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# $(n,d)$-INJECTIVE COVERS, $n$-COHERENT RINGS, AND $(n,d)$-RINGS

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*Abstract.* It is known that a ring $R$ is left Noetherian if and only if every left $R$-module has an injective (pre)cover. We show that (1) if $R$ is a right $n$-coherent ring, then every right $R$-module has an $(n,d)$-injective (pre)cover; (2) if $R$ is a ring such that every $(n,0)$-injective right $R$-module is $n$-pure extending, and if every right $R$-module has an $(n,0)$-injective cover, then $R$ is right $n$-coherent. As applications of these results, we give some characterizations of $(n,d)$-rings, von Neumann regular rings and semisimple rings.

*Keywords*: cover; envelope; $n$-coherent ring; $(n,d)$-injective; $(n,d)$-ring

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## 1. Introduction

Throughout this paper, $R$ is an associative ring with identity and all modules are unitary right $R$-modules. $\mathrm{Hom}(M,N)$ and $\mathrm{Ext}^m(M,N)$ mean $\mathrm{Hom}_R(M,N)$ and $\mathrm{Ext}^m_R(M,N)$, and $rD(R)$ and $wD(R)$ denote the usual right and weak, respectively, global dimension of a ring $R$.

Let $\mathscr{F}$ be a class of right $R$-modules and $M$ a right $R$-module. Following [7], a homomorphism $\phi\colon F \to M$ with $F \in \mathscr{F}$ is called an *$\mathscr{F}$-precover* of $M$ if for any homomorphism $f\colon F' \to M$ with $F' \in \mathscr{F}$, there is a homomorphism $g\colon F' \to F$ such that $\phi g = f$. Moreover, if the only such $g$ is an automorphism of $F$ when $F' = F$ and $f = \phi$, then the $\mathscr{F}$-precover $\phi$ is called an *$\mathscr{F}$-cover*. Dually, we have the definitions of an $\mathscr{F}$-preenvelope and an $\mathscr{F}$-envelope. We say that $\mathscr{F}$ is *(pre)covering* or *(pre)enveloping* provided every right $R$-module has an $\mathscr{F}$-(pre)cover or $\mathscr{F}$-(pre)envelope, respectively.

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Let $n$ and $d$ be non-negative integers. Following [4], we call a right $R$-module $P$ *$n$-presented* if it has an $n$-presentation, that is, there exists an exact sequence of right $R$-modules

$$F_n \to F_{n-1} \to \ldots \to F_1 \to F_0 \to P \to 0$$

where each $F_i$ is finitely generated free (equivalently projective), $i = 0, 1, \ldots, n$. It is clear that every $m$-presented $R$-module is $n$-presented for $m \geqslant n$. A ring $R$ is called right *$n$-coherent* [4] provided every $n$-presented right $R$-module is $(n+1)$-presented. It is easy to see that $R$ is right 0-coherent or 1-coherent if and only if $R$ is right Noetherian coherent, respectively. Following [4] and [16], $R$ is said to be a right $(n, d)$-ring if every $n$-presented right $R$-module has projective dimension at most $d$. A right $R$-module $M$ is called *$(n, d)$-injective* [16] if $\mathrm{Ext}^{d+1}(N, M) = 0$ for any $n$-presented right $R$-module $N$. The $(1, 0)$-injective modules are also known as *absolutely pure modules* [11] and *$FP$-injective modules* [13]. For unexplained concepts and notation we refer the reader to [17], [3], [12], and [14].

It is known that a ring $R$ is left Noetherian if and only if every left $R$-module has an injective (pre)cover (see [8], Theorem 5.4.1). Recently, Katherine Pinzon proved that if $R$ is left coherent, then every left $R$-module has a $(1, 0)$-injective (pre)cover (see [11], Theorem 2.6 and Corollary 2.7). On the other hand, Mao and Ding [10], Theorem 3.9, proved that the class of $(n, d)$-injective $R$-modules is preenveloping for any ring $R$. It is natural to ask what conditions on $R$ imply that the class of $(n, d)$-injective modules is precovering and what conditions on $R$ imply that the class of $(n, d)$-injective modules is covering?

