}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(t)\,\mathrm{d}t\\
&\leqslant \Big(\frac{1}{p} + \frac{T^{p/q}}{pa_2^p}\Big)[(2^{q\alpha}a_2^qM_1^q + 2^\alpha bmN)|\bar{u}_n|^{q\alpha} - qC_1] + \frac{2^{q\alpha}a_2^q}{q}M_1^q|\bar{u}|^{q\alpha}\\
&\quad + T^{(\alpha+1)/q}M_1[(2^{q\alpha}a_2^qM_1^q + 2^\alpha bmN)^{1/p}|\bar{u}_n|^{q\alpha/p} + C_2]^{\alpha+1}\\
&\quad + T^{1/q}M_2[(2^{q\alpha}a_2^qM_1^q + 2^\alpha bmN)^{1/p}|\bar{u}_n|^{q\alpha/p} + C_2] + \int_0^T F(t,\bar{u}_n)\,\mathrm{d}t\\
&\leqslant \Big(2^{q\alpha+1}a_2^qM_1^q + \frac{2^{\alpha+1}bmN}{p}\Big)|\bar{u}_n|^{q\alpha}\\
&\quad + T^{(\alpha+1)/q}M_1[(2^{q\alpha}a_2^qM_1^q + 2^\alpha bmN)^{1/p}|\bar{u}_n|^{q\alpha/p} + C_2]^{\alpha+1}\\
&\quad + T^{1/q}M_2[(2^{q\alpha}a_2^qM_1q^q + 2^\alpha bmN)^{1/p}|\bar{u}_n|^{q\alpha/p} + C_2] - \Big(\frac{1}{p} + \frac{T^{p/q}}{pa_2^p}\Big)qC_1\\
&\quad + \int_0^T F(t,\bar{u}_n)\,\mathrm{d}t\\
&\leqslant \Big(2^{q\alpha+1}a_2^qM_1^q + \frac{2^{\alpha+1}bmN}{p}\Big)|\bar{u}_n|^{q\alpha}\\
&\quad + T^{(\alpha+1)/q}M_1[2^{\alpha+1}(2^\alpha a_2^qM_1^q + 2^\alpha bmN)^{(\alpha+1)/p}|\bar{u}_n|^{q\alpha(\alpha+1)/p} + 2^{\alpha+1}C_2^{\alpha+1}]\\
&\quad + T^{1/q}M_2[(2^{q\alpha}a_2^qM_1^q + 2^\alpha bmN)^{1/p}|\bar{u}_n|^{q\alpha/p} + C_2] - \Big(\frac{1}{p} + \frac{T^{p/q}}{pa_2^p}\Big)qC_1\\
&\quad + \int_0^T F(t,\bar{u}_n)\,\mathrm{d}t\\
&\leqslant |\bar{u}_n|^{q\alpha}[|\bar{u}_n|^{-q\alpha}\int_0^T F(t,\bar{u}_n)\,\mathrm{d}t + \Big(2^{q\alpha+1}a_2^qM_1^q + \frac{2^{\alpha+1}bmN}{p}\Big)\\
&\quad + T^{(\alpha+1)/q}M_1 2^\alpha(a_2^qM_1^q + 2bmN)^{(\alpha+1)/p}|\bar{u}_n|^{\alpha(p-q\alpha)/p}\\
&\quad + T^{1/q}M_2[(a_2^qM_1^q + 2bmN)^{1/p}|\bar{u}_n|^{-\alpha}]\\
&\quad + T^{(\alpha+1)/q}2^\alpha M_2C_2^{\alpha+1} + T^{1/q}M_3C_2 - \Big(\frac{1}{p} + \frac{T^p}{pa_2^p}\Big)qC_1.
\end{aligned}
$$

The above inequality and (3.20) imply that $\{\bar{u}_n\}$ is bounded. Hence, $\{u_n\}$ is bounded by (3.24). Since $W_T^{1,p}$ is a reflexive Banach space, the boundedness and weak compactness are equivalent, and passing if necessary to a subsequence, we can assume that

