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Commentationes Mathematicae Universitatis Carolinae, Vol. 55 (2014), No. 4, 457--470

The upper triangular algebra loop of degree 4

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Abstract. A natural loop structure is defined on the set U_4 of unimodular upper-triangular matrices over a given field. Inner mappings of the loop are computed. It is shown that the loop is non-associative and nilpotent, of class 3. A detailed listing of the loop conjugacy classes is presented. In particular, one of the loop conjugacy classes is shown to be properly contained in a superclass of the corresponding algebra group.

Keywords: algebra group; quasigroup; loop; supercharacter; fusion scheme

Classification: 20N05

1. Introduction

1.1 Quasigroup schemes. Quasigroups may be defined combinatorially or equationally. Combinatorially, a *quasigroup* $(Q, \cdot)$ is a set Q equipped with a binary *multiplication* operation denoted by $\cdot$ or simple juxtaposition of the two arguments, in which specification of any two of x, y, z in the equation $x \cdot y = z$ determines the third uniquely. A *loop* is a quasigroup Q with an *identity* element 1 such that $1 \cdot x = x = x \cdot 1$ for all x in Q.

Equationally, a quasigroup $(Q, \cdot, /, \backslash)$ is a set Q equipped with three binary operations of multiplication, *right division* $/$ and *left division* $\backslash$, satisfying the identities:

$$\text{(SL)}\quad x \cdot (x\backslash z) = z\,; \qquad \text{(SR)}\quad z = (z/x) \cdot x\,;$$
$$\text{(IL)}\quad x\backslash(x \cdot z) = z\,; \qquad \text{(IR)}\quad z = (z \cdot x)/x\,.$$

In writing complex quasigroup words, it is often helpful to use the convention whereby juxtaposition binds stronger than $\cdot$, $/$, or $\backslash$, so that associativity, for example, takes the bracketless form $x \cdot yz = xy \cdot z$.

For each element x of a quasigroup Q, consider the *right multiplication*

$$R(x)\colon Q \to Q;\ y \mapsto y \cdot x$$

and *left multiplication*

$$L(x)\colon Q \to Q;\ y \mapsto x \cdot y\,.$$

The right and left multiplications are elements of the group $Q!$ of bijections from the set Q to itself. For example, the identity (SL) says that each $L(x)$ surjects, while (IL) gives the injectivity of $L(x)$. The *multiplication group* $\operatorname{Mlt} Q$ of a quasigroup Q is the subgroup of $Q!$ generated by $\{R(q), L(q) \mid q \in Q\}$. If Q is finite, the diagonal action of $\operatorname{Mlt} Q$ on Q^2 is multiplicity-free, so the *orbitals* of $\operatorname{Mlt} Q$, the orbits of this action, form an association scheme on Q [7, Ch. 6], which may conveniently be described here as a *quasigroup scheme*.

1.2 Loop conjugacy classes. Throughout this paragraph, consider a loop $(Q, \cdot, 1)$. The *inner multiplication group* $\operatorname{Inn} Q$ is the stabilizer $\operatorname{Mlt} Q_1$ in $\operatorname{Mlt} Q$ of the identity element 1. For example, if Q is a group, then $\operatorname{Inn} Q$ is the inner automorphism group of Q, although for a general loop Q, elements of $\operatorname{Inn} Q$ are not necessarily automorphisms of Q.

For elements q, r of Q, define the *conjugation*

$$T(q) = R(q)L(q)^{-1}\,, \tag{1.1}$$

the *right inner mapping*

$$R(q, r) = R(q)R(r)R(qr)^{-1}\,, \tag{1.2}$$

and the *left inner mapping*

$$L(q, r) = L(q)L(r)L(rq)^{-1} \tag{1.3}$$

in $\operatorname{Mlt} Q_1$. Collectively, (1.1)–(1.3) are known as *inner mappings*. Note that $\operatorname{Mlt} Q_1$ is generated by the subset

$$\{T(q), R(q, r), L(q, r) \mid q, r \in Q\}$$

of $\operatorname{Mlt} Q$ [4, Lemma IV.1.2], [7, §2.8].

