Toru Sasahara

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IDEAL CR SUBMANIFOLDS IN NON-FLAT COMPLEX SPACE FORMS

Toru Sasahara, Hachinohe

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Abstract. An explicit representation for ideal CR submanifolds of a complex hyperbolic space has been derived in T. Sasahara (2002). We simplify and reformulate the representation in terms of certain Kähler submanifolds. In addition, we investigate the almost contact metric structure of ideal CR submanifolds in a complex hyperbolic space. Moreover, we obtain a codimension reduction theorem for ideal CR submanifolds in a complex projective space.

Keywords: δ-invariants; CR submanifolds; ideal submanifolds

MSC 2010: 53C42, 53B25

1. Introduction

In [9], the author derived a representation formula for ideal CR submanifolds with rank one totally real distribution in a complex hyperbolic space $\mathbb{C}H^m$, under the condition that the shape operator with respect to the distinguished vector field has constant principal curvatures. The formula is described in terms of CR submanifolds whose second fundamental forms take certain special forms in a complex pseudo-Euclidean space. However, it is complicated. We simplify and reformulate the formula in terms of Kähler submanifolds. By virtue of the simplified representation formula, the geometric meaning of the formula derived in [9] is clarified, and moreover a rich family of ideal CR submanifolds in $\mathbb{C}H^m$ can be obtained.

On the other hand, a CR submanifold with rank one totally real distribution in a Kähler manifold M has an almost contact metric structure which is naturally induced from the almost complex structure of M. We prove that each ideal CR submanifold of $\mathbb{C}H^m$ investigated in [9] admits a Sasakian structure.

We also obtain a codimension reduction theorem for ideal CR submanifolds in a complex projective space $\mathbb{C}P^m$. As a corollary, we show that every 3-dimensional ideal proper CR submanifold in $\mathbb{C}P^m$ must be contained in $\mathbb{C}P^2$. This is contrary to the case of $\mathbb{C}H^m$ where there exist a great many linearly full 3-dimensional ideal proper CR submanifolds whose codimensions are greater than one.

2. Preliminaries

2.1. δ-invariants. Let M be an n-dimensional Riemannian manifold. Denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM$, $p \in M$. For any orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space T_pM, the scalar curvature τ at p is defined by

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j).$$

Let L be a subset of T_pM of dimension $r \geqslant 2$ and $\{e_1, \ldots, e_n\}$ an orthonormal basis of L. We define the scalar curvature $\tau(L)$ of the r-plane section L by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leqslant \alpha,\ \beta \leqslant r.$$

For an integer $k \geqslant 0$, denote by $\mathscr{S}(n,k)$ the finite set which consists of unordered k-tuples $(n_1, \ldots, n_k)$ of integers $\geqslant 2$ satisfying $n_1 < n$ and $n_1 + \ldots + n_k \leqslant n$. Denote by $\mathscr{S}(n)$ the set of k-tuples with $k \geqslant 0$ for a fixed n.

For each k-tuple $(n_1, \ldots, n_k) \in \mathscr{S}(n)$, B. Y. Chen introduced the notion of the *δ-invariants* $\delta(n_1, \ldots, n_k)$, as follows:

$$\delta(n_1, \ldots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \ldots + \tau(L_k)\},$$

where $L_1, \ldots, L_k$ runs over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j$, $j = 1, \ldots, k$.

Let $\overline{\mathrm{Ric}}$ denote the maximum Ricci curvature function on M defined by

$$\overline{\mathrm{Ric}}(p) = \max\{S(X,X);\ X \in T_p^1M\},$$

where S is the Ricci tensor and T_p^1M is the unit tangent vector space of M at p. Then we have $\delta(n-1)(p) = \overline{\mathrm{Ric}}(p)$.

We put $\delta_k(\lambda) = \delta(\lambda, \ldots, \lambda)$ (λ appears k times).

2.2. Kählerian δ-invariants. Let M be a (real) $2n$-dimensional Kähler manifold. For a k-tuple $(2n_1, \ldots, 2n_k) \in \mathscr{S}(2n)$, Chen also introduced the *Kählerian δ-invariants* $\delta^c(2n_1, \ldots, 2n_k)$ by

$$\delta^c(2n_1, \ldots, 2n_k)(p) = \tau(p) - \inf\{\tau(L_1^c) + \ldots + \tau(L_k^c)\},$$

where $L_1^c, \ldots, L_k^c$ run over all k mutually orthogonal complex subspaces of T_pM such that $\dim L_j = 2n_j$, $j = 1, \ldots, k$.