$$u_n \rightharpoonup u_0 \text{ in } W_T^{1,p}. \tag{3.25}$$

By Proposition 1.2 in [8], we have

$$u_n \to u_0 \text{ in } C([0,T],\mathbb{R}^N). \tag{3.26}$$

It follows from (2.3) and the Hölder inequality that

$$\begin{aligned}
(3.27)\quad & \langle \psi'(u_n) - \psi'(u_0), u_n - u_0 \rangle \\
&= \int_0^T |\dot u_n(t)|^{p-2}(\dot u_n(t), \dot u_n(t) - \dot u_0(t))\,\mathrm{d}t \\
&\quad - \int_0^T |\dot u_0(t)|^{p-2}(\dot u_0(t), \dot u_n(t) - \dot u_0(t))\,\mathrm{d}t \\
&\quad - \int_0^T (\nabla F(t,u_n(t)) - \nabla F(t,u_0(t)), u_n(t) - u_0(t))\,\mathrm{d}t \\
&= \|u_n\|^p + \|u_0\|^p \\
&\quad - \int_0^T |\dot u_n(t)|^{p-2}(\dot u_n(t), \dot u_0(t))\,\mathrm{d}t - \int_0^T |\dot u_0(t)|^{p-2}(\dot u_0(t), \dot u_k(t))\,\mathrm{d}t \\
&\quad - \int_0^T (\nabla F(t,u_n(t)) - \nabla F(t,u_0(t)), u_n(t) - u_0(t))\,\mathrm{d}t \\
&\geqslant \|u_n\|^p + \|u_0\|^p - \int_0^T |\dot u_n(t)|^{p-1}|\dot u_0(t)|\,\mathrm{d}t - \int_0^T |\dot u_0(t)|^{p-1}|\dot u_n(t)|\,\mathrm{d}t \\
&\quad - \int_0^T (\nabla F(t,u_n(t)) - \nabla F(t,u_0(t)), u_n(t) - u_0(t))\,\mathrm{d}t \\
&\geqslant \|u_n\|^p + \|u_0\|^p - \left(\int_0^T |\dot u_0(t)|^p\,\mathrm{d}t\right)^{1/p}\left(\int_0^T |\dot u_n(t)|^p\,\mathrm{d}t\right)^{1/q} \\
&\quad - \left(\int_0^T |\dot u_n(t)|^p\,\mathrm{d}t\right)^{1/p}\left(\int_0^T |\dot u_0(t)|^p\,\mathrm{d}t\right)^{1/q} \\
&\quad - \int_0^T (\nabla F(t,u_n(t)) - \nabla F(t,u_0(t)), u_n(t) - u_0(t))\,\mathrm{d}t \\
&\geqslant \|u_n\|^p + \|u_0\|^p \\
&\quad - \left(\int_0^T [|\dot u_0(t)|^p + |u_0(t)|^p]\,\mathrm{d}t\right)^{1/p}\left(\int_0^T [|\dot u_n(t)|^p + |u_n(t)|^p\,\mathrm{d}t]\,\mathrm{d}t\right)^{1/q} \\
&\quad - \left(\int_0^T [|\dot u_n(t)|^p + |u_n(t)|^p]\,\mathrm{d}t\right)^{1/p}\left(\int_0^T [|\dot u_0(t)|^p + |u_0(t)|^p]\,\mathrm{d}t\right)^{1/q} \\
&\quad - \int_0^T (\nabla F(t,u_n(t)) - \nabla F(t,u_0(t)), u_n(t) - u_0(t))\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \|u_n\|^p + \|u_0\|^p - \|u_0\|\|u_n\|^{p-1} - \|u_n\|\|u_0\|^{p-1} \\
&\quad - \int_0^T (\nabla F(t,u_n(t)) - \nabla F(t,u_0(t)), u_n(t) - u_0(t))\,\mathrm{d}t \\
&= (\|u_n\|^{p-1} - \|u_0\|^{p-1})(\|u_n\| - \|u_0\|) \\
&\quad - \int_0^T (\nabla F(t,u_n(t)) - \nabla F(t,u_0(t)), u_n(t) - u_0(t))\,\mathrm{d}t.
\end{aligned}
$$

It follows from (2.3) and (3.27) that

$$
\begin{aligned}
\langle \psi'(u_n) - \psi'(u_0), u_n - u_0\rangle &\geqslant (\|u_n\|^{p-1} - \|u_0\|^{p-1})(\|u_n\| - \|u_0\|) \\
&\quad - \int_0^T (\nabla F(t,u_n(t)) - \nabla F(t,u_0(t)), u_n(t) - u_0(t))\,\mathrm{d}t \\
&\quad - \sum_{j=1}^m \sum_{i=1}^N (I_{ij}(u_n^i(t_j)) - I_{ij}(u^i(t_j)))(u_n^i(t_j)) - u^i(t_j))).
\end{aligned} \tag{3.28}
$$

From (3.25)–(3.28), (A) and the continuity of $I_{ij}$, it follows that $\|u_n\| \to \|u\|$ in $W_T^{1,p}$. Thus, $\varphi$ satisfies the (PS) condition (see Definition 2.2).

In order to use the saddle point theorem ([12], Theorem 4.6]), we only need to verify the following conditions:

($A_1$) $\varphi(x) \to -\infty$ as $|x| \to \infty$ in $\mathbb{R}^N$.

($A_2$) $\varphi(u) \to \infty$ as $\|u\| \to \infty$ in $\widetilde{W}_T^{1,p}$, where $\widetilde{W}_T^{1,p} = \{u \in W_T^{1,p} \mid \bar{u} = 0\}$.

In fact, by (3.6), we get

$$
\int_0^T F(t,x)\,\mathrm{d}t \to -\infty \quad \text{as } |x| \to \infty \text{ in } \mathbb{R}^N. \tag{3.29}
$$

From (I3), (2.2), and (3.29), we have

$$
\varphi(x) = \int_0^T F(t,x)\,\mathrm{d}t + \varphi(x) \to -\infty \quad \text{as } |x| \to \infty \text{ in } \mathbb{R}^N.
$$

Thus ($A_1$) is easy to verify.

Next, for all $u \in \widetilde{W}_T^{1,p}$, by (F1) and Sobolev's inequality we have

$$
\begin{aligned}
\left|\int_0^T [F(t,u(t)) - F(t,0)]\,\mathrm{d}t\right| &= \left|\int_0^T\int_0^1 (\nabla F(t,su(t)), u(t))\,\mathrm{d}s\,\mathrm{d}t\right| \\
&\leqslant \int_0^T f(t)|u(t)|^{\alpha+1}\,\mathrm{d}t + \int_0^T g(t)|u(t)|\,\mathrm{d}t \\
&\leqslant M_1\|u\|_\infty^{\alpha+1} + M_2\|u\|_\infty \leqslant T^{(\alpha+1)/q} M_1 \|\dot{u}\|_{L^p}^{\alpha+1} + T^{1/q} M_2 \|\dot{u}\|_{L^p}.
\end{aligned} \tag{3.30}
$$

From (I2) we derive that

$$(3.31)\qquad \begin{aligned} |\varphi(u)| &= \Big|\sum_{j=1}^{m}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(t)\,\mathrm{d}t\Big| \\ &\leqslant \sum_{j=1}^{m}\sum_{i=1}^{N}\int_0^{u^i(t_j)} (a_{ij}+b_{ij}|t|^{\alpha\beta_{ij}})\,\mathrm{d}t \\ &\leqslant amN\|u\|_\infty + b\sum_{j=1}^{m}\sum_{i=1}^{N}\|u\|_\infty^{\alpha\beta_{ij}+1} \\ &\leqslant amNT^{1/q}\|\dot u(t)\|_{L^p} + b\sum_{j=1}^{m}\sum_{i=1}^{N}T^{(\alpha\beta_{ij}+1)/q}\|\dot u(t)\|_{L^p}^{(\alpha\beta_{ij}+1)/q}. \end{aligned}$$