The orbits of $\operatorname{Inn} Q$ on Q are defined as the (*loop*) *conjugacy classes* of Q [4, p. 63]. If Q is a group, the loop conjugacy classes of Q are just the usual group conjugacy classes. In a loop Q, the orbitals of $\operatorname{Mlt} Q$ are directly related to the loop conjugacy classes. Indeed, if O is an orbital of $\operatorname{Mlt} Q$, then

$$1^O = \{q \in Q \mid (1, q) \in O\}$$

is a loop conjugacy class. Conversely, if C is a loop conjugacy class, then

$$\backslash^{-1}(C) = \{(x, y) \in Q^2 \mid x\backslash y \in C\}$$

is an orbital of $\operatorname{Mlt} Q$ [7, §6.1].

1.3 Fusion schemes. Let $\Gamma = \{C_1 = \widehat{Q}, C_2, \ldots, C_r\}$ be the set of relations of an association scheme (Q, Γ) on a finite set Q of size n, with $\widehat{Q} = \{(q, q) \mid q \in Q\}$ as the diagonal or equality relation. The complex linear span of the linearly independent set $\{A_1 = I_n, A_2, \ldots, A_r\}$ of respective incidence matrices of the relations $C_1, C_2, \ldots, C_r$ forms a commutative subalgebra $V(Q, \Gamma)$ of the algebra $\mathbb{C}_n^n$ of complex $n \times n$ matrices. A scheme (Q, Δ) is said to be a *fusion* of a scheme (Q, Γ) if each relation D_i from Δ is a union $D_i = \bigcup_{1 \le j \le k_i} C_{ij}$ of relations C_{ij}

from Γ. Fusion schemes were studied over two decades ago [6] in connection with the specification of character tables of quasigroup isotopes (cf. [7, p. 5]).

1.4 Algebra groups. In the terminology of Diaconis and Isaacs [5], an *algebra group* is defined as a group of the form $1 + N$, where N is a (finite-dimensional) nilpotent algebra over a finite field F. An initial example studied by André took N as an algebra of $n \times n$ strictly upper-triangular matrices, obtaining the group $U_n = 1 + N$ of "unimodular" upper-triangular matrices [2]. Reprising many results from the quasigroup and association scheme theory of [6] using keywords such as "superclass," "supercharacter," etc., Diaconis and Isaacs studied schemes obtained by fusing the quasigroup schemes of algebra groups. More recently, supercharacters over the entire sequence of algebra groups U_n have been shown to form Hopf algebras related to algebras of symmetric functions in non-commuting variables [1]. A natural question concerns the extent to which these fused group schemes of algebra groups may be obtained directly as the quasigroup schemes of *algebra loops*, natural loop structures on the set $1 + N$.

1.5 Algebra loops. The goal of the current paper is to study the smallest nontrivial example of an algebra group, U_4, from the standpoint of quasigroup theory. In §2, the algebra loop structure on U_4 is defined, as the so-called (*upper triangular*) *algebra loop* of degree 4. The inner mappings of the loop are computed, and it is confirmed that the loop is neither commutative nor associative. In §3, the loop is shown to be nilpotent of class 3, and its ascending central series is identified (Theorem 3.2). Furthermore, the conjugacy classes of the loop are classified in detail (see Table 1 for a summary). Most significantly, it transpires that certain loop classes are properly contained in superclasses (Remark 3.7), so that the loop character theory of the algebra loop, i.e. the character theory of the quasigroup scheme of the loop, represents an intermediate stage between the group characters and the supercharacters.

Readers are referred to [7] and [8] for quasigroup-theoretic and general algebraic concepts and conventions that are not otherwise explicitly clarified here.

