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Nonconvex Lipschitz function in plane which is locally convex outside a discontinuum

DUŠAN POKORNÝ

Abstract. We construct a Lipschitz function on $\mathbb{R}^2$ which is locally convex on the complement of some totally disconnected compact set but not convex. Existence of such function disproves a theorem that appeared in a paper by L. Pasqualini and was also cited by other authors.

Keywords: convex function; convex set; exceptional set

Classification: 26B25, 52A20

1. Introduction

In his work from 1938 L. Pasqualini presents a theorem (see [4, Theorem 51, p. 43]) of which the following statement is a reformulation:

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous function and $M \subset \mathbb{R}^d$ a set not containing any continuum of topological dimension $(d - 1)$. Suppose that $f$ is locally convex on the complement of $M$. Then $f$ is convex on $\mathbb{R}^d$.

The proof however contains a gap. This result also appeared in the survey paper [1], where the (incorrect) proof was shortly repeated. Also V.G. Dmitriev mentions this result in [2], although he provides a wrong reference.

As a counterexample to the theorem of Pasqualini we present the following theorem:

Theorem 1.1. There is a Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$ and $M \subset \mathbb{R}^2$ such that

- $f$ is locally convex on $\mathbb{R}^2 \setminus M$,
- $f$ is not convex on $\mathbb{R}^2$,
- $M$ is compact and totally disconnected,
- $f$ has compact support.

Note that it is a simple observation that the set $M$ from Theorem 1.1 cannot be of one dimensional Hausdorff measure 0.
2. Preliminaries

In the paper we will use the following more or less standard notation and definitions. For $a,b \in \mathbb{R}^d$ and $r > 0$ we will denote by $B(a,r)$ the closed ball with center $a$ and radius $r$ and $[a,b]$ will denote the closed line segment with endpoints $a$ and $b$. For $A \subset \mathbb{R}^d$ the symbol $coA$ will mean the convex hull of $A$ and $A^c$ will mean the complement of $A$. If $l \subset \mathbb{R}^2$ is a line and $\varepsilon > 0$ then we define $l(\varepsilon) = \{x \in \mathbb{R}^2 : \text{dist}(x,l) < \varepsilon\}$.

A function $f$ defined on a set $A \subset \mathbb{R}^2$ is called $L$-Lipschitz, if for every $x, y \in A$, $x \not= y$, we have $|f(x) - f(y)| \leq L|x - y|$.

We will call $f$ locally convex on $A$ if for every $x, y$ such that $[x,y] \subset A$ and $\alpha \in [0,1]$ we have $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$.

Finally, $f$ will be called piecewise affine on $A$ if there is a locally finite triangulation $\Delta$ of $A$ such that $f$ is affine on every triangle from $\Delta$.

3. Construction of the function

Definition 3.1. Let $Q$ be the system of all unions of finite systems of (closed) polytopes in $\mathbb{R}^2$. Let $L > 0$, $f : \mathbb{R}^2 \to \mathbb{R}$ and $P \in Q$. We say that a pair $(P,f)$ is $L$-good if

1. $f$ is $L$-Lipschitz,
2. $f$ is piecewise affine on $P^c$,
3. $f$ is locally convex on $P^c$.

The key technical result is the following:

Lemma 3.2. Let $\delta, \varepsilon, L > 0$ and let $l$ be a line in $\mathbb{R}^2$. Let $(P,g)$ be an $L$-good pair. Then there is an $(L + \varepsilon)$-good pair $(Q,h)$ such that

1. $Q \subset P$,
2. $h = g$ on $P^c$,
3. if $x,y \in Q$ belong to different components of $\mathbb{R}^2 \setminus l(\delta)$ then they belong to different components of $Q$.

We first prove Theorem 1.1 using Lemma 3.2

Proof of Theorem 1.1: Choose a sequence $\{x_n\}_{n=1}^{\infty}$ dense in the plane and consider any sequence of lines $\{l_n\}_{n=1}^{\infty}$ with the property that for any $i,j \in \mathbb{N}$ there is some $k \in \mathbb{N}$ such that $x_i, x_j \in l_k$. Choose a sequence $\{\varepsilon_n\}_{n=1}^{\infty} \subset (0,\infty)$ such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. Then the sequence $\{l_n(\varepsilon_n)\}_{n=1}^{\infty}$ has the property that for every $x,y \in \mathbb{R}^2$, $x \not= y$, there is some $k \in \mathbb{N}$ such that $x$ and $y$ belong to the different component of $\mathbb{R}^2 \setminus l_k(\varepsilon_k)$.

In the proof we will proceed by induction and construct a sequence of functions $f_i : \mathbb{R}^2 \to \mathbb{R}$ and a sequence $\{P_i\} \subset Q$, $i = 0,1,\ldots$, such that for every $i$ the following conditions hold:

1. pair $(P_i,f_i)$ is $(1+\sum_{n=1}^{i} \varepsilon_n)$-good,
2. if $i > 0$ then $P_i \subset P_{i-1}$,
3. if $i > 0$ then $f_i = f_{i-1}$ on $(P_{i-1})^c$,
(4) if $i > 0$ and if $x, y \in P_i$ belong to the different component of $\mathbb{R}^2 \setminus l_i(\varepsilon_i)$ then they belong to the different component of $P_i$.

To do this let $f_0$ be an arbitrary $1$-Lipschitz function on $\mathbb{R}^2$ which is equal to 0 on $((-3, 3)^2)^c$ and equal to 1 on $[-1, 1]^2$ and put $P_0 := [-3, 3]^2 \setminus (-1, 1)^2$. Validity of conditions (1)–(4) follows directly from Lemma 3.2.