It follows from (2.2), (3.30), and (3.31) that

$$(3.32)\qquad \begin{aligned} \varphi(u) &= \frac{1}{p}\|\dot u\|_{L^p}^p + \int_0^T [F(t,u(t))-F(t,0)]\,\mathrm{d}t + \int_0^T F(t,0)\,\mathrm{d}t + \varphi(u) \\ &\geqslant \frac{1}{p}\|\dot u\|_{L^p}^p - T^{(\alpha+1)/q}M_1\|\dot u\|_{L^p}^{\alpha+1} - T^{1/q}M_2\|\dot u\|_{L^p} \\ &\quad - amNT^{1/q}\|\dot u\|_{L^p} - b\sum_{j=1}^{m}\sum_{i=1}^{N}T^{(\alpha\beta_{ij}+1)/q}\|\dot u\|_{L^p}^{(\alpha\beta_{ij}+1)/q} \\ &\quad + \int_0^T F(t,0)\,\mathrm{d}t \end{aligned}$$

for all $u\in\widetilde{W}_T^{1,p}$. By Lemma 2.1, $\|u\|\to\infty$ in $\widetilde{W}_T^{1,p}$ if and only if $\|\dot u\|_{L^p}\to\infty$. So we obtain $\varphi(u)\to\infty$ as $\|u\|\to\infty$ in $\widetilde{W}_T^{1,p}$ from (3.32), i.e. ($A_2$) is verified. The proof of Theorem 3.2 is complete. □

P r o o f of Theorem 3.3. By (f) and (F5), we can choose an $a_3\in\mathbb{R}$ such that

$$(3.33)\qquad a_3 > \frac{T^{1/q}}{(1-2^{p-1}pM_1T^{p/q})^{1/p}} > 0$$

and

$$(3.34)\qquad \liminf_{|x|\to\infty}|x|^{-p}\int_0^T F(t,x)\,\mathrm{d}t > \frac{2^p a_3^q}{q}M_1^q.$$

It follows from (F4), Lemma 2.1 and the Young inequality that

$$\begin{aligned}\Big|\int_0^T (F(t,u(t)) - F(t,\bar u))\,\mathrm{d}t\Big| &= \Big|\int_0^T\int_0^1 (\nabla F(t,\bar u + s\tilde u(t)), \tilde u(t))\,\mathrm{d}s\,\mathrm{d}t\Big|\\ &\leqslant \int_0^T\int_0^1 f(t)|\bar u + s\tilde u(t)|^{p-1}|\tilde u(t)|\,\mathrm{d}s\,\mathrm{d}t + \int_0^T\int_0^1 g(t)|\tilde u(t)|\,\mathrm{d}s\,\mathrm{d}t\\ &\leqslant 2^{p-1}\int_0^T f(t)(|\bar u|^{p-1} + |\tilde u(t)|^{p-1})|\tilde u(t)|\,\mathrm{d}t + \int_0^T g(t)|\tilde u(t)|\,\mathrm{d}t\\ &\leqslant 2^{p-1}(|\bar u|^{p-1}\|\tilde u\|_\infty + \|\tilde u\|_\infty^p)\int_0^T f(t)\,\mathrm{d}t + \|\tilde u\|_\infty\int_0^T g(t)\,\mathrm{d}t\\ &= 2^{p-1}M_1|\bar u|^{p-1}\|\tilde u\|_\infty + 2^{p-1}M_1\|\tilde u\|_\infty^p + M_2\|\tilde u\|_\infty\\ &\leqslant \frac{1}{pa_3^p}\|\tilde u\|_\infty^p + \frac{2^p a_3^q}{q}M_1^q|\bar u|^p + 2^{p-1}M_1\|\tilde u\|_\infty^p + M_2\|\tilde u\|_\infty\\ &\leqslant \frac{T^{p/q}}{pa_3^p}\|\dot u\|_{L^p}^p + \frac{2^p a_3^q}{q}M_1^q|\bar u|^p + 2^{p-1}T^{p/q}M_1\|\dot u\|_{L^p}^p + T^{1/q}M_2\|\dot u\|_{L^p}\\ &= \Big(\frac{T^{p/q}}{pa_3^p} + 2^{p-1}T^{p/q}M_1\Big)\|\dot u\|_{L^p}^p + \frac{2^p a_3^q}{q}M_1^q|\bar u|^p + T^{1/q}M_2\|\dot u\|_{L^p}.\end{aligned}$$

Hence, we have

$$\begin{aligned}\varphi(u) &= \frac{1}{p}\|\dot u\|_{L^p}^p + \int_0^T [F(t,u(t)) - F(t,\bar u)]\,\mathrm{d}t + \int_0^T F(t,\bar u)\,\mathrm{d}t + \varphi(u)\\ &\geqslant \Big(\frac{1}{p} - \frac{T^{p/q}}{pa_3^p} - 2^{p-1}T^{p/q}M_1\Big)\|\dot u\|_{L^p}^p - T^{1/q}M_2\|\dot u\|_{L^p}\\ &\quad + |\bar u|^p\Big(|\bar u|^{-p}\int_0^T F(t,\bar u)\,\mathrm{d}t - \frac{2^p a_3^q}{q}M_1^q\Big).\end{aligned} \tag{3.35}$$

As $\|u\| \to \infty$ if and only if $(|\bar u|^p + \|\dot u\|_{L^p})^{1/p} \to \infty$, the above inequality implies that $\varphi(u) \to \infty$ as $\|u\| \to \infty$. Similarly to the proof of Theorem 3.1, $\varphi$ has a minimum point on $W_T^{1,p}$, which is a critical point of $\varphi$. The proof of Theorem 3.3 is complete. □