We put $\delta_k^c(\lambda) = \delta^c(\lambda, \ldots, \lambda)$ (λ appears k times).

2.3. General inequalities for submanifolds in complex space forms. Denote by $\tilde{M}^m(4\varepsilon)$ a complex space form of constant holomorphic sectional curvature 4ε and complex dimension m. Every complete simply connected complex space form $\tilde{M}^m(4\varepsilon)$ is holomorphically isometric to the complex projective space $\mathbb{C}P^m(4\varepsilon)$, complex Euclidean space $\mathbb{C}^m$ or complex hyperbolic space $\mathbb{C}H^m(4\varepsilon)$ according as $\varepsilon > 0$, $\varepsilon = 0$ or $\varepsilon < 0$, respectively.

Let M be an n-dimensional submanifold in $\tilde{M}^m(4\varepsilon)$ and let J be the complex structure of $\tilde{M}^m(4\varepsilon)$. For any vector X tangent to M, we put $JX = PX + FX$, where PX and FX are the tangential and normal components of JX, respectively. For a subspace $L \subset T_pM$ of dimension r, we put

$$\Psi(L) = \sum_{1 \leqslant i < j \leqslant r} \langle Pu_i, u_j \rangle^2,$$

where $\{u_1, \ldots, u_r\}$ is an orthonormal basis of L.

For each $(n_1, \ldots, n_k) \in \mathscr{S}(n)$, let $c(n_1, \ldots, n_k)$ and $b(n_1, \ldots, n)$ be the constants given by

$$c(n_1, \ldots, n_k) = \frac{n^2\Big(n + k - 1 - \sum\limits_{j=1}^{k} n_j\Big)}{2\Big(n + k - \sum\limits_{j=1}^{k} n_j\Big)},$$

$$b(n_1, \ldots, n_k) = \frac{1}{2}\Big(n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1)\Big).$$

Denote by H the mean curvature vector field of M in $\tilde{M}^m(4\varepsilon)$. Then we have the following general inequalities involving the δ-invariants and the squared mean curvature $|H|^2$ (cf. [3]):

Proposition 2.1. *Given an n-dimensional submanifold M in a complex space form $\tilde{M}^m(4\varepsilon)$, we have*

$$\tau - \sum_{i=1}^{k} \tau(L_i) \leqslant c(n_1, \dots, n_k)|H|^2 + b(n_1, \dots, n_k) + \frac{3}{2}|P|^2\varepsilon - 3\varepsilon \sum_{i=1}^{k} \Psi(L_i) \tag{2.1}$$

for any k-tuple $(n_1, \dots, n_k) \in \mathscr{S}(n)$. The equality case of inequality (2.1) *holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \dots, e_{2m}\}$ at p such that*

(a) $L_j = \mathrm{Span}\{e_{n_1+\dots+n_{j-1}+1}, \dots, e_{n_1+\dots+n_j}\}$

(b) *the shape operators of M in $\tilde{M}^m(4\varepsilon)$ at p take the forms*

$$A_{e_r} = \begin{pmatrix} A_1^r & \dots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \dots & A_k^r & \\ & 0 & & \mu_r I \end{pmatrix}, \quad r = n+1, \dots, 2m,$$

where I is the identity matrix and each A_j^r is a symmetric $n_j \times n_j$ submatrix such that

$$\mathrm{trace}(A_1^r) = \dots = \mathrm{trace}(A_k^r) = \mu_r.$$

By using Proposition 2.1, we have the following general inequalities for Kähler submanifolds in complex space forms (cf. [4]).

Proposition 2.2. *Let M be a $2n$-dimensional Kähler submanifold in a complex space form $\tilde{M}^m(4\varepsilon)$. Then we have*

$$\delta^c(2n_1, \dots, 2n_k) \leqslant 2\Big(n(n+1) - \sum_{j=1}^{k} n_j(n_j+1)\Big)\varepsilon. \tag{2.2}$$

The equality case of inequality (2.2) *holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \dots, e_{2m}\}$ at p such that $e_1, \dots, e_{2n}$ are tangent to M and $e_{2l} = Je_{2l-1}$ $(1 \leqslant l \leqslant k)$ and, moreover, the shape operators of M in $\tilde{M}^m(4\varepsilon)$ at p take the forms*

$$A_{e_r} = \begin{pmatrix} A_1^r & \dots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \dots & A_k^r & \\ & 0 & & 0 \end{pmatrix}, \quad r = 2n+1, \dots, 2m,$$

where each A_j^r is a symmetric $(2n_j) \times (2n_j)$ submatrix satisfying $\mathrm{trace}(A_j^r) = 0$.