Now, if we have constructed $f_{i-1}$ and $P_{i-1}$ we obtain $f_i$ and $P_i$ simply by applying Lemma 3.2 with $\varepsilon = \delta = \varepsilon_i$, $L = (1 + \sum_{n=1}^{i-1} \varepsilon_n)$, $l = l_i$, $P = P_{i-1}$ and $g = f_{i-1}$. The function $f_i$ will be then equal to $h$ from the statement of Lemma 3.2 and $P_i$ will be equal to the corresponding $Q$. Validity of conditions (1)–(4) follows directly from Lemma 3.2.

Put $M := \bigcap P_i$. Due to property (2) $M$ is compact and nonempty. To prove that $M$ is totally disconnected consider $x, y \in M$, $x \neq y$. By the choice of the sequences $\{l_n\}_{n=1}^\infty$ and $\{\varepsilon_n\}_{n=1}^\infty \subset \mathbb{R}^+$ there is some $i$ such that $x$ and $y$ belong to the different component of $\mathbb{R}^2 \setminus l_i(\varepsilon_i)$. By property (3) we have that $x$ and $y$ belong to the different component of $P_i$. Using property (2) again we then obtain that $x$ and $y$ belong to the different component of $M$ as well.

Define $\tilde{f} : M^c \to \mathbb{R}$ in such a way that $\tilde{f}(x) = f_i(x)$ whenever $x \in (P_i)^c$. It is easy to see that the definition of $\tilde{f}$ is correct due to properties (2) and (3) and the definition of $M$, and also that by property (1) the function $\tilde{f}$ is $(1 + \sum_{n=1}^\infty \varepsilon_n)$-Lipschitz and locally convex on $M^c$. By Kirszbraun’s theorem (see [3]) there is a $(1 + \sum_{n=1}^\infty \varepsilon_n)$-Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f = \tilde{f}$ on $M^c$. Therefore $f$ is locally convex on $M^c$ as well. Also, $f$ has compact support due to properties (2) and (3), the fact that $P_0$ is compact and that $f_0$ is supported in $P_0$.

It remains to show that $f$ is not convex on $\mathbb{R}^2$, but this is easy since

$$\frac{f(-3, 0) + f(3, 0)}{2} = 0 < 1 = f(0, 0).$$

The proof of Lemma 3.2 is divided into several lemmas.

**Lemma 3.3.** Let $H \subset \mathbb{R}^2$ be a closed halfplane, $x \in \mathbb{R}^2 \setminus H$, $w \in \partial H$ and $L > 0$. If $f : H \cup \{x\} \to \mathbb{R}$ is $L$-Lipschitz and affine on $H$, then the function

$$g_w(u) = \begin{cases} f(u), & \text{if } u \in H, \\ \alpha f(x) + (1 - \alpha)f(w), & \text{for } u = \alpha x + (1 - \alpha)w, \alpha \in [0, 1], \end{cases}$$

is $L$-Lipschitz as well.

**Proof:** Without any loss of generality we can suppose that $f(w) = 0$ and $w = (0, 0)$. This means that $g_w$ is in fact linear on both $H$ and $[x, w]$. Choose $a \in H$
and \( b = \alpha x \) for some \( \alpha \in [0, 1] \). Now,

\[
|g_w(a) - g_w(b)| = \alpha \left| g_w \left( \frac{1}{\alpha} \right) - g_w \left( \frac{1}{\alpha} \right) \right| = \alpha \left| g_w \left( \frac{1}{\alpha} \right) - g_w \left( \frac{1}{\alpha} \alpha x \right) \right|
\]

\[
= \alpha \left| g_w \left( \frac{1}{\alpha} \right) - g_w (x) \right| \leq \alpha L \left| \frac{1}{\alpha} a - x \right| = \alpha L \left| \frac{1}{\alpha} a - \frac{1}{\alpha} \alpha x \right|
\]

\[
= L|a - \alpha x| = L|a - b|.
\]

Similarly, if \( a = \alpha x \) and \( b = \beta x \) for some \( \alpha, \beta \in [0, 1] \), \( \alpha \neq \beta \) we have

\[
|g_w(a) - g_w(b)| = |\alpha f(x) - \beta f(x)| = |f(x)| \cdot |\alpha - \beta| \leq L|x| \cdot |\alpha - \beta| = L|a - b|.
\]

\[\square\]

**Lemma 3.4.** Let \( \varepsilon, L, K > 0 \). Let \( f \) be an \( L \)-Lipschitz function on \([-K, K]^2\), which is equal to an affine function \( f_1 \) on \([-K, 0] \times [-K, K]\), and \( z \in (0, K) \times (-K, K) \). Then there is an \( x \in [(0, 0), z] \) and \( \gamma > 0 \) such that for every \( y \in B(x, \gamma) \) and every \( w \in B((0, 0), \gamma) \cap \{0\} \times (-K, K) \) the function

\[
g_{y,w}(u) = \begin{cases} 
  f(u), & \text{if } u \in [-K, 0] \times [-K, K], \\
  \alpha f(w) + (1 - \alpha)f(x), & \text{for } u = \alpha w + (1 - \alpha)y, \alpha \in [0, 1],
\end{cases}
\]

is \((L + \varepsilon)\)-Lipschitz and \( |g_{y,w} - f| < \varepsilon \) on \([-K, 0] \times [-K, K] \cup [w, y]\).