P r o o f of Theorem 3.4. First we prove that $\varphi$ satisfies the (PS) condition. Suppose that $\{u_n\} \subset W_T^{1,p}$ is a (PS) sequence of $\varphi$, that is $\varphi'(u_n) \to 0$ as $n \to \infty$ and $\{\varphi(u_n)\}$ is bounded. By (f) and (F6), we can choose an $a_4 \in \mathbb{R}$ such that

$$a_4 > \frac{T^{1/q}}{(1 - 2^{p-1}pM_1T^{p/q})^{1/p}} > 0, \tag{3.36}$$

and

$$\limsup_{|x|\to\infty} |x|^{-p} \int_0^T F(t,x)\,\mathrm{d}t \tag{3.37}$$
$$< -\Big[\frac{2^pT(1+2^{p-1}T^{p/q}M_1)}{(1-2^{p-1}T^{p/q}M_1)(1-2^{p-1}pT^{p/q}M_1)} + \frac{2^pT(\frac{q}{p})^{q/p}}{q}\Big]M_1^q - \frac{2^{p-1}(1+2^{p-1}T^{p/q}M_1)qbmN}{1-2^{p-1}T^{p/q}M_1}.$$

In a way similar to the proof of Theorem 3.3, we have

$$\Big|\int_0^T (\nabla F(t,u_n(t)),\tilde{u}(t))\,\mathrm{d}t\Big| \leqslant \Big(\frac{T^{p/q}}{pa_4^p} + \Big(1+\frac{1}{q}\Big)2^{p-1}T^{p/q}M_1\Big)\|\dot{u}\|_{L^p}^p + \frac{2^pa_4^q}{q}M_1^q|\bar{u}|^p + T^{1/q}M_2\|\dot{u}\|_{L^p}.$$

By (I4), we have

$$\begin{aligned} \|\tilde{u}_n\| &\geqslant \langle\varphi'(u_n),\tilde{u}_n\rangle \\ &= \|\dot{u}_n\|_{L^p}^p + \int_0^T (\nabla F(t,u_n(t)),\tilde{u}_n(t))\,\mathrm{d}t + \sum_{j=1}^m\sum_{i=1}^N I_{ij}(u_n^i(t))\tilde{u}_n^i(t) \\ &\geqslant \Big(1-\frac{T^{p/q}}{pa_4^p} - \Big(1+\frac{1}{q}\Big)2^{p-1}T^{p/q}M_1\Big)\|\dot{u}_n\|_{L^p}^p - \frac{2^pa_4^q}{q}M_1^q|\bar{u}_n|^p \\ &\quad - T^{1/q}M_2\|\dot{u}_n\|_{L^p} - \sum_{j=1}^m\sum_{i=1}^N (a_{ij}+b_{ij}|u_n^i(t)|^{\gamma\beta_{ij}})\tilde{u}_n^i(t) \\ &\geqslant \Big(1-\frac{T^{p/q}}{pa_4^p} - \Big(1+\frac{1}{q}\Big)2^{p-1}T^{p/q}M_1\Big)\|\dot{u}_n\|_{L^p}^p - \frac{2^pa_4^q}{q}M_1^q|\bar{u}_n|^p \\ &\quad - T^{1/q}M_2\|\dot{u}_n\|_{L^p} - \sum_{j=1}^m\sum_{i=1}^N (a_{ij}+b_{ij}|\bar{u}_n^i(t)+\tilde{u}_n^i(t)|^{\gamma\beta_{ij}})|\tilde{u}_n^i(t)| \\ &\geqslant \Big(1-\frac{T^{p/q}}{pa_4^p} - \Big(1+\frac{1}{q}\Big)2^{\gamma}T^{p/q}M_1\Big)\|\dot{u}_n\|_{L^p}^p - \frac{2^pa_4^q}{q}M_1^q|\bar{u}_n|^p \\ &\quad - T^{1/q}M_2\|\dot{u}_n\|_{L^p} - amN\|\tilde{u}_n\|_\infty \\ &\quad - b\sum_{j=1}^m\sum_{i=1}^N 2^{p-1}(|\bar{u}_n|^{\gamma\beta_{ij}} + |\tilde{u}_n|^{\gamma\beta_{ij}})\|\tilde{u}_n\|_\infty \end{aligned} \tag{3.38}$$

$$
\begin{aligned}
&\geqslant \Big(1-\frac{T^{p/q}}{pa_4^p}-\Big(1+\frac{1}{q}\Big)2^{\gamma}T^{p/q}M_1\Big)\|\dot{u}_n\|_{L^p}^p-\frac{2^p a_4^q}{q}M_1^q|\bar{u}_n|^p\\
&\quad -T^{1/q}M_2\|\dot{u}_n\|_{L^p}-amNT^{1/q}\|\dot{u}_n\|_{L^p}-2^{p-1}b\frac{\gamma\beta_{ij}}{p}\sum_{j=1}^{m}\sum_{i=1}^{N}\beta_{ij}|\bar{u}_n|^p\\
&\quad -2^{p-1}b\sum_{j=1}^{m}\sum_{i=1}^{N}\frac{p-\gamma\beta_{ij}}{p}\|\tilde{u}_n\|_\infty^{p/(p-\gamma\beta_{ij})}-2^{p-1}b\sum_{j=1}^{m}\sum_{i=1}^{N}\|\tilde{u}_n\|_\infty^{\gamma\beta_{ij}+1}\\
&\geqslant \Big(1-\frac{T^{p/q}}{pa_4^p}-\Big(1+\frac{1}{q}\Big)2^{\gamma}T^{p/q}M_1\Big)\|\dot{u}_n\|_{L^p}^p-\frac{2^p a_4^q}{q}M_1^q|\bar{u}_n|^p\\
&\quad -T^{1/q}M_2\|\dot{u}_n\|_{L^p}-amNT^{1/q}\|\dot{u}_n\|_{L^p}-\frac{2^{p-1}\gamma}{p}bmN|\bar{u}_n|^p\\
&\quad -2^{p-1}b\sum_{j=1}^{m}\sum_{i=1}^{N}\frac{p-\gamma\beta_{ij}}{p}(T^{p/q}\|\dot{u}_n\|_{L_p}^p)^{1/(p-\gamma\beta_{ij})}\\
&\quad -2^{p-1}b\sum_{j=1}^{m}\sum_{i=1}^{N}(T^{p/q}\|\dot{u}_n\|_{L_p}^p)^{(\gamma\beta_{ij}+1)/p}.
\end{aligned}
$$