In Section 3, we show that (1) if $R$ is a right $n$-coherent ring, then every right $R$-module has an $(n, d)$-injective (pre)cover; (2) if $R$ is a ring such that every $(n, 0)$-injective right $R$-module is $n$-pure extending, and if every right $R$-module has an $(n, 0)$-injective cover, then $R$ is right $n$-coherent $(n \geqslant 1)$; (3) $R$ is a right $n$-coherent ring if and only if every $n$-pure submodule of an $(n, 1)$-injective right $R$-module is $(n, 1)$-injective $(n \geqslant 1)$.

In Section 4, as applications of the previous results, we give some characterizations of $(n, d)$-rings. We show that $R$ is a right $(n, d)$-ring if and only if $R$ is a right $(n, d)$-$FC$ ring and the kernel of any $(n, d)$-injective cover of a right $R$-module is $(n, d)$-injective if and only if $R$ is a right $(n, d)$-$FC$ ring and the cokernel of any $(n, d)$-injective preenvelope of a right $R$- module is $(n, d)$-injective if and only if $R$ is a right $(n, d)$-$FC$ ring and $R$ is a right $(n, d + m)$-ring for some $m \geqslant 0$ if and only if $R$ is a right $(n, d)$-$FC$ ring and every right $R$-module has an $(n, d + m)$-injective cover with the unique mapping property for some $m \geqslant 0$. Some known results are extended or obtained as corollaries. For example, we get that $R$ is von Neumann regular if and only if $R$ is a right $FC$ ring and the kernel of any $(1, 0)$-injective cover of a right $R$-module is $(1,0)$-injective if and only if $R$ is a right $FC$ ring and every right $R$-module has a $(1,m)$-injective cover with the unique mapping property for some $m \geqslant 0$ if and only if $R$ is a right $FC$ ring and $wD(R) < \infty$; $R$ is semisimple if and only if $R$ is a $QF$ ring and every right $R$-module has a $(0,m)$-injective cover with the unique mapping property for some $m \geqslant 0$ if and only if $R$ is a $QF$ ring and $rD(R) < \infty$.

## 2. Preliminaries

The results listed in this section will be important ingredients in proving our main results.

**Proposition 2.1** ([10], Theorem 4.1). *The following assertions are equivalent for a ring $R$ and $n \geqslant 1$:*

(1) *$R$ is right $n$-coherent.*

(2) *For any short exact sequence $0 \to A \to B \to C \to 0$ of right $R$-modules, if $A$ and $B$ are $(n,0)$-injective, then $C$ is $(n,0)$-injective.*

Recall that a short exact sequence $0 \to A \to B \to C \to 0$ is said to be *$n$-pure* [10], Definition 3.5, if the sequence $\mathrm{Hom}(M,B) \to \mathrm{Hom}(M,C) \to 0$ is exact for any $n$-presented $R$-module $M$. A submodule $A \subset B$ is called *$n$-pure* if the sequence $0 \to A \to B \to B/A \to 0$ is $n$-pure. It is clear that $A$ is 1-pure in $B$ if and only if it is pure, and if $A$ is $n$-pure in $B$, then $A$ is $m$-pure for any $m \geqslant n$.

**Proposition 2.2** ([10], Proposition 3.6). *A module $M$ is $(n,0)$-injective if and only if it is an $n$-pure submodule of an $(n,0)$-injective module $N$.*

In the following, we assume that $n$ and $d$ are non-negative integers.

**Proposition 2.3** ([16], Proposition 3.1). *Let $R$ be a right $n$-coherent ring. Then every direct limit of $(n,d)$-injective right $R$-modules is $(n,d)$-injective.*

The next proposition says that the class of $(n,d)$-injective modules is closed under extensions.