A $2n$-dimensional Kähler submanifold is said to be *$\delta^c(2n_1, \ldots, 2n_k)$-ideal* if it satisfies the equality case of (2.2) identically for some k-tuple $(2n_1, \ldots, 2n_k) \in \mathscr{S}(2n)$.

For more information about δ-invariants and ideal submanifolds, see [4].

2.4. General inequalities for CR submanifolds in non-flat complex space forms. Let M be a pseudo-Riemannian submanifold of a pseudo-Kähler manifold $\tilde{M}$ and let J be the complex structure of $\tilde{M}$. A submanifold M is called a *CR submanifold* if there exists a differentiable holomorphic distribution $\mathscr{H}$ (i.e. $J\mathscr{H} = \mathscr{H}$) on M such that its orthogonal complement $\mathscr{H}^\perp$ is totally real, i.e., $J\mathscr{H}^\perp \subset T^\perp M$, where $T^\perp M$ denotes the normal bundle of M. A unit normal vector field $N \in J\mathscr{H}^\perp$ is called the *distinguished normal vector field* if $\dim \mathscr{H}^\perp = 1$. A CR submanifold is said to be *proper* if $\mathrm{rank}\mathscr{H} > 0$ and $\mathrm{rank}\mathscr{H}^\perp > 0$. Denote by ν the orthogonal complement of $J\mathscr{H}^\perp$ in $T^\perp M$.

By using Proposition 2.1, we have the following general inequalities for CR submanifolds in non-flat complex space forms.

Proposition 2.3. *Let M be an n-dimensional* CR *submanifold with* $\dim \mathscr{H} = 2h$ *in $\mathbb{C}H^m(-4)$. Then we have*

$$\delta(n_1, \ldots, n_k) \leqslant c(n_1, \ldots, n_k)|H|^2 - b(n_1, \ldots, n_k) - 3h + \frac{3}{2}\sum_{j=1}^{k} n_j. \tag{2.3}$$

Equality sign in (2.3) *holds at a point $p \in M$ for some $(n_1, \ldots, n_k) \in \mathscr{S}(n)$ if and only if there exists an orthonormal basis $\{e_1, \ldots, e_{2m}\}$ at p such that*

(a) $L_j := \mathrm{Span}\{e_{n_1+\ldots+n_{j-1}+1}, \ldots, e_{n_1+\ldots+n_j}\}$ *satisfy* $\Psi(L_j) = n_j/2$ *for* $1 \leqslant j \leqslant k$,

(b) *the shape operators of M in $\mathbb{C}H^m(-4)$ at p take the forms*

$$A_{e_r} = \begin{pmatrix} A_1^r & \ldots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \ldots & A_k^r & \\ & 0 & & \mu_r I \end{pmatrix}, \quad r = n+1, \ldots, 2m, \tag{2.4}$$

where I is the identity matrix and each A_j^r is a symmetric $n_j \times n_j$ submatrix such that

$$\mathrm{trace}(A_1^r) = \ldots = \mathrm{trace}(A_k^r) = \mu_r. \tag{2.5}$$

Proposition 2.4. *Let M be an n-dimensional* CR *submanifold in $\mathbb{C}P^m(4)$. Then we have*

$$\delta(n_1, \ldots, n_k) \leqslant c(n_1, \ldots, n_k)|H|^2 + b(n_1, \ldots, n_k) + 3h. \tag{2.6}$$

Equality sign in (2.6) *holds at a point* $p \in M$ *for some* $(n_1, \ldots, n_k) \in \mathscr{S}(n)$ *if and only if there exists an orthonormal basis* $\{e_1, \ldots, e_{2m}\}$ *at* p *such that*

(a) $L_j := \mathrm{Span}\{e_{n_1+\ldots+n_{j-1}+1}, \ldots, e_{n_1+\ldots+n_j}\}$ *satisfy* $\Psi(L_j) = 0$ *for* $1 \leqslant i \leqslant k$,

(b) *the shape operators of* M *in* $\mathbb{C}P^m(4)$ *at* p *satisfy* (2.4) *and* (2.5).