**Proof:** Without any loss of generality we can suppose that \( \varepsilon < 1, L = 1 \) and that \( f(0,0) = 0 \). Indeed, if \( f(0,0) \neq 0 \) we can just consider the function \( u \mapsto f(u) - f(0,0) \) in the place of \( f \) and then add \( f(0,0) \) to the resulting function \( g_{y,w} \).

If \( L \neq 1 \) then we can just consider the function \( u \mapsto f(u) \frac{L}{L} \) in the place of \( f \) and \( \frac{\varepsilon}{L} \) in the place of \( \varepsilon \) and multiply the resulting function \( g_{y,w} \) by \( L \).

Since \( f \) is 1-Lipschitz we can find a sequence \( \{x_i\}_{i=1}^{\infty} \subset [(0,0), z] \) converging to \((0,0)\) such that for some \( s \in [-1,1] \)

\[
s_i := \frac{f(x_i)}{|x_i|} \to s \quad \text{as} \quad i \to \infty.
\]

Denote \( \tilde{z} := \frac{z}{|z|} \). Consider now the sequence of functions \( h_i : [-K, 0] \times [-K, K] \cup \{\tilde{z}\} \to \mathbb{R} \) defined as

\[
h_i(u) := \frac{1}{|x_i|} f \left( \frac{|x_i|}{|x_i|} \cdot u \right).
\]

Then \( h_i \) is 1-Lipschitz for every \( i \). Since \( f \) is equal to an affine function \( f_1 \) on \([-K, 0] \times [-K, K]\) and \( f(0,0) = 0 \) we have \( h_i = f_1 \) on \( [-K, 0] \times [-K, 0] \cup [-K, K] \). Also \( h_i(\tilde{z}) = s_i \), because \( \tilde{z} = \frac{\tilde{z}}{|\tilde{z}|} = \frac{s_i}{|x_i|} \). Therefore by (3.1) the function \( h := \lim h_i : H \cup \{\tilde{z}\} \to \mathbb{R} \) which is equal to \( f_1 \) on \( H := (-\infty, 0] \times (-\infty, \infty) \) and such that \( h(\tilde{z}) = s \), is also 1-Lipschitz.
Consider \( \tilde{\gamma} > 0 \) such that \( \tilde{\gamma} < \frac{\varepsilon \tilde{z}_1}{4} \) (here by \( \tilde{z}_1 \) we mean the first coordinate of \( \tilde{z} \)). This choice then implies

\[
\frac{|v - \tilde{z}|}{|v - \tilde{z}| - \tilde{\gamma}} = 1 + \frac{\tilde{\gamma}}{|v - \tilde{z}| - \tilde{\gamma}} < 1 + \frac{\varepsilon \tilde{z}_1}{\tilde{z}_1 - \varepsilon \tilde{z}_1} = 1 + \frac{\varepsilon}{4 - \varepsilon}
\]

for \( v \in H \), which gives us inequality

\[
\frac{|v - \tilde{z}|}{|v - \tilde{z}| - \tilde{\gamma}} < 1 + \frac{\varepsilon}{2},
\]

as \( \varepsilon < 1 \). Now, for every \( \tilde{s} \in [s - \tilde{\gamma}, s + \tilde{\gamma}] \), \( v \in H \) and \( t \in B(\tilde{z}, \tilde{\gamma}) \)

\[
\frac{f_1(v) - \tilde{s}}{|v - t|} \leq \frac{|f_1(v) - s|}{|v - t|} + \frac{|s - \tilde{s}|}{|v - t|} \leq \frac{|f_1(v) - s|}{|v - \tilde{z}|} + \frac{\tilde{\gamma}}{|v - \tilde{z}| - \tilde{\gamma}} + \frac{2\tilde{\gamma}}{\tilde{z}_1} \leq \left(1 + \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} = 1 + \varepsilon.
\]

Therefore, by Lemma 3.3 for every \( \tilde{s} \in [s - \tilde{\gamma}, s + \tilde{\gamma}] \), \( w \in \{0\} \times (-\infty, \infty) \) and \( t \in B(\tilde{z}, \tilde{\gamma}) \) the function

\[
\tilde{h}_{w, t, \tilde{s}}(u) = \begin{cases} f_1(u), & \text{if } u \in H, \\ (1 - \alpha)\tilde{s} + \alpha f_1(w), & \text{for } u = (1 - \alpha)t + \alpha w, \alpha \in [0, 1], \end{cases}
\]

is \((1 + \varepsilon)\)-Lipschitz as well.

Choose \( i \) such that \( s_i \in [s - \tilde{\gamma}, s + \tilde{\gamma}] \) and put \( x = x_i \) and \( \gamma = \frac{|x|^2}{2} \). Now, consider some \( y \in B(x, \gamma) \) and some \( w \in B((0,0), \gamma) \cap \{0\} \times (-K, K) \) and let \( g_{y, w} \) be as in the statement of the lemma. First we will prove that \( g_{y, w} \) is \((1 + \varepsilon)\)-Lipschitz.