**Proposition 2.4.** *For any short exact sequence $0 \to A \to B \to C \to 0$ of right $R$-modules, if $A$ and $C$ are $(n,d)$-injective, then $B$ is $(n,d)$-injective.*

Proof. It is straightforward. □

**Proposition 2.5** ([10], Lemma 3.4). *Let $R$ be a right $n$-coherent ring. Then the class of $(n,d)$-injective right $R$-modules is closed under cokernel of monomorphisms.*

**Proposition 2.6.** *Let $R$ be a ring. Let $\mathscr{F}$ be the class of right $R$-modules closed under summands and isomorphisms. If $\mathscr{F}$ is precovering, then $\mathscr{F}$ is closed under direct sums.*

Proof. Let $(F_i)_{i\in I}$ be a family of right $R$-modules such that each $F_i \in \mathscr{F}$. Then we get an $\mathscr{F}$-precover $f\colon F \to \bigoplus F_i$. For each $j \in I$, let $l_j\colon F_j \to \bigoplus F_i$ be the canonical injection. Then there exists a homomorphism $g_j\colon F_j \to F$ such that $l_j = fg_j$. In addition, there is a homomorphism $\varphi\colon \bigoplus F_i \to F$ such that $g_j = \varphi l_j$, and hence $l_j = f\varphi l_j$. So $f\varphi$ is an isomorphism. Thus, $\bigoplus F_i$ is isomorphic to a summand of $F$ by [1], Lemma 5.1, and so $\bigoplus F_i \in \mathscr{F}$. $\square$

## 3. $(n,d)$-injective covers and $n$-coherent rings

To show that over a right $n$-coherent ring $R$ the class of $(n,d)$-injective right $R$-modules is covering, we need the following four lemmas.

**Lemma 3.1.** *Let $R$ be a right $n$-coherent ring. Then every $m$-pure submodule of an $(n,d)$-injective right $R$-module is $(n,d)$-injective, for any non-negative integer $m$.*

Proof. Let $N$ be an $m$-pure submodule of an $(n,d)$-injective right $R$-module $M$, and $P$ an $n$-presented right $R$-module. Since $R$ is right $n$-coherent, $P$ has an $(m+d)$-presentation

$$F_{m+d} \to F_{m+d-1} \to \ldots \to F_1 \to F_0 \to P \to 0.$$

Let $K = \ker(F_{d-1} \to F_{d-2})$, then $K$ is $m$-presented. Since $M$ is $(n,d)$-injective, $\operatorname{Ext}^1(K,M) \cong \operatorname{Ext}^{d+1}(P,M) = 0$. In addition, the short exact sequence $0 \to N \to M \to M/N \to 0$ induces a long exact sequence

$$0 \to \operatorname{Hom}(K,N) \to \operatorname{Hom}(K,M) \to \operatorname{Hom}(K,M/N) \to \operatorname{Ext}^1(K,N) \to \operatorname{Ext}^1(K,M) = 0.$$

Note that $N$ is an $m$-pure submodule of $M$. So $\operatorname{Hom}(K,M) \to \operatorname{Hom}(K,M/N) \to 0$ is exact and hence $\operatorname{Ext}^{d+1}(P,N) \cong \operatorname{Ext}^1(K,N) = 0$, that is, $N$ is $(n,d)$-injective. $\square$

A deep result of Robort El Bashir [2], Theorem 5, is that given a ring $R$ and given a cardinal $\lambda$, there is a cardinal $\kappa$ such that if $\operatorname{Card} M \geqslant \kappa$ and $\operatorname{Card} M/L \leqslant \lambda$ then $L$ contains a nonzero pure submodule of $M$.

**Lemma 3.2.** *Let $R$ be a right $n$-coherent ring, and $M$ an $(n,d)$-injective right $R$-module. Given a cardinal $\lambda$, there is a cardinal $\kappa$ such that if $\operatorname{Card} M \geqslant \kappa$ and $\operatorname{Card} M/L \leqslant \lambda$ then $L$ contains a nonzero $(n,d)$-injective submodule of $M$.*

P r o o f. This follows from Lemma 3.1 and [2], Theorem 5. □

The proof of the following lemma is similar to that of [11], Lemma 2.5.