2. The algebra loop U_4

2.1 The loop multiplication. Consider matrices

$$x = \begin{bmatrix} 1 & x_{12} & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$y = \begin{bmatrix} 1 & y_{12} & y_{13} & y_{14} \\ 0 & 1 & y_{23} & y_{24} \\ 0 & 0 & 1 & y_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with entries x_{ij}, y_{ij} from a field F. Then the quasigroup product is

$$\begin{bmatrix} 1 & x_{12} & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & y_{12} & y_{13} & y_{14} \\ 0 & 1 & y_{23} & y_{24} \\ 0 & 0 & 1 & y_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x_{12}+y_{12} & x_{13}+y_{13}+x_{12}y_{23} & [x \cdot y]_{14} \\ 0 & 1 & x_{23}+y_{23} & x_{24}+y_{24}+x_{23}y_{34} \\ 0 & 0 & 1 & x_{34}+y_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.1}$$

with

$$[x \cdot y]_{14} = x_{14} + y_{14} + x_{12}y_{24} + x_{13}y_{34} + x_{12}x_{23}y_{34} + x_{12}y_{23}y_{34} \tag{2.2}$$

as the last entry in the top row of the product. The summands in (2.2) correspond to paths of respective lengths $1, 2, 3$ from 1 to 4 in the chain $1 < 2 < 3 < 4$, with labels chosen from x over the former part of the path, and y over the latter part. The other entries in the product have a similar (but simpler) structure. Note that the product (2.1) has the matrix I_4 as its identity element.

2.2 The right division. With matrices x and y as above, consider the right division $z = y/x$, namely a solution

$$z = \begin{bmatrix} 1 & z_{12} & z_{13} & z_{14} \\ 0 & 1 & z_{23} & z_{24} \\ 0 & 0 & 1 & z_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

to the equation $z \cdot x = y$.

Lemma 2.1. *There is a unique solution $z = yR(x)^{-1}$ to $z \cdot x = y$.*

Proof: The entries z_{ij}, for $1 \le i < j \le 4$, are obtained by recursion on the length $j - i$ of the path from i to j in the chain $1 < 2 < 3 < 4$.

Paths of length 1:

$$z_{12} = (y_{12} - x_{12}),\ z_{23} = (y_{23} - x_{23}),\ z_{34} = (y_{34} - x_{34}).$$

Paths of length 2:

$$\begin{aligned} z_{13} + x_{13} + z_{12}x_{23} &= y_{13}, \text{ so} \\ z_{13} = y_{13} - x_{13} - z_{12}x_{23} &= (y_{13} - x_{13}) - (y_{12} - x_{12})x_{23}. \\ \text{Similarly, } z_{24} &= (y_{24} - x_{24}) - (y_{23} - x_{23})x_{34}. \end{aligned}$$

The path of length 3:

$$\begin{aligned}
&z_{14} + x_{14} + z_{12}x_{24} + z_{13}x_{34} + z_{12}z_{23}x_{34} + z_{12}x_{23}x_{34} = y_{14}\,, \text{ so} \\
&z_{14} = y_{14} - x_{14} - z_{12}x_{24} - z_{13}x_{34} - z_{12}z_{23}x_{34} - z_{12}x_{23}x_{34} \\
&\quad = y_{14} - x_{14} - (y_{12} - x_{12})x_{24} - (y_{13} - x_{13} - (y_{12} - x_{12})x_{23})x_{34} \\
&\qquad - (y_{12} - x_{12})(y_{23} - x_{23})x_{34} - (y_{12} - x_{12})x_{23}x_{34} \\
&\quad = (y_{14} - x_{14}) \\
&\quad + (y_{12} - x_{12})(-x_{24}) + (y_{13} - x_{13})(-x_{34}) \\
&\quad + (y_{12} - x_{12})(y_{23} - x_{23})(-x_{34})\,.
\end{aligned}$$

Note that in each case, the coefficient z_{ij} is uniquely specified in terms of x and y. □

2.3 The left division. With matrices x and y as above, consider the left division $z = x\backslash y$, namely a solution

$$z = \begin{bmatrix} 1 & z_{12} & z_{13} & z_{14} \\ 0 & 1 & z_{23} & z_{24} \\ 0 & 0 & 1 & z_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

to the equation $x \cdot z = y$.