An n-dimensional CR submanifold in $\mathbb{C}H^m(-4)$ or $\mathbb{C}P^m(4)$ is said to be $\delta(n_1, \ldots, n_k)$*-ideal* if it satisfies the equality case of (2.3) or (2.6) identically for some k-tuple $(n_1, \ldots, n_k) \in \mathscr{S}(n)$, respectively.

A submanifold is said to be *linearly full* in $\tilde{M}^m(4\varepsilon)$ if it does not lie in any totally geodesic Kähler hypersurfaces of $\tilde{M}^m(4\varepsilon)$.

3. Ideal CR submanifolds in a complex hyperbolic space

3.1. Explicit representation. Let $\mathbb{C}_1^{m+1}$ be the complex number $(m+1)$-space endowed with the complex coordinates $(z_0, \ldots, z_m)$, the pseudo-Euclidean metric given by $\tilde{g} = -\,\mathrm{d}z_0\,\mathrm{d}\overline{w}_0 + \sum\limits_{i=1}^{m} dz_i\,\mathrm{d}\overline{w}_i$ and the standard complex structure J_0. For $\varepsilon < 0$, we put $H_1^{2m+1}(\varepsilon) = \{z \in \mathbb{C}_1^{m+1};\ \langle z, z\rangle = 1/\varepsilon\}$, where $\langle\,,\,\rangle$ denotes the inner product on $\mathbb{C}_1^{m+1}$ induced from $\tilde{g}$. On $H_1^{2m+1}(\varepsilon)$ we consider the following tensor fields: $\varphi = s \circ J_0$, $\xi = \sqrt{-\varepsilon}J_0 z$, $\eta(X) = \sqrt{-\varepsilon}g(J_0 z, X)$, where $s\colon T_z\mathbb{C}_1^{m+1} \to T_z H_1^{2m+1}(\varepsilon)$ denotes the orthogonal projection and g is the induced metric from $\mathbb{C}_1^{m+1}$. Then the quadruplet (φ, ξ, η, g) defines an almost contact structure on $H_1^{2m+1}(\varepsilon)$. The Hopf fibration is given by

$$\Pi_{\{m,\varepsilon\}}\colon\ H_1^{2m+1}(\varepsilon) \to \mathbb{C}H^m(4\varepsilon)\colon\ z \mapsto z \cdot \mathbb{C}^*.$$

Let $z\colon\ M \to H_1^{2m+1}(\varepsilon) \subset \mathbb{C}_1^{m+1}$ be an isometric immersion such that $\mathrm{i}z$ is tangent to M. Then M is a CR submanifold with $\mathscr{H}^\perp = \mathrm{Span}\{\mathrm{i}z\}$ in $\mathbb{C}_1^{m+1}$ if and only if M is an invariant submanifold in $H_1^{2m+1}(\varepsilon)$, i.e., $\varphi(TM) \subset TM$. For a vector field X tangent to $\mathbb{C}H^m(4\varepsilon)$, we denote the horizontal lift of X by X^*. Since $(JX)^* = \varphi X^*$ holds, we have the following:

Lemma 3.1. *Let* N *be a submanifold in* $\mathbb{C}H^m(4\varepsilon)$. *Then* $\Pi_{\{m,\varepsilon\}}^{-1}(N)$ *is a* CR *submanifold in* $\mathbb{C}_1^{m+1}$ *with* $\mathscr{H}^\perp = \mathrm{Span}\{\mathrm{i}z\}$ *if and only if* N *is a Kähler submanifold in* $\mathbb{C}H^m(4\varepsilon)$, *where* z *is the position vector of* $\Pi_{\{m,\varepsilon\}}^{-1}(N)$ *in* $\mathbb{C}_1^{m+1}$.

Denote by h and $\tilde{h}$ the second fundamental forms of the immersions $i\colon N \to \mathbb{C}H^m(4\varepsilon)$ and $\tilde{i}\colon \Pi_{\{m,\varepsilon\}}^{-1}(N) \to \mathbb{C}_1^{m+1}$, respectively. Then we have the following (cf. [5]):

$$\tilde{h}(X^*, Y^*) = (h(X, Y))^* - \varepsilon\,\langle X, Y\rangle\, z,\ \tilde{h}(X^*, \mathrm{i}z) = (FX)^*,\ \tilde{h}(\mathrm{i}z, \mathrm{i}z) = -z, \tag{3.1}$$

for all vectors X and Y tangent to N.