To do this we first observe that \( \frac{1}{|x|}g_{y, w}(|x| \cdot \xi) \) is equal to \( \tilde{h}_{w, t, \tilde{s}}(\xi) \), whenever the first function (as a function of \( \xi \)) is defined. Now, we have \( \frac{|y|}{|x|} \in \{0\} \times (-\infty, \infty) \),

\[
\left| \frac{y}{|x|} - \frac{x}{|x|} \right| = \left| \frac{y - x}{|x|} \right| \leq \frac{|y - x|}{2|x|} \leq \frac{|x| \gamma}{2|x|} \leq \tilde{\gamma},
\]

which means \( \frac{|y|}{|x|} \in B(\tilde{z}, \tilde{\gamma}) \) and finally \( f(\frac{|x|}{|x|}) = s_i \in [s - \tilde{\gamma}, s + \tilde{\gamma}] \) and we are done since \( \frac{|y|}{|x|}g_{y, w}(|x| \cdot \xi) \) (as a function of \( \xi \)) and \( g_{y, w} \) have the same Lipschitz constant.

To finish the proof it is now sufficient to observe that if we additionally choose \( x_i \) small enough we obtain also \( |g_{y, w} - f| < \varepsilon \) on \([-K, 0] \times [-K, K] \cup [w, y] \). \( \square \)

**Lemma 3.5.** Let \( L, \varepsilon, \delta > 0, a < b \) and \( c < d \) be given. Let

\[
P = \text{co}\{(−1, a), (−1, b), (1, c), (1, d)\}
\]

and

\[
P^\varepsilon = \text{co}\{(−1, a - \varepsilon), (−1, b + \varepsilon), (1, c - \varepsilon), (1, d + \varepsilon)\}.
\]
Suppose that $f$ is an $L$-Lipschitz function defined on $\mathbb{R}^2$ which is locally affine on $P^\varepsilon \setminus P$. Then there are

$$\frac{a + c}{2} =: a_0 < a_1 < \cdots < a_{n-1} < a_n := \frac{b + d}{2}$$

and $\frac{1}{2} > \kappa > 0$ such that, using the notation introduced below, the function $g_\kappa : P^\varepsilon \setminus (P \setminus [-\kappa, \kappa] \times \mathbb{R}) \to \mathbb{R}$ defined as $g_\kappa(z^\pm_i) = f(z^\pm_i)$ for $i = 0, n$, $g_\kappa(z^\pm_i) = f(z_i)$ for $i = 1, \ldots, n - 1$ and

$$g_\kappa(u) = \begin{cases} f(u), & \text{if } u \in P^\varepsilon \setminus P, \\
\alpha g(z^+_i) + \beta g(z^-_i) + \gamma g(z^+_i+1), & \text{for } u = \alpha z^+_i + \beta z^-_i + \gamma z^+_i+1, \\
\alpha g(z^-_i) + \beta g(z^+_i) + \gamma g(z^+_i+1), & \text{for } u = \alpha z^-_i + \beta z^+_i+1 + \gamma z^+_i+1,
\end{cases}$$

is $(L + \delta)$-Lipschitz and such that $|f - g_\kappa| < \delta$ on $\mathbb{R}^2$. Here we denoted $z^\pm_0 := (\pm \kappa, \frac{a + c}{2} \pm \frac{\kappa(e - a)}{2})$, $z^\pm_n := (\pm \kappa, \frac{b + d}{2} \pm \frac{\kappa(d - b)}{2})$, $z^\pm_i := (\pm \kappa, a_i)$ for $i = 1, \ldots, n - 1$ and $z_i := (0, a_i)$ for $i = 0, \ldots, n$.

**Proof:** Without any loss of generality we can suppose $L = 1$. Denote $P_i^\varepsilon$ the connectivity component of $P^\varepsilon \setminus P$ containing $z_i$, $i = 0, n$. When we have found $a_i$ we denote $P_i = \text{co}\{z^\pm_i, z^\pm_{i+1}\}$ for $i = 0, \ldots, n - 1$. Put $S = \text{co}\{z^\pm_1, z^\pm_n\}$ and $\alpha = \text{dist}(S, P^\varepsilon \setminus P)$. We always assume $\kappa$ to be small enough that $1 > \alpha > 0$.

First, we will use Lemma 3.4 twice to find points $a_1 \in B(a_0, \frac{\min(|a_0 - a_n|)}{2})$, $a_{n-1} \in B(a_n, \min(|a_0 - a_n|))$ and $\kappa_1 > 0$ such that for every $\kappa_1 > \kappa > 0$ the functions $g_\kappa|_{P_0^\varepsilon \cup P_0}$ and $g_\kappa|_{P_n^\varepsilon \cup P_n}$ are both $(1 + \delta)$-Lipschitz and such that $|f - g_\kappa| < \delta$ on $P_0^\varepsilon \cup P_0 \cup P_0 \cup P_{n-1}$. Here, in the notation of the points $z_i$, the point $z_1$ corresponds to the point $x$ guaranteed by Lemma 3.4 (when we identify $0$ with the origin) and similarly the point $z_{n-1}$ corresponds to $x$ in the case when we apply Lemma 3.4 centred in $z_n$. Note that although Lemma 3.4 guarantees $(1 + \delta)$-Lipschitzness on $P_0$ (or on $P_{n-1}$) only on line segments with one endpoint in $P_0^\varepsilon$ (or in $P_n^\varepsilon$), this is enough for our purposes. Indeed, if for instance $a, b \in \text{co}\{z^-_0, z^+_0, z^+_1\}$, we can always find $\tilde{a}, \tilde{b}$ with $\tilde{a} \in P_0^\varepsilon$ and such that the vector $a - b$ is parallel to the vector $\tilde{a} - \tilde{b}$. In such situation of course

$$\frac{|g_\kappa(a) - g_\kappa(b)|}{|a - b|} = \frac{|g_\kappa(\tilde{a}) - g_\kappa(\tilde{b})|}{|\tilde{a} - \tilde{b}|}.$$