Lemma 2.2. *There is a unique solution $z = yL(x)^{-1}$ to $x \cdot z = y$.*

Proof: The entries z_{ij}, for $1 \le i < j \le 4$, are again obtained by recursion on the length $j - i$ of the path from i to j in the chain $1 < 2 < 3 < 4$.

Paths of length 1:

$$z_{12} = (y_{12} - x_{12})\,,\ z_{23} = (y_{23} - x_{23})\,,\ z_{34} = (y_{34} - x_{34})\,.$$

Paths of length 2:

$$\begin{aligned}
x_{13} + z_{13} + x_{12}z_{23} &= y_{13}\,, \text{ so} \\
z_{13} = y_{13} - x_{13} - x_{12}z_{23} &= (y_{13} - x_{13}) - x_{12}(y_{23} - x_{23})\,. \\
\text{Similarly, } z_{24} &= (y_{24} - x_{24}) - x_{23}(y_{34} - x_{34})\,.
\end{aligned}$$

The path of length 3:

$$\begin{aligned}
&x_{14}+z_{14}+x_{12}z_{24}+x_{13}z_{34}+x_{12}x_{23}z_{34}+x_{12}z_{23}z_{34}=y_{14}\,,\ \text{so}\\
&z_{14}=y_{14}-x_{14}-x_{12}z_{24}-x_{13}z_{34}-x_{12}x_{23}z_{34}-x_{12}z_{23}z_{34}\\
&\quad=y_{14}-x_{14}-x_{12}(y_{24}-x_{24}-x_{23}(y_{34}-x_{34}))-x_{13}(y_{34}-x_{34})\\
&\qquad-x_{12}x_{23}(y_{34}-x_{34})-x_{12}(y_{23}-x_{23})(y_{34}-x_{34})\\
&\quad=(y_{14}-x_{14})\\
&\quad+(-x_{12})(y_{24}-x_{24})+(-x_{13})(y_{34}-x_{34})\\
&\quad+(-x_{12})(y_{23}-x_{23})(y_{34}-x_{34})\,.
\end{aligned}$$

Note that in each case, the coefficient z_{ij} is once more uniquely specified in terms of x and y. □

2.4 The algebra loop. Lemmas 2.1 and 2.2 yield the following.

Proposition 2.3. *With the product* (2.1)*, the algebra group* U_4 *over a field* F *forms a loop.*

Definition 2.4. The loop U_4 is known as the *upper triangular algebra loop* of degree 4.

2.5 Inner mappings. Within the loop U_4, the effects of the inner mappings (1.1)–(1.3) may be computed using the work of §§2.1–2.3. Consider elements $x=[x_{ij}]$, $q=[q_{ij}]$, $r=[r_{ij}]$ of U_4. Then

$$\begin{aligned}
&xT(q)=xR(q)L(q)^{-1}=q\backslash xq\\
&=\begin{bmatrix}
1 & x_{12} & x_{13}+\begin{vmatrix}x_{12} & q_{12}\\ x_{23} & q_{23}\end{vmatrix} & [xT(q)]_{14}\\
0 & 1 & x_{23} & x_{24}+\begin{vmatrix}x_{23} & q_{23}\\ x_{34} & q_{34}\end{vmatrix}\\
0 & 0 & 1 & x_{34}\\
0 & 0 & 0 & 1
\end{bmatrix}
\end{aligned}$$

with $[xT(q)]_{14}=$

$$x_{14}+\begin{vmatrix}x_{12} & q_{12}\\ x_{24} & q_{24}\end{vmatrix}+\begin{vmatrix}x_{13} & q_{13}\\ x_{34} & q_{34}\end{vmatrix}+\begin{vmatrix}x_{12} & q_{12}\\ x_{23}x_{34} & x_{23}q_{34}\end{vmatrix}+\begin{vmatrix}x_{12} & q_{12}\\ x_{23}q_{34} & q_{23}q_{34}\end{vmatrix}\,.$$