Let M be a linearly full $(2n+1)$-dimensional $\delta_k(2n/k)$-ideal CR submanifold in $\mathbb{C}H^m(-4)$ such that $\dim \mathscr{H}^\perp = 1$, $m > n+1$ and $n/k \in \mathbb{Z} - \{1\}$. Assume that the shape operator with respect to the distinguished normal vector field has constant principal curvatures. Then, up to rigid motions of $\mathbb{C}H^m(-4)$, the immersion of M into $\mathbb{C}H^m(-4)$ is given by (see [9, Theorem 1])

$$\Pi_{\{m,-1\}}\Big(f(x_1,y_1,\ldots,x_n,y_n)\mathrm{e}^{-(1-\alpha^2)\mathrm{i}s}, \frac{\alpha}{\sqrt{1-\alpha^2}}\mathrm{e}^{\mathrm{i}t}\Big),$$

where $\alpha = \sqrt{k/(2n-k)}$ and $z_1(x_1,y_1,\ldots,x_n,y_n,s) := f(x_1,y_1,\ldots,x_n,y_n)\times \mathrm{e}^{-(1-\alpha^2)\mathrm{i}s}$ is a CR submanifold in $\mathbb{C}^m_1$ which satisfies

$$\langle f,f\rangle = \alpha^2 - 1 \tag{3.2}$$

and the following condition: There exists an orthonormal frame $\{E_1,\ldots,E_{2n},E_{2n+1}\}$ on z_1 such that $E_{2r} = iE_{2r-1}$ for $r\in\{1,\ldots,n\}$, $E_{2n+1} = (1/\sqrt{1-\alpha^2})\partial/\partial s$ and the second fundamental form $\tilde{h}$ of z_1 in $\mathbb{C}^{m-1}_1$ satisfies

$$\begin{aligned}
\tilde{h}(E_{2r-1},E_{2r-1}) &= \sqrt{1-\alpha^2}iE_{2n+1} + \tilde{\varphi}_r\tilde{\xi}_r,\\
\tilde{h}(E_{2r},E_{2r}) &= \sqrt{1-\alpha^2}iE_{2n+1} - \tilde{\varphi}_r\tilde{\xi}_r,\\
\tilde{h}(E_{2r-1},E_{2r}) &= i\tilde{\varphi}_r\tilde{\xi}_r,\\
\tilde{h}(E_{2n+1},E_{2n+1}) &= -\sqrt{1-\alpha^2}iE_{2n+1},\\
\tilde{h}(X_i,X_j) &= \tilde{h}(X_i,E_{2n+1}) = 0 \quad (i\neq j),
\end{aligned} \tag{3.3}$$

where $\tilde{\varphi}_r = \tilde{\varphi}_r(x_1,y_1\ldots,x_n,y_n,s)$ are functions, $\tilde{\xi}_r$ are normal vector fields perpendicular to iE_{2n+1}, and $X_i \in \mathrm{Span}\{E_{(2n(i-1)/k)+1},\ldots,E_{2ni/k}\}$ for $i\in\{1,\ldots,k\}$.

By Proposition 2.2, Lemma 3.1, (3.1), (3.2) and (3.3), we see that $\Pi_{\{m-1,\alpha^2-1\}}\circ z_1$ is a $2n$-dimensional $\delta^c_k(2n/k)$-ideal Kähler submanifold in $\mathbb{C}H^{m-1}(4\alpha^2-4)$. Therefore, we can simplify and reformulate (2) of Theorem 1 in [9] as (2) of the following theorem.

Theorem 3.1. *Let M be a linearly full $(2n+1)$-dimensional $\delta(n_1,\ldots,n_k)$-ideal* CR *submanifold in $\mathbb{C}H^m(-4)$ such that $\dim\mathscr{H}^\perp = 1$, $k\geqslant 1$ and $m > n+1$. Assume that the shape operator with respect to the distinguished normal vector field has constant principal curvatures. Then, up to rigid motions of $\mathbb{C}H^m(-4)$, the immersion of M into $\mathbb{C}H^m(-4)$ is given by the composition $\Pi_{\{m,-1\}}\circ z$, where z is one of the following:*

(1) *$k=n$, $n_1=\ldots=n_n=2$, and*

$$z = \Big(-1-\frac{1}{2}|\Psi|^2+\mathrm{i}u, -\frac{1}{2}|\Psi|^2+\mathrm{i}u, \Psi\Big)\mathrm{e}^{\mathrm{i}t},$$

where Ψ is a $2n$-dimensional $\delta^c_n(2)$-ideal Kähler submanifold in $\mathbb{C}^{m-1}$.