Also, if $a, b \in \text{co}\{z^-_0, z^-_1, z^+_1\}$ one can always consider $\tilde{a} = z^-_1$ or $\tilde{a} = z^+_1$ such that

$$\frac{|g_\kappa(a) - g_\kappa(b)|}{|a - b|} \leq \frac{|g_\kappa(\tilde{a}) - g_\kappa(z^-_0)|}{|\tilde{a} - z^-_0|}.$$
Observe that for every $u_0 \in P_0^c \cup P_0$ and every $u_n \in P_n^c \cup P_{n-1}$ we have

\[
\frac{|g_\kappa(u_0) - g_\kappa(u_n)|}{|u_0 - u_n|} \leq \frac{|g_\kappa(u_0) - g_\kappa(z_0)|}{|u_0 - u_n|} + \frac{|g_\kappa(z_0) - g_\kappa(z_n)|}{|u_0 - u_n|} + \frac{|g_\kappa(z_n) - g_\kappa(u_n)|}{|u_0 - u_n|}.
\]

and since the last expression can be smaller than $1 + \delta$ when we assume $|a_0 - a_1|$ and $|a_{n-1} - a_n|$ to be small enough, we can additionally assume that $g|_{P_0^c \cup P_0 \cup P_{n-1}}$ is $(1 + \delta)$-Lipschitz.

Next, note that the function $g_\kappa|_{[z_1,z_{n-1}]}$ is actually independent on $\kappa$ and that it is 1-Lipschitz for any choice of $a_2, \ldots, a_{n-2}$ (this is true because in one dimension the affine extension never increases the Lipschitz constant). This also means that for $S = \text{co}\{z_1^\pm, z_{n-1}^\pm\}$ we have $g_\kappa|_S$ is 1-Lipschitz for any choice of $a_2, \ldots, a_{n-2}$ as well. Put $\alpha = \text{dist}(S, P^c \setminus P)$, we can assume $\kappa_2$ to be small enough that $1 > \alpha > 0$ (here we used the fact that $|a_0 - a_1|, |a_{n-1} - a_n| \leq \frac{1}{2}$). Consider $n$ big enough such that $\frac{|a_i| - a_{n-1}}{n - 1} \leq \frac{\alpha}{4}$, put $a_i = a_1 + \frac{i(a_1 - a_{n-1})}{n - 1}$ and pick $\kappa_3 < \min(\kappa_2, \frac{\alpha}{4})$.

Then for $\kappa < \kappa_3$ and $a \in S$

\[
|g_\kappa(a) - f(a)| \leq |g_\kappa(a) - g_\kappa(z_i)| + |g_\kappa(z_i) - f(z_i)| + |f(z_i) - f(a)|
\]

\[
\leq |a - z_i| + 0 + |a - z_i| \leq \frac{\delta}{2} < \delta,
\]

where $i$ is chosen such that $a \in P_i$.

To finish the proof we need to observe that for $\kappa < \kappa_3$ the function $g_\kappa$ is $(1 + \delta)$-Lipschitz. Since $S \cup P_0 \cup P_{n-1}$ is convex, the remaining case we have to consider is $a \in S$ and $b \in P^c \setminus P$. Find $i$ such that $a \in P_i$. With this choice we have $|a - z_i| \leq \frac{\alpha \delta}{2}$ and therefore

\[
|b - z_i| \leq |a - b| + |a - z_i| \leq |a - b| + \frac{\alpha \delta}{2} \leq (1 + \delta)|a - b|.
\]

Now, we have

\[
|g_\kappa(a) - g_\kappa(b)| \leq |g_\kappa(a) - g_\kappa(z_i)| + |g_\kappa(z_i) - g_\kappa(b)|
\]

\[
\leq \frac{\delta \alpha}{2} + |f(z_i) - f(b)| \leq \frac{\delta}{2}|a - b| + |b - z_i|
\]

\[
\leq \frac{\delta}{2}|a - b| + \left(1 + \frac{\delta}{2}\right) \cdot |a - b| \leq (1 + \delta)|a - b|.
\]

\[\square\]

**Lemma 3.6.** Let $1 > \varepsilon > 0$ and $\alpha, L > 0$. Let $f$ be a $L$-Lipschitz function on $[-1, 1]^2$ which is affine on both $[-1, 1] \times [-1, 0]$ and $[-1, 1] \times [0, 1]$ (and equal to affine functions $f_1$ and $f_2$, respectively). Put

\[
A_1 = [-1, -1/2] \times [-1, 0], A_2 = [1/2, 1] \times [0, 1],
\]
Proof: It follows from a direct computation. □