Furthermore, one has

$$[xR(q,r)]_{14}=[(xq\cdot r)/qr]_{14}=x_{14}+q_{12}x_{23}r_{34}-x_{12}r_{23}q_{34}$$

and

$$[xL(q,r)]_{14}=[rq\backslash(r\cdot qx)]_{14}=x_{14}+r_{12}x_{23}q_{34}-q_{12}r_{23}x_{34}\,,$$

with

$$[xR(q,r)]_{ij} = [xL(q,r)]_{ij} = x_{ij}$$

for $1 \leq i < j \leq 4$ and $j - i < 3$.

2.6 Properties of the algebra loop.

Definition 2.5. For given $1 \leq l < m \leq 4$, the *elementary* element E^{lm} of U_4 is defined by

$$[E^{lm}]_{ij} = \begin{cases} 1 & \text{if } i = l \text{ and } j = m, \text{ or if } i = j\,; \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.6. *Over a given field F, the upper triangular algebra loop U_4 of degree 4 is neither commutative nor associative.*

Proof: It was already observed in Proposition 2.3 that U_4 forms a loop. Consider elementary elements $x = E^{23}$, $q = E^{12}$, $r = E^{34}$ of U_4. The computations of §2.5 show that

$$[xT(q)]_{13} = x_{13} + \begin{vmatrix} x_{12} & q_{12} \\ x_{23} & q_{23} \end{vmatrix} = -1 \neq 0 = x_{13}\,,$$

so the loop is not commutative, and

$$[xR(q,r)]_{14} = x_{14} + q_{12}x_{23}r_{34} - x_{12}r_{23}q_{34} = 1 \neq 0 = x_{14}\,,$$

so the loop is not associative. □

3. Nilpotence and conjugacy classes

Consider the upper triangular algebra loop U_4 of degree 4 over a given field F. The goal of this section is to demonstrate that U_4 is nilpotent, and to determine the loop conjugacy class of each element $x = [x_{ij}]$ of U_4. These conjugacy classes are the orbits of the natural action of the inner multiplication group $\mathrm{Inn}(U_4)$.

3.1 Nilpotence and the center. Recall that the *center* Z or $Z(Q)$ of a loop Q is the set

$$\{z \in Q \mid \forall\, q, r \in Q\,,\ zT(q) = zR(q,r) = zL(q,r) = z\}$$

[3], [7, (3.31)]. In other words, the center $Z(Q)$ consists precisely of the set of elements of Q which lie in singleton conjugacy classes.

Proposition 3.1. *The set*

$$\left\{x = [x_{ij}] \in U_4 \;\middle|\; x_{ij} = 0 \text{ if } 1 \leq j - i < 3\right\} \tag{3.1}$$

forms the center of U_4.

PROOF: Consider a central element $x = [x_{ij}]$ of U_4. If $x_{23} \neq 0$, one would have

$$[xR(E^{12}, E^{34})]_{14} = x_{14} + x_{23} \neq x_{14} \,,$$

which would exclude x from the center, so $x_{23} = 0$. Thus for q in U_4, one has

$$[xT(q)]_{14} = x_{14} + \begin{vmatrix} x_{12} & q_{12} \\ x_{24} & q_{24} \end{vmatrix} + \begin{vmatrix} x_{13} & q_{13} \\ x_{34} & q_{34} \end{vmatrix} + \begin{vmatrix} x_{12} & q_{12} \\ 0 & q_{23}q_{34} \end{vmatrix} .$$

If $x_{12} \neq 0$, one would have

$$[xT(E^{24})]_{14} = x_{14} + x_{12} \neq x_{14} \,,$$

which would exclude x from the center, so $x_{12} = 0$ and

$$[xT(q)]_{14} = x_{14} + \begin{vmatrix} 0 & q_{12} \\ x_{24} & q_{24} \end{vmatrix} + \begin{vmatrix} x_{13} & q_{13} \\ x_{34} & q_{34} \end{vmatrix} .$$