(2) $n/k \in \mathbb{Z} - \{1\}$, $n_1 = \ldots = n_k = 2n/k$, *and*

$$z = \left(\Pi^{-1}_{\{m-1, \frac{2k-2n}{2n-k}\}}(\Psi), \sqrt{\frac{k}{2n-k}}\mathrm{e}^{\mathrm{i}t}\right),$$

where Ψ *is a* $2n$*-dimensional* $\delta^c_k(2n/k)$*-ideal Kähler submanifold in* $\mathbb{C}H^{m-1}(\frac{8k-8n}{2n-k})$.

If $n > 1$, $k = 1$ and $n_1 = 2n$, then (3.3) is satisfied automatically. By noting that $\delta(2n)(p) = \overline{\mathrm{Ric}}(p)$, we reobtain the representation formula in [8].

Corollary 3.1. *Let* M *be a linearly full* $(2n+1)$*-dimensional* $\delta(2n)$*-ideal* CR *submanifold in* $\mathbb{C}H^m(-4)$ *such that* $\dim \mathscr{H}^{\perp} = 1$, $n > 1$ *and* $m > n+1$. *Assume that the shape operator with respect to the distinguished normal vector field has constant principal curvatures. Then, up to rigid motions of* $\mathbb{C}H^m(-4)$, *the immersion of* M *into* $\mathbb{C}H^m(-4)$ *is given by*

$$\Pi_{\{m,-1\}}\left(\Pi^{-1}_{\{m-1, \frac{2-2n}{2n-1}\}}(\Psi), \sqrt{\frac{1}{2n-1}}\mathrm{e}^{\mathrm{i}t}\right),$$

where Ψ *is a* $2n$*-dimensional Kähler submanifold in* $\mathbb{C}H^{m-1}(\frac{8-8n}{2n-1})$.

Let N be a $2n$-dimensional Kähler hypersurface in a complex space form. Let V and JV be normal vector fields of N. By virtue of $A_{JV} = JA_V$ and $JA_V = -A_V J$, we can choose an orthonormal basis $\{e_1, Je_1, \ldots, e_n, Je_n\}$ of T_pN with respect to which the shape operators A_V and A_{JV} take the following form:

$$A_V = \begin{pmatrix} \lambda_1 & & & & 0 \\ & -\lambda_1 & & & \\ & & \ddots & & \\ & & & \lambda_n & \\ 0 & & & & -\lambda_n \end{pmatrix}, \quad A_{JV} = \begin{pmatrix} 0 & \lambda_1 & & & 0 \\ \lambda_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \lambda_n \\ 0 & & & \lambda_n & 0 \end{pmatrix}. \tag{3.4}$$

By Proposition 2.2 and (3.4), we see that every Kähler hypersurface in a complex space form is $\delta^c_k(2n/k)$-ideal for any natural number k such that $n/k \in \mathbb{Z}$. Accordingly, there exist many CR submanifolds which are described in Theorem 3.1.

3.2. Almost contact metric structure. A differentiable manifold M is called an *almost contact manifold* if it admits a unit vector field ξ, a one-form η and a $(1,1)$-tensor field φ satisfying

$$\eta(\xi) = 1, \quad \varphi^2 = -I + \eta \otimes \xi.$$

Every almost contact manifold admits a pseudo-Riemannian metric g satisfying

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

The quadruplet (φ, ξ, η, g) is called an *almost contact metric structure*. An almost contact metric structure is said to be *normal* if the tensor field S defined by

$$S(X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + 2\,\mathrm{d}\eta(X,Y)\xi$$

vanishes identically. A normal almost contact structure is said to be *Sasakian* if it satisfies

$$\mathrm{d}\eta(X,Y) = \frac{1}{2}(X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y])) = g(X, \varphi Y).$$

Let M be a CR submanifold with $\dim \mathscr{H}^\perp = 1$ in a complex space form. We define a one-form η by $\eta(X) = g(U, X)$, where U is a unit tangent vector field lying in $\mathscr{H}^\perp$, and g is an induced metric on M. We put $\overline{U} = (1/\sqrt{r})U$, $\overline{\eta} = \sqrt{r}\eta$ and $\overline{g} = rg$ for a positive constant r. Then the quadruplet $(P, \overline{U}, \overline{\eta}, \overline{g})$ defines an almost contact structure on M (cf. [6, p. 96]).