Lemma 3.7. Let $L, \alpha > 0$ and $1 > \gamma > \varepsilon > 0$. Let $f$ be a $L$-Lipschitz function on $[-4, 4]^2 \cup [1, 2] \times [4, 5]$ which is affine on both $[-4, 4] \times [-4, 0]$ and $[-4, 4] \times [0, 4] \cup [1, 2] \times [4, 5]$ (and equal to affine functions $f_1$ and $f_2$, respectively). Put

$$A_1 = [-3, -2] \times [0, \gamma], A_2 = [-3, 0] \times [\gamma, \gamma + \varepsilon], A_3 = [-1, 2] \times [\gamma - \varepsilon, \gamma],$$

$$A_4 = [1, 2] \times [\gamma, 4], B_1 = [-4, 4] \times [-4, 0], B_2 = [1, 2] \times [4, 5],$$

and

$$A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup B_1 \cup B_2.$$

Then either $f$ is locally convex on $[-4, 4]^2 \cup [1, 2] \times [4, 5]$ or the function

$$g(u) = \begin{cases} f_1(u), & \text{if } u \in A_1 \cup A_2 \cup B_1, \\ f_2(u) + \frac{f_1(0, \gamma) - f_2(0, \gamma)}{\gamma - 4}(u \cdot (0, 1) - 4), & \text{if } u \in A_3 \cup A_4, \\ f_2(u), & \text{if } u \in B_2, \end{cases}$$

is $(L + \alpha)$-Lipschitz, locally convex on $A$ and $|f - g| < \alpha$ on $A$, if $\varepsilon$ and $\gamma$ are small enough.

Proof: Without any loss of generality we can suppose $L = 1$. First we prove that $g$ is continuous on $A$. To do this we need to prove that

$$f_1(a, \gamma) = f_2(a, \gamma) + \frac{f_1(0, \gamma) - f_2(0, \gamma)}{\gamma - 4}((a, \gamma) \cdot (0, 1) - 4) \quad (3.3)$$

whenever $(\gamma, a) \in A_2 \cap A_3$ and that

$$f_2(a, 4) = f_2(a, 4) + \frac{f_1(0, \gamma) - f_2(0, \gamma)}{\gamma - 4}((a, 4) \cdot (0, 1) - 4) \quad (3.4)$$

whenever $(a, 4) \in A$. Define an affine function $f_3$ on $\mathbb{R}^2$ as

$$f_3(u, v) = \frac{f_1(0, \gamma) - f_2(0, \gamma)}{\gamma - 4}((u, v) \cdot (0, 1) - 4).$$
To prove (3.3) we can write
\[ g(a, \gamma) = f_2(a, \gamma) + f_3(a, \gamma) = f_2(a, \gamma) + \frac{f_1(0, \gamma) - f_2(0, \gamma)}{\gamma - 4} \cdot (\gamma - 4) = f_2(a, \gamma) + f_1(0, \gamma) - f_2(0, \gamma) + f_1(0, 0) \gamma - 4 \cdot (\gamma - 4) = f_2(a, \gamma) + f_1(0, \gamma) - f_1(0, 0) - f_2(0, \gamma) + f_2(0, 0) = f_2(a, \gamma) + f_1(0, \gamma) - f_1(a, 0) - f_2(a, \gamma) + f_1(a, 0) = f_1(a, \gamma). \]

To prove (3.4) we can write
\[ g(a, 4) = f_2(a, 4) + f_3(a, 4) = f_2(a, 4) + \frac{f_1(0, \gamma) - f_1(0, 0) - f_2(0, \gamma) + f_1(0, 0)}{\gamma - 4}(4 - 4) = f_2(a, 4). \]

Next note that since both \( f_1 \) and \( f_2 \) are 1-Lipschitz we have
\[ g \text{ is 1-Lipschitz on } B_1 \cup A_1 \cup A_2, \]
and
\[ g \text{ is 1-Lipschitz on } B_2. \]

Since additionally \( f_3 \) is constant on all lines parallel to \( x \)-axis and since
\[ \frac{f_3(0, \gamma) - f_3(0, 4)}{4 - \gamma} \leq \frac{f_1(0, \gamma) - f_1(0, 0) - f_2(0, \gamma) + f_2(0, 0)}{3} \leq \frac{2\gamma}{3} \leq \gamma. \]
we have
\[ g \text{ is } (1 + \gamma)\text{-Lipschitz on } A_4 \cup A_3 \]
and
\[ |g - f_2| \leq 4\gamma \text{ on } A_4 \cup A_3. \]

Now, if \( x \in B_1 \) and \( y \in A_3 \) then \( g(x) = f_1(x) \), \( |g(y) - f_1(y)| \leq 3\varepsilon \) and \( |x - y| \geq \gamma - \varepsilon \)
and therefore
\[ |g(x) - g(y)| \leq |g(x) - f_1(y)| + |f_1(y) - g(y)| \leq |x - y| + 3\varepsilon \leq \frac{\gamma + 2\varepsilon}{\gamma - \varepsilon}. \]
So we have
\[ g \text{ is } \frac{\gamma + 2\varepsilon}{\gamma - \varepsilon}\text{-Lipschitz on } B_1 \cup A_3. \]

If \( x \in B_1 \) and \( y \in A_4 \) then \( g(x) = f_1(x) \), \( f(y) \leq g(y) \leq f_1(y) \) and therefore
\[ g \text{ is 1-Lipschitz on } B_1 \cup A_4. \]
Using (3.6) and (3.7) and continuity of $g$ we obtain that

\begin{equation}
\text{(3.11)} \quad g \text{ is } (1 + \gamma)\text{-Lipschitz on } A_2 \cup A_3 \text{ and on } B_2 \cup A_4.
\end{equation}

Finally, if $x \in A_1 \cup A_2$ and $y \in A_4 \cup B_2$ or $x \in A_1$ and $y \in A_3 \cup A_4 \cup B_2$ we have

\begin{equation}
\text{(3.12)} \quad |g(x) - f_2(x)| \leq 2(\gamma + \epsilon) \leq 4\gamma, \quad |g(y) - f_2(y)| \leq 4\gamma
\end{equation}

and $|x - y| \geq 1$. This implies

\begin{equation}
\text{(3.13)} \quad |g(x) - g(y)| \leq |g(x) - f_2(x)| + |f_2(x) - f_2(y)| + |f_2(y) - g(y)|
\leq 4\gamma + |x - y| + 4\gamma \leq (1 + 8\gamma)|x - y|.
\end{equation}