If $x_{24} \neq 0$, one would have

$$[xT(E^{12})]_{14} = x_{14} - x_{24} \neq x_{14} \,,$$

which would exclude x from the center, so $x_{24} = 0$ and

$$[xT(q)]_{14} = x_{14} + \begin{vmatrix} x_{13} & q_{13} \\ x_{34} & q_{34} \end{vmatrix} .$$

If $x_{13} \neq 0$, one would have

$$[xT(E^{34})]_{14} = x_{14} + x_{13} \neq x_{14} \,,$$

which would exclude x from the center, so $x_{13} = 0$ and

$$[xT(q)]_{14} = x_{14} + \begin{vmatrix} 0 & q_{13} \\ x_{34} & q_{34} \end{vmatrix} .$$

Finally, if $x_{34} \neq 0$, one would have

$$[xT(E^{13})]_{14} = x_{14} - x_{34} \neq x_{14} \,,$$

which would exclude x from the center, so $x_{34} = 0$. Thus the center of U_4 is contained in the set (3.1). The opposite containment is apparent from the computations of §2.5. □

Recall the recursive definition $Z_0(Q) = \{1\}$ and $Z_{r+1}(Q)/Z_r(Q) = Z(Q/Z_r(Q))$ for the *ascending central series*

$$Z_0(Q) \leq Z_1(Q) \leq \cdots \leq Z_r(Q) \leq \dots$$

of a loop Q. The loop Q is *nilpotent* (of class at most c) if $Z_c(Q) = Q$.

Theorem 3.2. *The loop U_4 is nilpotent, of class 3. Indeed,*

$$Z_{4-k}(U_4) = \{x = [x_{ij}] \in U_4 \mid x_{ij} = 0 \text{ if } 1 \leq j - i < k\} \tag{3.2}$$

for $1 \leq k \leq 4$.

PROOF: The equation (3.2) certainly holds for $k = 4$, and Proposition 3.1 implies that it holds for $k = 3$. Now consider the quotient $U_4/Z_1(U_4)$. For brevity, cosets of the center will be denoted simply by any one of their representative elements. Note that $U_4/Z_1(U_4)$ is associative, since by §2.5 the right and left inner mappings only affect the $(1, 4)$-entries of the matrices in U_4. Consider cosets x and q, with $q_{ij} = 0$ if $j - i = 1$. By §2.5, one has $xT(q) = 0$. Indeed, $xT(q) = x$ holds for all cosets x if and only if $q_{ij} = 0$ when $j - i = 1$. Thus (3.2) is verified for $k = 2$. Finally, $U_4/Z_2(U_4)$ is abelian, so $Z_3(U_4) = U_4$. □

3.2 Zeroes on the superdiagonal.

Proposition 3.3. *If the vector (x_{13}, x_{24}) is non-zero, the conjugacy class of*

$$x = \begin{bmatrix} 1 & 0 & x_{13} & x_{14} \\ 0 & 1 & 0 & x_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is

$$\left\{ \begin{bmatrix} 1 & 0 & x_{13} & a \\ 0 & 1 & 0 & x_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \;\middle|\; a \in F \right\}. \tag{3.3}$$

PROOF: For $q = [q_{ij}]$ in U_4, one has

$$xT(q) = \begin{bmatrix} 1 & 0 & x_{13} & x_{14} + \begin{vmatrix} x_{13} & x_{24} \\ q_{12} & q_{34} \end{vmatrix} \\ 0 & 1 & 0 & x_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

by §2.5. Since the vector (x_{13}, x_{24}) is non-zero, each element a of F may be realized as

$$a = x_{14} + \begin{vmatrix} x_{13} & x_{24} \\ q_{12} & q_{34} \end{vmatrix}$$

for a suitable vector (q_{12}, q_{34}). Thus (3.3) is contained in the conjugacy class of x.