Each almost contact structure (P, U, η, g) of the CR submanifold described in Theorem 3.1 is normal (cf. [9]). Moreover, we have the following:

Proposition 3.1. *An almost contact structure $(P, \overline{U}, \overline{\eta}, \overline{g})$ with $r = \sqrt{k/(2n-k)}$ on a* CR *submanifold in Theorem* 3.1 *becomes a Sasakian structure. In particular, in the case of* (1), *the structure is Sasakian with respect to the induced metric.*

Proof. A unit normal vector field JU of a CR submanifold in Theorem 3.1 is parallel (see [9]). Hence, we have (see [6, (15.27)])

$$\nabla_X U = PA_{JU}X, \tag{3.5}$$

where ∇ is the Levi-Civita connection of M.

By Lemma 7 of [9], we know that there exists an orthonormal frame $\{e_1, \ldots, e_{2m}\}$ such that $e_{2r} = Je_{2r-1}$ for $r \in \{1, \ldots, n\}$, $e_{2n+1} \in \mathscr{H}^\perp$ and the second fundamental form takes the following form:

$$\begin{aligned} h(e_{2r-1}, e_{2r-1}) &= \sqrt{\frac{k}{2n-k}} Je_{2n+1} + \varphi_r \xi_r, \\ h(e_{2r}, e_{2r}) &= \sqrt{\frac{k}{2n-k}} Je_{2n+1} - \varphi_r \xi_r, \\ h(e_{2r-1}, e_{2r}) &= \varphi_r J\xi_r, \\ h(e_{2n+1}, e_{2n+1}) &= \frac{2n}{\sqrt{k(2n-k)}} Je_{2n+1}, \\ h(u_i, u_j) &= h(u_i, e_{2n+1}) = 0 \quad (i \neq j), \end{aligned}$$

where φ_r are functions, $\xi_r \in \nu$ and $u_j \in \mathrm{Span}\{e_{n_1+\ldots+n_{j-1}+1}, \ldots, e_{n_1+\ldots+n_j}\}$.

From this and (3.5), we get $\mathrm{d}\overline{\eta}(X,Y) = \overline{g}(X, PY)$ for all vector fields X, Y tangent to the CR submanifold. $\square$

4. Ideal CR submanifolds in a complex projective space

Let M be an n-dimensional $\delta(n_1, \ldots, n_k)$-ideal CR submanifold in $\mathbb{C}P^m(4)$. Let L_j be subspaces of T_pM defined in (a) of Proposition 2.4. Define the subspace L_{k+1} by $L_{k+1} = \mathrm{Span}\{e_{n_1+\ldots+n_k+1}, \ldots, e_n\}$. Obviously, we have $T_pM = L_1 \oplus \ldots \oplus L_{k+1}$. We denote by $\mathscr{L}_i$ the distribution generated by L_i.

We have the following codimension reduction theorem.

Theorem 4.1. *Let M be an n-dimensional $\delta(n_1, \ldots, n_k)$-ideal* CR *submanifold with* $\dim \mathscr{H}^\perp = 1$ *in* $\mathbb{C}P^m(4)$. *If* $\mathscr{H}^\perp \subset \mathscr{L}_i$ *for some* $i \in \{1, \ldots, k+1\}$, *then M is contained in a totally geodesic complex submanifold* $\mathbb{C}P^{(n+1)/2}(4)$ *in* $\mathbb{C}P^m(4)$.

Proof. Let M be an n-dimensional $\delta(n_1, \ldots, n_k)$-ideal CR submanifold with $\dim \mathscr{H}^\perp = 1$ in $\mathbb{C}P^m(4)$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal frame of M satisfying (a) and (b) in Proposition 2.4 at each point. Assume that $\mathscr{H}^\perp \subset \mathscr{L}_i$ for some $i \in \{1, \ldots, k+1\}$.