On the other hand, we have

$$\|\tilde{u}_n\|\leqslant(1+T^p)^{1/p}\|\dot{u}_n\|_{L^p} \tag{3.39}$$

which, together with (3.38), implies that

$$
\begin{aligned}
\Big(\frac{2^p a_4^q}{q}M_1^q+\frac{2^{p-1}\gamma}{p}bmN\Big)|\bar{u}_n|^p &\geqslant \Big(1-\frac{T^p}{pa_4^p}-\Big(1+\frac{1}{q}\Big)2^{p-1}T^{p/q}M_1\Big)\|\dot{u}_n\|_{L^p}^p\\
&\quad -[(1+T^p)^{1/p}+T^{1/q}M_2+amNT^{1/q}]\|\dot{u}_n\|_{L^p}\\
&\quad -2^{p-1}b\sum_{j=1}^{m}\sum_{i=1}^{N}\frac{p-\gamma\beta_{ij}}{p}(T^{p/q}\|\dot{u}_n\|_{L_p}^p)^{1/(p-\gamma\beta_{ij})}\\
&\quad -2^{p-1}b\sum_{j=1}^{m}\sum_{i=1}^{N}(T^{p/q}\|\dot{u}_n\|_{L_p}^p)^{(\gamma\beta_{ij}+1)/p}\\
&\geqslant \frac{1}{q}(1-2^{p-1}T^{p/q}M_1)\|\dot{u}_n\|_{L^p}^p+C_4,
\end{aligned}
$$

where

$$C_4=\min_{s\in[0,\infty)}\{H(s)\},$$

and

$$H(s) = \Big(\frac{1}{p} - \frac{T^p}{pa_4^p} - 2^{p-1}T^{p/q}M_1\Big)s^p$$
$$- 2^{p-1}b\sum_{j=1}^{m}\sum_{i=1}^{N}\frac{p-\gamma\beta_{ij}}{p}(T^{p/q}s^p)^{1/(p-\gamma\beta_{ij})}$$
$$- 2^{p-1}b\sum_{j=1}^{m}\sum_{i=1}^{N}(T^{p/q}s^p)^{(\gamma\beta_{ij}+1)/p}$$
$$- [(1+T^p)^{1/p} + T^{1/q}M_3^{1/p}M_3 + amNT^{1/q}]s.$$

The fact that

$$a_4 > \frac{T^{1/q}}{(1-2^{p-1}pM_1T^{p/q})^{1/p}} > 0$$

implies that $-\infty < C_4 < 0$. So we obtain

$$\|\dot{u}_n\|_{L^p}^p \leqslant \frac{2^p a_4^q M_1^q + 2^{p-1}qbmN}{1-2^{p-1}T^{p/q}M_1}|\bar{u}_n|^p - \frac{qC_4}{2^p a_4^q M_1^q + 2^{p-1}qbmN}, \tag{3.40}$$

and so

$$\|\dot{u}_n\|_{L^p} \leqslant \frac{\sqrt[p]{2^p a_4^q M_1^q + 2^{p-1}qbmN}}{\sqrt[p]{1-2^{p-1}T^{p/q}M_1}}|\bar{u}_n| + C_5, \tag{3.41}$$

where $C_5 > 0$. By the proof of Theorem 3.3, we have

$$\begin{aligned}\Big|\int_0^T (F(t,u_n(t)) - F(t,\bar{u}_n))\,\mathrm{d}t\Big| \qquad (3.42)\\ &\leqslant 2^{p-1}M_1|\bar{u}_n|^{p-1}\|\tilde{u}_n\|_{L^p} + 2^{p-1}M_1\|\tilde{u}_n\|_\infty^p + M_2\|\tilde{u}_n\|_\infty \\ &\leqslant \Big(\frac{1}{q} + 2^{p-1}T^{p/q}M_1\Big)\|\dot{u}_n\|_{L^p}^p + \frac{2^pT(q/p)^{q/p}}{q}M_1^q|\bar{u}_n|^p \\ &\quad + T^{1/q}M_2\|\dot{u}_n\|_{L^p}.\end{aligned}$$

It follows from the boundedness of $\varphi(u_n)$, (I3), and (3.42) that

$$
\begin{aligned}
C_6 &\leqslant \varphi(u_n)\\
&= \frac{1}{p}\|\dot{u}_n\|_{L^p}^p + \int_0^T [F(t,u_n(t)) - F(t,\bar{u}_n)]\,\mathrm{d}t + \int_0^T F(t,\bar{u}_n)\,\mathrm{d}t + \varphi(u_n)\\
&\leqslant (1+2^{p-1}T^{p/q}M_1)\|\dot{u}_n\|_{L^p}^p + \frac{2^pT(\frac{q}{p})^{q/p}}{q}M_1^q|\bar{u}_n|^p\\
&\quad + T^{1/q}M_2\|\dot{u}_n\|_{L^p} + \int_0^T F(t,\bar{u}_n)\,\mathrm{d}t\\
&\leqslant (1+2^{p-1}T^{p/q}M_1)\Big(\frac{2^pa_4^qM_1^q + 2^{p-1}qbmN}{1-2^{p-1}T^{p/q}M_1}|\bar{u}_n|^p - \frac{qC_4}{2^pa_4^qM_1^q+2^{p-1}qbmN}\Big)\\
&\quad + \frac{2^pT(\frac{q}{p})^{q/p}}{q}M_1^q|\bar{u}_n|^p + T^{1/q}M_2\left(\frac{\sqrt[p]{2^pa_4^qM_1^q+2^{p-1}qbmN}}{\sqrt[p]{1-2^{p-1}T^{p/q}M_1}}|\bar{u}_n| + C_5\right)\\
&\quad + \int_0^T F(t,\bar{u}_n)\,\mathrm{d}t\\
&= \Big[\frac{(1+2^{p-1}T^{p/q}M_1)2^pa_4^q}{1-2^{p-1}T^{p/q}M_1}M_1^q + \frac{2^pT(\frac{q}{p})^{q/p}}{q}M_1^q\\
&\quad + \frac{2^{p-1}(1+2^{p-1}T^{p/q}M_1)qbmN}{1-2^{p-1}T^{p/q}M_1}\Big]|\bar{u}_n|^p\\
&\quad + T^{1/q}M_2\frac{\sqrt[p]{2^pa_4^qM_1^q+2^{p-1}qbmN}}{\sqrt[p]{1-2^{p-1}T^{p/q}M_1}}|\bar{u}_n| - \frac{(1+2^{p-1}T^{p/q}M_1)qC_4}{2^pa_4^qM_1^q+2^{p-1}qbmN}\\
&\quad + T^{1/q}M_2C_5 + \int_0^T F(t,\bar{u}_n)\,\mathrm{d}t\\
&= |\bar{u}_n|^p\bigg\{|\bar{u}_n|^{-p}\int_0^T F(t,\bar{u}_n)\,\mathrm{d}t + \frac{(1+2^{p-1}T^{p/q}M_1)2^pa_4^q}{1-2^{p-1}T^{p/q}M_1}M_1^q + \frac{2^pT(q/p)^{q/p}}{q}M_1^q\\
&\quad + \frac{2^{p-1}(1+2^{p-1}T^{p/q}M_1)qbmN}{1-2^{p-1}T^{p/q}M_1} + T^{1/q}M_2\frac{\sqrt[p]{2^pa_4^qM_1^q+2^{p-1}qbmN}}{\sqrt[p]{1-2^{p-1}T^{p/q}M_1}}|\bar{u}_n|^{1-p}\bigg\}\\
&\quad - \frac{(1+2^{p-1}T^{p/q}M_1)qC_4}{2^pa_4^qM_1^q+2^{p-1}qbmN} + T^{1/q}M_2C_5.
\end{aligned}
$$