For the converse, set

$$y=\begin{bmatrix}1&0&x_{13}&a\\0&1&0&x_{24}\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

with an element a of F. Then for $q=[q_{ij}]$ in U_4, one has

$$yT(q)=\begin{bmatrix}1&0&x_{13}&a+\begin{vmatrix}x_{13}&x_{24}\\q_{12}&q_{34}\end{vmatrix}\\0&1&0&x_{24}\\0&0&1&0\\0&0&0&1\end{bmatrix},$$

which is contained in (3.3). Similarly, one has $yR(r,s)$ and $yL(r,s)$ in (3.3) for r,s in U_4, since the right and left inner mappings only affect the entry y_{14} of the matrix y. It follows that (3.3) contains the conjugacy class of x, and therefore coincides with it. □

3.3 The remaining cases.

Proposition 3.4. *If x_{23} is non-zero, the conjugacy class of x is*

$$\left\{\begin{bmatrix}1&x_{12}&b&a\\0&1&x_{23}&c\\0&0&1&x_{34}\\0&0&0&1\end{bmatrix}\,\middle|\,a,b,c\in F\right\}. \tag{3.4}$$

Proof: For $q=[q_{ij}]$ in U_4, one has $xT(q)=$

$$\begin{bmatrix}1&x_{12}&x_{13}+\begin{vmatrix}x_{12}&q_{12}\\x_{23}&q_{23}\end{vmatrix}&[xT(q)]_{14}\\0&1&x_{23}&x_{24}+\begin{vmatrix}x_{23}&q_{23}\\x_{34}&q_{34}\end{vmatrix}\\0&0&1&x_{34}\\0&0&0&1\end{bmatrix}$$

by §2.5. Set $q_{23}=0$. For elements b,c of F, choose q_{12} so that $b=x_{13}-q_{12}x_{23}$, and q_{34} so that $c=x_{24}+x_{23}q_{34}$. Then

$$xT(q)=\begin{bmatrix}1&x_{12}&b&k\\0&1&x_{23}&c\\0&0&1&x_{34}\\0&0&0&1\end{bmatrix}$$

for some element k of F. Now for r, s in U_4, one has

$$xT(q)R(r,s) = \begin{bmatrix} 1 & x_{12} & b & k + r_{12}x_{23}s_{34} - x_{12}s_{23}r_{34} \\ 0 & 1 & x_{23} & c \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

by §2.5. Consider an element a of F. Setting $r_{12} = 1$ and $r_{34} = 0$, choose s_{34} so that $a = k + x_{23}s_{34}$. Then

$$xT(q)R(r,s) = \begin{bmatrix} 1 & x_{12} & b & a \\ 0 & 1 & x_{23} & c \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

whence (3.4) is contained in the conjugacy class of x.

For the converse, set

$$y = \begin{bmatrix} 1 & x_{12} & b & a \\ 0 & 1 & x_{23} & c \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with elements a, b, c of F. Then for $q = [q_{ij}]$ in U_4, one has $yT(q) =$

$$\begin{bmatrix} 1 & x_{12} & b + \begin{vmatrix} x_{12} & q_{12} \\ x_{23} & q_{23} \end{vmatrix} & [yT(q)]_{14} \\ 0 & 1 & x_{23} & c + \begin{vmatrix} x_{23} & q_{23} \\ x_{34} & q_{34} \end{vmatrix} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

which lies in (3.4). Again, one has $yR(r,s)$ and $yL(r,s)$ in (3.4) for r, s in U_4, since the right and left inner mappings only affect the entry y_{14} of the matrix y. It follows that (3.4) contains the conjugacy class of x, and therefore coincides with it. □

Proposition 3.5. *Suppose that $x_{23} = 0$, while the vector (x_{12}, x_{34}) is non-zero. Then the conjugacy class of x is*

$$\left\{ \begin{bmatrix} 1 & x_{12} & x_{13} + bx_{12} & a \\ 0 & 1 & 0 & x_{24} - bx_{34} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| \, a, b \in F \right\}. \tag{3.5}$$