Now, according to (3.5)–(3.12) it is sufficient to choose $2 \frac{\gamma}{\delta} > \gamma > \epsilon > 0$ small enough such that

\[
\max \left(1 + 8\gamma, \frac{\gamma + 2\epsilon}{\gamma - \epsilon} \right) < 1 + \alpha
\]

to obtain that $g$ is $(1 + \alpha)$-Lipschitz on $A$ and $|f - g| < \alpha$ on $A$. \hfill \Box

**Lemma 3.8.** Under the assumptions of Lemma 3.5 there is $\frac{1}{2} > \kappa > 0$, $R \subset P \cap (-\kappa, \kappa) \times \mathbb{R}$ and a function $h : \overline{P} \setminus \overline{\mathcal{P}} \cup R \to \mathbb{R}$ such that:

(a) $R \in \mathcal{Q}$,
(b) $h = f$ on $\overline{P} \setminus \overline{P}$,
(c) $h$ is locally convex on $\overline{P} \setminus \mathcal{P} \cup R$,
(d) $\overline{P} \setminus \overline{P} \cup R$ is connected,
(e) $h$ is piecewise affine on $\overline{P} \setminus \mathcal{P} \cup R$,
(f) $h$ is $(L + \delta)$-Lipschitz.

**Proof:** Without any loss of generality we can suppose $L = 1$. Let $\kappa, z_i$ and $g_\kappa$ be as in Lemma 3.5, but with $\frac{\delta}{2}$ in the place of $\delta$. Consider the sets

\[X = [-4, 4]^2 \cup [1, 2] \times [4, 5]\]
\[Y = [-1, 1]^2.\]

Find homotheties $\Psi_i : x \mapsto \rho_i x + v_i$, $\rho_i > 0$, $v_i \in \mathbb{R}^2$, $i = 1, \ldots, n - 1$ and orientation preserving similarities $\Psi_0$ and $\Psi_n$, with scaling ratios $\rho_0$ and $\rho_n$, such that if we put $M_i = \Psi_i(X)$, $i = 0, n$ and $M_i = \Psi_i(Y)$, $i = 1, \ldots, n - 1$ we have

(A) $M_i \cap M_j = \emptyset$ if $i \neq j$,
(B) $\Psi_0([-4, 4] \times [-4, 0]) \subset \overline{P} \setminus \overline{P}$,
(C) $\Psi_n([-4, 4] \times [-4, 0]) \subset \overline{P} \setminus \overline{P}$,
(D) $M_i \subset (-\kappa, \kappa) \times \mathbb{R}$,
(E) $[z_i^-, z_i^+] \subset \Psi_i(\mathbb{R} \times \{0\})$.

Put $\Omega = \min_{i \neq j} \text{dist} (M_i, M_j)$ and note that $\Omega > 0$ due to property (A). Define

\[T_i := \text{co} \{ \Psi_i(\frac{1}{2}, 1), \Psi_i(1, 1), \Psi_{i+1}(\frac{1}{2}, -1), \Psi_{i+1}(-1, -1) \}, \]
for \(i = 1, \ldots, n - 2\),
\[
T_0 := \text{co}\{\Psi_0(1,5), \Psi_0(2,5), \Psi_1(-\frac{1}{2}, -1), \Psi_1(-1, -1)\}
\]
and
\[
T_{n-1} := \text{co}\{\Psi_n(1,5), \Psi_n(2,5), \Psi_{n-1}(\frac{1}{2}, 1), \Psi_{n-1}(1, 1)\}.
\]
Put
\[
(3.14) \quad R := \left(\bigcup_{i=0}^{n-1} T_i\right) \cup \left(\bigcup_{i=0}^{n} M_i\right).
\]

Let \(g_i, i = 1, \ldots, n - 1\) be the function \(g\) from Lemma 3.6 with \(\alpha = \frac{\Omega \delta \rho_i}{4}\) (and corresponding \(\varepsilon\)) and with \(f_1(x) = \rho_i g_\kappa \circ \Psi_i\) and \(f_2(x) = \rho_i g_\kappa \circ \Psi_i\) (with the exception when \(g_\kappa\) is already convex on \(M_i\), in which case we put \(g_i = g_\kappa|_{M_i}\). Let \(g_0\) be the function \(g\) from Lemma 3.7 with \(\gamma = \frac{\Omega \delta \rho_i}{4}\) (and corresponding \(\varepsilon\) and \(\gamma\)) and with \(f_1 = \rho_0 g_\kappa \circ \Psi_0\) and \(f_2 = \rho_0 g_\kappa \circ \Psi_0\) and finally, let \(g_n\) be the function \(g\) from Lemma 3.7 with \(\gamma = \frac{\Omega \delta \rho_i}{4}\) (and corresponding \(\varepsilon\) and \(\gamma\)) and with \(f_1 = \rho_n g_\kappa \circ \Psi_n\) and \(f_2 = \rho_n g_\kappa \circ \Psi_n\).

Consider now the function \(h\) defined by the formula
\[
h = \begin{cases} \frac{1}{\rho_i} g_i \circ \Psi_i^{-1} & \text{on } M_i \\ g_\kappa & \text{otherwise.} \end{cases}
\]

Property (a) follows from (3.14) and the fact that every \(M_i\) and every \(T_i\) is a polygon. Properties (b), (c) and (e) follow directly from the construction and corresponding properties of the functions \(g_i\) and property (d) is obvious. We will now finish the proof by proving property (f).