The above inequality and (F6) imply that $\{\bar{u}_n\}$ is bounded. Hence, $\{u_n\}$ is bounded. Arguing then as in Proposition 4.1 in [8], we conclude that the (PS) condition is satisfied.

Similarly to the proof of Theorem 3.2, we only need to verify ($A_1$) and ($A_2$). It is easy to verify ($A_1$) by (3.9). In what follows, we verify that ($A_2$) holds as well. For all $u \in \widetilde{W}_T^{1,p}$, by (3.7) and Sobolev's inequality, we have

$$\begin{aligned}
(3.43) \qquad \left|\int_0^T [F(t,u(t)) - F(t,0)]\,\mathrm{d}t\right| &= \left|\int_0^T\int_0^1 (\nabla F(t,su(t)),u(t))\,\mathrm{d}s\,\mathrm{d}t\right| \\
&\leqslant \frac{1}{p}\int_0^T f(t)|u(t)|^p\,\mathrm{d}t + \int_0^T g(t)|u(t)|\,\mathrm{d}t \\
&\leqslant \frac{2^{p-1}M_1}{p}\|u\|_\infty^p + M_2\|u\|_\infty \\
&\leqslant \frac{2^{p-1}T^{p/q}M_1}{p}\|\dot{u}\|_{L^p}^p + T^{1/q}M_2\|\dot{u}\|_{L^p}.
\end{aligned}$$

Like in the proof of Theorem 3.2, we have

$$\begin{aligned}
(3.44) \qquad |\varphi(u)| &= \left|\sum_{j=1}^m\sum_{i=1}^N\int_0^{u^i(t_j)} I_{ij}(t)\,\mathrm{d}t\right| \\
&\leqslant amNT^{1/q}\|\dot{u}\|_{L^p} + b\sum_{j=1}^m\sum_{i=1}^N T^{(\gamma\beta_{ij}+1)/q}\|\dot{u}\|_{L^p}^{(\gamma\beta_{ij}+1)/q}.
\end{aligned}$$

It follows from (2.3), (3.43) and (3.44) that

$$\begin{aligned}
(3.45) \qquad \varphi(u) &= \frac{1}{p}\|\dot{u}\|_{L^p}^p + \int_0^T [F(t,u(t)) - F(t,0)]\,\mathrm{d}t + \int_0^T F(t,0)\,\mathrm{d}t + \varphi(u) \\
&\geqslant \frac{1-2^{p-1}T^{p/q}M_1}{p}\|\dot{u}\|_{L^p}^p - T^{1/q}M_2\|\dot{u}\|_{L^p} + \int_0^T F(t,0)\,\mathrm{d}t \\
&\quad - amNT^{1/q}\|\dot{u}\|_{L^p} - b\sum_{j=1}^m\sum_{i=1}^N T^{(\gamma\beta_{ij}+1)/q}\|\dot{u}\|_{L^p}^{(\gamma\beta_{ij}+1)/q}
\end{aligned}$$

for all $u \in \widetilde{W}_T^{1,p}$. By Wirtinger's inequality, $\|u\| \to \infty$ in $\widetilde{W}_T^{1,p}$ if and only if $\|\dot{u}\|_{L^p} \to \infty$. So we obtain $\varphi(u) \to \infty$ as $\|u\| \to \infty$ in $\widetilde{W}_T^{1,p}$, i.e. ($A_2$) is verified. The proof of Theorem 3.4 is complete. □

## 4. Examples

In this section we give some examples to illustrate our results.

Example 4.1. Let $T=1.4$, $N=3$, $t_1=1$, $p=3/2$, $q=3$, consider the second-order Hamiltonian systems with impulsive effects

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(|\dot u(t)|^{1/2}) = \nabla F(t,u(t)), \quad \text{a.e. } t\in[0,T], \\ u(0)-u(1.4)=\dot u(0)-\dot u(1.4)=0, \\ \Delta \dot u^i(1)=\dot u^i(1^+)-\dot u^i(1^-)=(u^i(1))^{1/3}, \quad i=1,2,3, \end{cases} \tag{4.1}$$

let

$$F(t,x)=\Big(\frac{T}{2}-t\Big)|x|^{10/7}+\Big(\frac{2}{3}T^2-t^2\Big)|x|^{9/7}+(h(t),x), \tag{4.2}$$