Proof: For $q = [q_{ij}]$ in U_4, one has $xT(q) =$

$$\begin{bmatrix} 1 & x_{12} & x_{13} + x_{12}q_{23} & [xT(q)]_{14} \\ 0 & 1 & 0 & x_{24} - q_{23}x_{34} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with $[xT(q)]_{14} = x_{14} + \begin{vmatrix} x_{12} & q_{12} \\ x_{24} & q_{24} \end{vmatrix} + \begin{vmatrix} x_{13} & q_{13} \\ x_{34} & q_{34} \end{vmatrix} + x_{12}q_{23}q_{34}$, by §2.5. Set $q_{23} = b$ for a given field element b. Then with $q_{12} = q_{34} = 0$, choose the vector (q_{13}, q_{24}) so that $a = x_{14} + \begin{vmatrix} x_{12} & q_{13} \\ x_{34} & q_{24} \end{vmatrix}$ for a given field element a. Since

$$xT(q) = \begin{bmatrix} 1 & x_{12} & x_{13} + bx_{12} & a \\ 0 & 1 & 0 & x_{24} - bx_{34} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

the set (3.5) is contained in the conjugacy class of x.

For the converse, take a typical element

$$y = \begin{bmatrix} 1 & x_{12} & x_{13} + bx_{12} & a \\ 0 & 1 & x_{23} & x_{24} - bx_{34} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

of (3.5), with a, b in F. Then $yT(q) =$

$$\begin{bmatrix} 1 & x_{12} & x_{13} + (b + q_{23})x_{12} & [yT(q)]_{14} \\ 0 & 1 & x_{23} & x_{24} - (b + q_{23})x_{34} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

for $q = [q_{ij}]$ in U_4, again giving an element of (3.5). As before, one has $yR(r, s)$ and $yL(r, s)$ in (3.5) for r, s in U_4, since the right and left inner mappings only affect the entry y_{14} of the matrix y. It follows that (3.5) contains the conjugacy class of x, and therefore coincides with it. □

Corollary 3.6. *Suppose that $x_{23} = 0$, while the vector (x_{12}, x_{34}) is non-zero. Then the conjugacy class of x has cardinality $|F|^2$.*

Remark 3.7. Corollary 3.6 shows that if F is finite, the size of the loop conjugacy class of $E^{12} + E^{34} - 1$ is $|F|^2$. On the other hand, the computations at the end of §3 of [5] show that the superclass of $E^{12} + E^{34} - 1$ has size $|F|^3$. Thus the loop conjugacy classes in U_4 do not necessarily coincide with the superclasses.

3.4 Summary. Table 1 lists the sizes and number of each kind of loop conjugacy class in U_4. The element types are identified by the pattern of matrix entries above the diagonal, in conjunction with the reference to the proposition giving the full description of the type. The symbol $*$ is used as a "wild card" to denote a (potentially) non-zero field element. As a pattern entry, the symbol F denotes arbitrary elements of F that appear in the class. The symbol q stands for the cardinality of the underlying field F.

Type of element	Size of class	Number of classes	Reference
$\begin{matrix} 0 & 0 & * \\ & 0 & 0 \\ & & 0 \end{matrix}$	1	q	Prop. 3.1
$\begin{matrix} 0 & * & F \\ & 0 & * \\ & & 0 \end{matrix}$	q	q^2-1	Prop. 3.3
$\begin{matrix} * & F & F \\ & *\neq 0 & F \\ & & * \end{matrix}$	q^3	$q^2(q-1)$	Prop. 3.4
$\begin{matrix} * & F & F \\ & 0 & F \\ & & * \end{matrix}$	q^2	$q(q^2-1)$	Prop. 3.5

TABLE 1. Conjugacy classes of U_4.

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(*Received* December 12, 2013)