So suppose that \(a, b \in (P^c \setminus P) \cup R\). We need to prove that \(|h(a) - h(b)| \leq (1 + \delta)|a - b|\). We can additionally suppose that either \(a\) or \(b\) belongs to some \(M_i\) since otherwise there is nothing to prove. We will prove only the case \(a \in M_i, b \in M_j, i \neq j\), the other cases can be proved following the same lines. By Lemma 3.6 (for \(i = 1, \ldots, n - 1\)) and Lemma 3.7 (for \(i = 0, n\)) we can now write
\[
|h(a) - h(b)| \leq |h(a) - g_\kappa(a)| + |g_\kappa(a) - g_\kappa(b)| + |g_\kappa(b) - h(b)|
\]
\[
< \frac{1}{\rho_i} \cdot \frac{\Omega \delta \rho_i}{4} + \left(1 + \frac{\delta}{2}\right) \cdot |a - b| + \frac{1}{\rho_j} \cdot \frac{\Omega \delta \rho_j}{4}
\]
\[
\leq \frac{\delta}{2} |a - b| + \left(1 + \frac{\delta}{2}\right) \cdot |a - b| = (1 + \delta)|a - b|,
\]
which is what we need.

\[\square\]

**Proof of Lemma 3.2:** Without any loss of generality we can suppose \(L = 1\). Let \(V\) be the set of all points \(v \in \partial P\) with the property that there is some \(\varepsilon_v > 0\) such that \(P \cap B(v, \varepsilon_v)\) is similar to \(\{(x, y) : x \geq 0\} \cap B(0, 1)\) and that \(g\) is affine
on $P \cap B(v, \varepsilon_v)$. Since $P \in Q$, the set $\partial P \setminus V$ is finite and without any loss of generality we can assume that $l(\delta) \cap (\partial P \setminus V) = \emptyset$. We can also assume that $l = \{0\} \times \mathbb{R}$ and that $\delta = 1$.

This means that the closure of every component $P_i$ of $P \cap l(\delta)$ is of the form
\[
\text{co}\{(-1, a_i), (-1, b_i), (1, c_i), (1, d_i)\}
\]
for some $a_i < b_i$, $c_i < d_i$ and such that, for some $\varepsilon_i > 0$, $g$ is locally affine on $P_i^{\varepsilon_i} \setminus P_i$, where
\[
P_i^{\varepsilon_i} := \text{co}\{(-1, a_i - \varepsilon_i), (-1, b_i + \varepsilon_i), (1, c_i - \varepsilon_i), (1, d_i + \varepsilon_i)\}.
\]
Then we have
\[
\alpha = \min_{i \neq j} \text{dist} (P_i, P_j) > 0.
\]
Let $\kappa_i$, $R_i$ and $h_i$ be equal to $\kappa$, $R$ and $h$ obtained from Lemma 3.8 for $\varepsilon_i$ in the place of $\varepsilon$, $P_i$ in the place of $P$, $g$ in the place of $f$ and $\frac{\min(\alpha, \varepsilon_i, 1)}{4}$ in the place of $\delta$.

Put $Q = P \setminus (\bigcup R_i)$ and define $\tilde{h} : Q^c \to \mathbb{R}$ by
\[
\tilde{h}(u) = \begin{cases} h_i(u) & \text{on } R_i \\ g(u) & \text{otherwise}. \end{cases}
\]

Let $K$ be the Lipschitz constant of $\tilde{h}$. Using the Kirszbraun theorem we can find a $K$-Lipschitz function $h$ on $\mathbb{R}^2$ such that $h = \tilde{h}$ on $P^c$.

Now, property (1) follows directly from the definition of $Q$ and (a) in Lemma 3.8, property (2) from the definition of $h$ and (b) in Lemma 3.8 and property (3) from (d) in Lemma 3.8.

It remains to prove that the pair $(Q, h)$ is $(1 + \varepsilon)$-good. The local convexity and piecewise affinity of $h$ on $Q^c$ follow from (c) and (e) in Lemma 3.8 and the corresponding properties of $g$, so the proof will be finished, if we verify that $K \leq (1 + \varepsilon)$.

To do this pick $a, b \in \mathbb{R}^2$, we need to prove that $|h(a) - h(b)| \leq (1 + \varepsilon)|a - b|$. We can additionally suppose that either $a$ or $b$ belongs to some $R_i$ since otherwise there is nothing to prove. We will prove only the case $a \in R_i$, $b \in R_j$, $i \neq j$, the other cases can be proved following the same lines.

Using the definition of $h$, namely property (f) from Lemma 3.8 we can now write
\[
|h(a) - h(b)| = |h_i(a) - h_j(b)| \leq |h_i(a) - f(a)| + |f(a) - f(b)| + |f(b) - h_j(b)|
\]
\[
\leq \frac{\min(\alpha, \varepsilon_i, 1)}{4} + (1 + \frac{\varepsilon}{4}) \cdot |a - b| + \frac{\min(\alpha, \varepsilon_j, 1)}{4} \varepsilon
\]
\[
\leq \frac{2\varepsilon}{4} |a - b| + (1 + \frac{\varepsilon}{2}) \cdot |a - b| < (1 + \varepsilon)|a - b|.
\]

\hfill \Box
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