**Lemma 3.3.** *Let $R$ be right $n$-coherent and let $\operatorname{Card} M = \lambda$ for a right $R$-module $M$. There is a cardinal $\kappa$ such that any homomorphism $E \to M$ with $E$ $(n,d)$-injective has a factorization $E \to E' \to M$ with $E'(n,d)$-injective and $\operatorname{Card} E' < \kappa$.*

P r o o f. For any homomorphism $E \to M$ with $E$ $(n,d)$-injective, by Lemma 3.2, we get a cardinal $\kappa$ such that if $\operatorname{Card} E \geqslant \kappa$ and $\operatorname{Card} E/L \leqslant \lambda$ then $L$ contains a nonzero $(n,d)$-injective submodule of $E$. If $\operatorname{Card} E < \kappa$, let $E' = E$ and we are done. So assume $\operatorname{Card} E \geqslant \kappa$. We can choose a submodule $A \subset E$ maximal with respect to the two properties that $A$ is $(n,d)$-injective and that $A \subset \ker(E \to M)$. Let $E' = E/A$. Then it is easy to see that the homomorphism $E \to M$ has a factorization $E \to E' \to M$. Note that $R$ is right $n$-coherent. So by Proposition 2.5, the exactness of the sequence $0 \to A \to E \to E' \to 0$ implies that $E'$ is $(n,d)$-injective. Next we argue that $\operatorname{Card} E' < \kappa$. Suppose $\operatorname{Card} E' \geqslant \kappa$. Let $K = \ker(E' \to M)$. Clearly, $\operatorname{Card} E'/K \leqslant \operatorname{Card} M = \lambda$. Again by Lemma 3.2, there is a nonzero $(n,d)$-injective submodule $B/A$ of $E/A$ contained in $K$, and so $B \subset \ker(E \to M)$. Considering the exact sequence $0 \to A \to B \to B/A \to 0$, we see that $B$ is also $(n,d)$-injective by Proposition 2.4. This contradicts the choice of $A$. Hence, $E \to M$ has a factorization $E \to E' \to M$ with $E'$ $(n,d)$-injective and $\operatorname{Card} E' < \kappa$. □

**Lemma 3.4** ([11], Lemma 2.4). *Let $\mathscr{F}$ be a class of $R$-modules that is closed under direct sums. If $\mathscr{X} \subset \mathscr{F}$, for some set $\mathscr{X}$, is such that any homomorphism $F \to M$ with $F \in \mathscr{F}$ can be factored $F \to X \to M$ for some $X \in \mathscr{X}$, then $M$ has an $\mathscr{F}$-precover.*

**Theorem 3.5.** *Let $R$ be a right $n$-coherent ring. Then every right $R$-module has an $(n,d)$-injective precover.*

P r o o f. Let $M$ be any right $R$-module with $\operatorname{Card} M = \lambda$. Then by Lemma 3.3, there is a cardinal $\kappa$ such that any homomorphism $E \to M$ with $E$ $(n,d)$-injective has a factorization $E \to E' \to M$ with $E'$ $(n,d)$-injective and $\operatorname{Card} E' < \kappa$. Let $\mathbf{X}$ be any set with $\operatorname{Card} \mathbf{X} = \kappa$. Let $\mathbf{A}$ be all $(n,d)$-injective right $R$-modules such that $\mathbf{A} \subset \mathbf{X}$ (as sets). Hence, replacing $E'$ by an isomorphic copy we may assume $E' \subset \mathbf{X}$ (as a set), and so we can apply Lemma 3.4. Thus, the conclusion follows. □

**Theorem 3.6.** *Let $R$ be a right $n$-coherent ring, then every right $R$-module has an $(n,d)$-injective cover.*

Proof. By Theorem 3.5, we get that every right $R$-module has an $(n,d)$-injective precover. But for a right $n$-coherent ring $R$, the class of $(n,d)$-injective right $R$-modules is closed under well ordered inductive limits by Proposition 2.3. Hence, the result follows from [8], Corollary 5.2.7. □

**Corollary 3.7** ([8], Theorem 5.4.1). *$R$ is right Noetherian if and only if every right $R$-module has an injective precover if and only if every right $R$-module has an injective cover.*

Proof. This follows from Theorem 3.5, Theorem 3.6, Proposition 2.6, and the fact that $R$ is right Noetherian if and only if the class of injective right $R$-modules is closed under direct sums. □