$I_{ij}(t)=t^{1/3}$, $\alpha=3/7$. It is easy to see that

$$\begin{aligned} |\nabla F(t,x)| &\leqslant \frac{10}{7}\Big|\frac{T}{2}-t\Big||x|^{3/7}+\frac{9}{7}\Big|\frac{2}{3}T^2-t^2\Big||x|^{3/7}+|h(t)| \\ &\leqslant \frac{10}{7}\Big(\Big|\frac{T}{2}-t\Big|+\varepsilon\Big)|x|^{3/7}+\frac{T^6}{\varepsilon^2}+|h(t)|. \end{aligned}$$

This shows (3.2) holds with $\alpha=3/7$ and

$$f(t)=\frac{10}{7}\Big(\Big|\frac{T}{2}-t\Big|+\varepsilon\Big), \quad g(t)=\frac{T^6}{\varepsilon^2}+|h(t)|, \tag{4.3}$$

and

$$\begin{aligned} \frac{T^3}{3}\Big(\int_0^T f(t)\,\mathrm{d}t\Big)^3 &= \Big(\frac{10}{7}\Big)^3\frac{T^3}{3}\int_0^T\Big(\Big|\frac{T}{2}-t\Big|+\varepsilon\Big)^3\,\mathrm{d}t \\ &= \frac{1000T^4}{1039}\Big(\frac{5T^3}{32}+\frac{T^2}{4}\varepsilon+\frac{3T}{2}\varepsilon^2+\varepsilon^3\Big). \end{aligned}$$

If $T^4<2744/1250=2.1952$, we choose $\varepsilon>0$ sufficiently small such that

$$\liminf_{|x|\to\infty}|x|^{-3\alpha}\int_0^T F(t,x)\,\mathrm{d}t=\frac{T^3}{3}>\frac{1000T^4}{1039}\Big(\frac{5T^3}{32}+\frac{T^2}{4}\varepsilon+\frac{3T}{2}\varepsilon^2+\varepsilon^3\Big).$$

This shows that (3.3) holds. By Theorem 3.1, problem (1.1) has at least one solution.

E x a m p l e 4.2. Let $T = 0.3$, $N = 5$, $t_1 = 0.2$, $p = 3/2$, $q = 3$, consider the second-order Hamiltonian system with impulsive effects

$$(4.4)\qquad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(|\dot{u}(t)|^{1/2}) = \nabla F(t, u(t)), \quad \text{a.e. } t \in [0, \pi], \\ u(0) - u(0.3) = \dot{u}(0) - \dot{u}(0.3) = 0, \\ \Delta \dot{u}^i(0.2) = \dot{u}^i(0.2^+) - \dot{u}^i(0.2^-) = I_{i1}(u^i(0.2))^{1/9}, \quad i = 1, 2, 3, 4, 5. \end{cases}$$

Let

$$(4.5)\qquad F(t, x) = \Big(\frac{T}{2} - t\Big)|x|^{10/7} + \Big(\frac{1}{4}T^2 - t^2 - \frac{4bmN}{pT}\Big)|x|^{9/7} + (h(t), x),$$

$I_{i1}(t) = -t^{1/7}$, $\alpha = 3/7$, $\beta_{i1} = 1/3$, $h \in L^1([0, T], \mathbb{R}^N)$. It is easy to see that

$$\begin{aligned} |\nabla F(t, x)| &\leqslant \frac{10}{7}\Big|\frac{T}{2} - t\Big||x|^{3/7} + \frac{9}{7}\Big|\frac{1}{4}T^2 - t^2 - \frac{4bmN}{pT}\Big||x|^{2/7} + |h(t)| \\ &\leqslant \frac{10}{7}\Big(\Big|\frac{T}{2} - t\Big| + \varepsilon\Big)|x|^{3/7} + \frac{T^6}{\varepsilon^2} + \frac{36bmN}{7pT}|x|^{2/7} + |h(t)| \\ &\leqslant \frac{10}{7}\Big(\Big|\frac{T}{2} - t\Big| + 2\varepsilon\Big)|x|^{3/7} + \frac{T^6}{\varepsilon^2} + \frac{(4bmN)^3}{p^3T^3\varepsilon^2} + |h(t)| \end{aligned}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$, where $\varepsilon > 0$. This shows (1.7) holds with $\alpha = 3/7$ and

$$(4.6)\qquad f(t) = \frac{10}{7}\Big(\Big|\frac{T}{2} - t\Big| + 2\varepsilon\Big)|x|^{3/7}, \quad g(t) = \frac{T^6}{\varepsilon^2} + \frac{(4bmN)^3}{p^3T^3\varepsilon^2} + |h(t)|.$$

However, $F(t, x)$ satisfies neither (1.8) nor (1.9). In fact,

$$\begin{aligned} &|x|^{-3\alpha}\int_0^T F(t, x)\,\mathrm{d}t \\ &\quad = |x|^{-9/7}\int_0^T \Big[\Big(\frac{T}{2} - t\Big)|x|^{10/7} + \Big(\frac{1}{4}T^2 - t^2 - \frac{4bmN}{pT}\Big)|x|^{9/7} + (h(t), x)\Big]\,\mathrm{d}t \\ &\quad = -T^3/12 - \frac{4bmN}{p} + \Big(\int_0^T h(t)\,\mathrm{d}t, |x|^{-9/7}x\Big). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \Big(\int_0^T f(t)\,\mathrm{d}t\Big)^3 &= \int_0^T \Big(\frac{10}{7}\Big(\Big|\frac{T}{2} - t\Big| + 2\varepsilon\Big)|x|^{3/7}\Big)^3\,\mathrm{d}t \\ &= \frac{1000T}{343}\Big(\frac{5}{32}T^3 + \frac{\varepsilon T^2}{2} + 3\varepsilon^2 T + 8\varepsilon^3\Big). \end{aligned}$$

It is easy to check that the conditions of Theorem 3.2 hold true, By Theorem 3.2, problem (1.1) has at least one solution.