Case (i): $\mathscr{H}^\perp \subset \mathscr{L}_i$ for some $i \in \{1, \ldots, k\}$. In this case, we may assume that $\mathscr{H}^\perp \subset \mathscr{L}_1$ and $e_1 \in \mathscr{H}^\perp$. Due to [1], [2] we have

$$A_V X = A_{JV} JX \tag{4.1}$$

for vector fields $X \in \mathscr{H}$ and $V \in \nu$. Since $e_s \in \mathscr{H}$ for $s \neq 1$ and $\Psi(L_j) = 0$ for $j = 1, \ldots, k$, it follows from (4.1) that

$$\langle A_V e_s, e_t \rangle = \langle A_{JV} J e_s, e_t \rangle = 0 \tag{4.2}$$

for any $s, t \in \{2, \ldots, n\}$ and $V \in \nu$. By Proposition 2.4 and (4.2), we get

$$\langle A_V e_1, e_1 \rangle = 0. \tag{4.3}$$

Since $\tilde{\nabla} J = 0$ for the Levi-Civita connection $\tilde{\nabla}$ of $\mathbb{C}P^m(4)$ holds, by using the formula of Gauss, we obtain that for $r \in \{2, \ldots, n_1\}$

$$\begin{aligned} 0 &= (\tilde{\nabla}_{e_1} J)(e_r) = (\tilde{\nabla}_{e_1} J e_r) - J(\tilde{\nabla}_{e_1} e_r) \\ &= \nabla_{e_1} J e_r + h(e_1, J e_r) - J(\nabla_{e_1} e_r) - J h(e_1, e_r) \\ &= \nabla_{e_1} J e_r - J(\nabla_{e_1} e_r) - J h(e_1, e_r), \end{aligned}$$

where h is the second fundamental form. This implies that $h(e_1, e_r) \in J\mathscr{H}^\perp$. Hence, it follows from (4.2) and (4.3) that $A_V = 0$ for any $V \in \nu$.

On the other hand, by the formulas of Gauss and Weingarten, we have

$$-A_{Je_1}X + D_X(Je_1) = \tilde{\nabla}_X(Je_1) = J(\nabla_X e_1) + Jh(X, e_1).$$

This yields that $D_X(Je_1) \in J\mathscr{H}^\perp$ for any $X \in TM$. Since $J\mathscr{H}^\perp$ is of rank one and Je_1 is of unit length, we obtain that $D(Je_1) = 0$. Therefore, by applying the codimension reduction theorem for real submanifolds of a complex projective space [7], we conclude that M must be contained in $\mathbb{C}P^{(n+1)/2}(4)$.

Case (ii): $\mathscr{H}^\perp \subset \mathscr{L}_{k+1}$. In this case, we may assume that $e_n \in \mathscr{H}^\perp$. Similarly to the case of (i), by applying (4.1), Proposition 2.4 and the formulas of Gauss and Weingarten, we have $A_V = 0$ for any $V \in \nu$ and $D(Je_n) = 0$, which implies that M must be contained in $\mathbb{C}P^{(n+1)/2}(4)$. □

Corollary 4.1. *Let M be a 3-dimensional $\delta(2)$-ideal proper* CR *submanifold in $\mathbb{C}P^m(4)$. Then M is contained in $\mathbb{C}P^2(4)$.*

Proof. Let M be a 3-dimensional $\delta(2)$-ideal proper CR submanifold in $\mathbb{C}P^m(4)$. Clearly, $\dim \mathscr{H}^\perp = 1$. Let $\{e_1, e_2, e_3\}$ be an orthonormal frame of M satisfying (a) and (b) in Proposition 2.4 at each point. For a vector field $U \in \mathscr{H}^\perp$, we put $U = \alpha e_1 + \beta e_2 + \gamma e_3$ for some functions α, β and γ. It follows from $\langle Je_1, e_2\rangle = 0$ that $\alpha^2 + \beta^2 \neq 0$ and $\gamma \langle Je_3, e_1\rangle = \gamma \langle Je_3, e_2\rangle = 0$, which implies $\gamma = 0$. Therefore, by applying Theorem 4.1, we obtain the statement. □

Remark 4.1. A real hypersurface M in a complex space form is called a *Hopf* hypersurface if JV is a principal curvature vector, where V is a unit normal vector of M. All the $\delta_k(2)$-ideal Hopf hypersurfaces in non-flat complex space forms have been determined in [3].

Remark 4.2. In contrast to the case of $\mathbb{C}P^m(4)$, there exist a great many linearly full 3-dimensional $\delta(2)$-ideal proper CR submanifolds in $\mathbb{C}H^m(-4)$ with $m > 2$ (see (1) of Theorem 3.1).

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Author's address: Toru Sasahara, Division of Mathematics, Center for Liberal Arts and Sciences, Hachinohe Institute of Technology, 88-1, Ohbiraki Myo, Hachinohe Aomori, 031-8501, Japan, e-mail: `sasahara@hi-tech.ac.jp`.