By Theorem 3.5, Theorem 3.6, and Corollary 3.7, we have

**Corollary 3.8.** *$R$ is right Noetherian if and only if every right $R$-module has a $(0,d)$-injective precover if and only if every right $R$-module has a $(0,d)$-injective cover for any non-negative integer $d$.*

**Corollary 3.9** ([11], Theorem 2.6). *If $R$ is a right coherent ring, then every right $R$-module has an absolutely pure precover.*

An $(n,d)$-injective (pre)cover is not necessarily an epimorphism. It is known that if $R$ is right coherent or Noetherian and $R_R$ (as a right $R$-module) is $FP$-injective or injective, respectively, then every right $R$-module has an epimorphic $FP$-injective or injective cover. In general, we have

**Corollary 3.10.** *The following are equivalent for a right $n$-coherent ring $R$:*

(1) *Every right $R$-module has an $(n,d)$-injective (pre)cover which is an epimorphism;*
(2) *$R_R$ has an $(n,d)$-injective (pre)cover which is an epimorphism;*
(3) *$R_R$ is $(n,d)$-injective;*
(4) *every $(n,d)$-injective (pre)cover of a right $R$-module is an epimorphism.*

Proof. Obvious. □

**Definition 3.11.** Let $n \geqslant 1$. An *$n$-pure* monomorphism is a monomorphism $A \to B$ whose image is an $n$-pure submodule of $B$. A module $B$ is called *$n$-pure extending* if for any $n$-pure submodule $A \subset B$, any $n$-pure monomorphism $A \to B$ can be extended to $B \to B$.

We note that the class of $n$-pure extending modules contains all *quasi-injective* modules [15], *pure injective* modules [9] and simple modules, and a homomorphism is an 1-pure monomorphism if and only if it is a *pure monomorphism* [9].

**Lemma 3.12.** *Let $n \geqslant 1$. Let $B$ be an $n$-pure extending right $R$-module and $A \subset B$ an $n$-pure submodule. Let $f\colon A \to B$ be an $n$-pure monomorphism. Then there is a homomorphism $h\colon B \to B$ such that $hf = 1_A$ where $1_A$ is the identity homomorphism of $A$.*

Proof. Since $f$ is a monomorphism, we get a homomorphism $f^{-1}\colon \operatorname{Im}(f) \to A$. By hypothesis, there exists a homomorphism $h\colon B \to B$ such that the restriction $h|_{\operatorname{Im}(f)} = f^{-1}$. So $hf = 1_A$. □

Let $\mathscr{F}$ be a class of $R$-modules. We will denote by $\mathscr{F}^{\perp} = \{C\colon \operatorname{Ext}^1(F,C) = 0$ for all $F \in \mathscr{F}\}$ the right orthogonal class of $\mathscr{F}$.

A question posed by Pinzon [11], Remark 2.8, is whether $R$ must necessarily be right coherent in order that every right $R$-module have a $(1,0)$-injective cover. The following theorem gives a partial answer to this question.

**Theorem 3.13.** *Let $n \geqslant 1$. Let $R$ be a ring such that every $(n,0)$-injective right $R$-module is $n$-pure extending. If every right $R$-module has an $(n,0)$-injective cover, then $R$ is right $n$-coherent.*

Proof. For convenience, we let $\mathscr{F}$ denote the class of $(n,0)$-injective right $R$-modules. Let $0 \to A \to B \to C \to 0$ be any exact sequence of right $R$-modules where $A$ and $B \in \mathscr{F}$. We want to show that $C \in \mathscr{F}$. By hypothesis, $C$ has an $\mathscr{F}$-cover $F \to C$. Then it is easy to see that $F \to C$ is an epimorphism. Using a pullback construction for

$$\begin{array}{ccc} & & F \\ & & \downarrow \\ B & \longrightarrow & C \end{array}$$