E x a m p l e 4.3. Let $T=0.6$, $N=3$, $t_1=0.5$, consider the second-order Hamiltonian system with impulsive effects

$$\begin{cases} \ddot{u}(t)=\nabla F(t,u(t)), \quad \text{a.e. } t\in[0,T], \\ u(0)-u(0.6)=\dot{u}(0)-\dot{u}(0.6)=0, \\ \Delta\dot{u}^i(1)=\dot{u}^i(0.5^+)-\dot{u}^i(0.5^-)=(u^i(0.5))^{1/3}, \quad i=1,2,3. \end{cases} \tag{4.7}$$

Let

$$F(t,x)=(0.6T-t)|x|^2-t|x|^{3/2}+(h(t),x), \tag{4.8}$$

where $h\in L^1([0,T],\mathbb{R}^N)$, $I_{ij}(t)=t^{1/3}$. It is easy to see that

$$\begin{aligned} |\nabla F(t,x)| &\leqslant 2|0.6T-t||x|+\frac{3t}{2}|x|^{1/2}+|h(t)| \\ &\leqslant 2(|0.6T-t|+\varepsilon)|x|+\frac{T^2}{2\varepsilon}+|h(t)| \end{aligned}$$

for all $x\in\mathbb{R}^N$ and a.e. $t\in[0,T]$, where $\varepsilon>0$. This shows (3.9) holds with

$$f(t)=2(|0.6T-t|+\varepsilon), \quad g(t)=\frac{T^2}{2\varepsilon}+|h(t)|. \tag{4.9}$$

Observe that

$$\begin{aligned} |x|^{-2}\int_0^T F(t,x)\,\mathrm{d}t &= |x|^{-2}\int_0^T [(0.6T-t)|x|^2-t|x|^{3/2}+(h(t),x)]\,\mathrm{d}t \\ &= 0.1T^2-0.5T^2|x|^{-1/2}+\Big(\int_0^T h(t)\,\mathrm{d}t, |x|^{-2}x\Big). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \int_0^T f(t)\,\mathrm{d}t &= 2\int_0^T(|0.6T-t|+\varepsilon)\,\mathrm{d}t=0.52T^2+2\varepsilon T, \\ \Big(\int_0^T f(t)\,\mathrm{d}t\Big)^2 &= 4\int_0^T(|0.6T-t|+\varepsilon)^2\,\mathrm{d}t=\frac{28}{75}T^3+2.08\varepsilon T^2+4\varepsilon^2T, \end{aligned}$$

and

$$\frac{3T^2\int_0^T f^2(t)\,\mathrm{d}t}{2\pi^2(12-T\int_0^T f(t)\,\mathrm{d}t)}=\frac{T^3(1.12T^2+6.24\varepsilon T+12\varepsilon^2)}{2\pi^2[12-T^2(0.52T+2\varepsilon)]}.$$

If $T^3<0.4808$, we choose $\varepsilon>0$ sufficiently small such that

$$\int_0^T f(t)\,\mathrm{d}t=0.52T^2+2\varepsilon T<\frac{1}{4T}$$

and

$$\liminf_{|x|\to\infty} |x|^{-2} \int_0^T F(t,x)\,\mathrm{d}t = 0.1T^2$$
$$> \frac{4T^2(\frac{28}{75}T^3 + 2.08\varepsilon T^2 + 4\varepsilon^2 T)}{[1 - 4T(0.52T^2 + 2\varepsilon T)]} = \frac{4T^2\big(\int_0^T f(t)\,\mathrm{d}t\big)^2}{1 - 4T\int_0^T f(t)\,\mathrm{d}t}.$$

This shows that (3.9) and (3.10) hold. By Theorem 3.3, problem (1.1) has at least one solution.

Example 4.4. Let $T = 0.2$, $N = 2$, $t_1 = 0.1$, consider the second-order Hamiltonian system with impulsive effects

$$(4.10)\quad \begin{cases} \ddot{u}(t) = \nabla F(t,u(t)), \quad \text{a.e. } t \in [0,T], \\ u(0) - u(0.2) = \dot{u}(0) - \dot{u}(0.2) = 0, \\ \Delta \dot{u}^i(0.1) = \dot{u}^i(0.1^+) - \dot{u}^i(0.1^-) = I_{ij}(u^i(0.1)), \quad i = 1,2;\ j = 1,2,\ldots,m, \end{cases}$$

$$(4.11)\qquad F(t,x) = (0.4T - t)|x|^2 + t|x|^{3/2} + (h(t),x),$$

where $h \in L^1([0,T],\mathbb{R}^N)$. It is easy to see that

$$|\nabla F(t,x)| \leqslant 2|0.4T - t||x| + \frac{3t}{2}|x|^{1/2} + |h(t)|$$
$$\leqslant 2(|0.4T - t| + \varepsilon)|x| + \frac{T^2}{2\varepsilon} + |h(t)|$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$, where $\varepsilon > 0$. This shows (1.12) holds with

$$(4.12)\qquad f(t) = 2(|0.4T - t| + \varepsilon), \quad g(t) = \frac{T^2}{2\varepsilon} + |h(t)|.$$

Observe that

$$|x|^{-2}\int_0^T F(t,x)\,\mathrm{d}t = |x|^{-2}\int_0^T [(0.4T - t)|x|^2 + t|x|^{3/2} + (h(t),x)]\,\mathrm{d}t$$
$$= -0.1T^2 + 0.5T^2|x|^{-1/2} + \Big(\int_0^T h(t)\,\mathrm{d}t, |x|^{-2}x\Big).$$

On the other hand, we have

$$\int_0^T f(t)\,\mathrm{d}t = 2\int_0^T (|0.4T - t| + \varepsilon)\,\mathrm{d}t = 0.52T^2 + 2\varepsilon T,$$
$$\Big(\int_0^T f(t)\,\mathrm{d}t\Big)^2 = 4\int_0^T (|0.4T - t| + \varepsilon)^2\,\mathrm{d}t = \frac{28}{75}T^3 + 2.08\varepsilon T^2 + 4\varepsilon^2 T.$$

If $T < 0.5$, we choose $\varepsilon > 0$ sufficiently small such that

$$\int_0^T f(t)\,\mathrm{d}t = 0.52T^2 + 2\varepsilon T < \frac{1}{4T}.$$

It is easy to show that all conditions of Theorem 3.4 hold. By Theorem 3.4, problem (1.1) has at least one solution.

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