we get a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & K & \longrightarrow & K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & M & \longrightarrow & F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns. Since both $A \in \mathscr{F}$ and $F \in \mathscr{F}$, we have $M \in \mathscr{F}$ by Proposition 2.4. Since $F \to C$ is an $\mathscr{F}$-cover, $K = \ker(F \to C) \in \mathscr{F}^{\perp}$ by [8], Corollary 7.2.3, page 156. Note that $B \in \mathscr{F}$. So $\operatorname{Ext}^1(B, K) = 0$. Thus, the middle column of the diagram above is split exact, and hence $K \in \mathscr{F}$. We claim: $K = 0$. Let

$$0 \longrightarrow K \xrightarrow{\iota} E \xrightarrow{\pi} L \longrightarrow 0$$

be exact with $E$ injective. By hypothesis, $L$ has an $\mathscr{F}$-cover $\gamma\colon D \to L$. Then $\gamma$ is an epimorphism. We construct the pullback diagram of $(L, \gamma, \pi)$ and get the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & K_1 & \longrightarrow & K_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \xrightarrow{i} & N & \xrightarrow{f_2} & D & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \xrightarrow{\iota} & E & \xrightarrow{\pi} & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns. Similarly to the proof above, we have $N \in \mathscr{F}$ and the middle column of the diagram above is split exact. Hence $N \cong K_1 \oplus E$. That is, $E \cong G$ for an injective submodule $G \subset N$. Since $\operatorname{Im}(i) \cong K \cong \operatorname{Im}(\iota) \subset E \cong G \subset N$, we have a monomorphism $h_1\colon \operatorname{Im}(i) \to G$ which induces an exact sequence $0 \longrightarrow \operatorname{Im}(i) \xrightarrow{h_1} N \xrightarrow{h_2} P \longrightarrow 0$, where $P = N/\operatorname{Im}(h_1)$. Note that $\operatorname{Im}(i)$ is $(n, 0)$-injective. So both $\operatorname{Im}(i)$ and $\operatorname{Im}(h_1)$ are $n$-pure in $N$ by Proposition 2.2. Thus, by Lemma 3.12, there is a homomorphism $\theta\colon N \to N$ such that $\theta h_1 = f_1$ where $f_1$ is the identity homomorphism of $\operatorname{Im}(i)$. On the other hand, the sequence $0 \longrightarrow \operatorname{Im}(i) \xrightarrow{f_1} N \xrightarrow{f_2} D \longrightarrow 0$ is exact. Thus, we obtain the rows exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Im}(i) & \xrightarrow{h_1} & N & \xrightarrow{h_2} & P & \longrightarrow & 0 \\
 & & \| & & \downarrow \theta & & \downarrow \psi & & \\
0 & \longrightarrow & \operatorname{Im}(i) & \xrightarrow{f_1} & K_1 & \xrightarrow{f_2} & D & \longrightarrow & 0
\end{array}$$

Next we construct an exact sequence $0 \longrightarrow N \xrightarrow{\alpha} P \oplus N \xrightarrow{\beta} D \longrightarrow 0$. Define

$\alpha\colon N \to P \oplus N$ such that $\alpha(x) = (h_2(x), \theta(x))$ for any $x \in N$;

$\beta\colon P \oplus N \to D$ such that $\beta(p, x) = f_2(x) - \psi(p)$ for any $(p, x) \in P \oplus N$.

If $\alpha(x) = 0 = (h_2(x), \theta(x))$, then $h_2(x) = 0$. So there is $y \in \mathrm{Im}(i)$ such that $h_1(y) = x$. Hence $0 = \theta(x) = \theta h_1(y) = f_1(y)$. Noting that $f_1$ is monomorphic, we have $y = 0$. Thus, $x = h_1(y) = 0$, and we see that $\alpha$ is monomorphic. Clearly, $\beta$ is epimorphic and $\beta\alpha = 0$.

Let $(p, x) \in \ker(\beta)$. Then $\beta(p, x) = f_2(x) - \psi(p) = 0$. Since $h_2$ is epimorphic, there is $x' \in N$ such that $h_2(x') = p$. So $f_2\theta(x') = \psi h_2(x') = \psi(p) = f_2(x)$, and $f_2(\theta(x') - x) = 0$. Thus, there is $y \in \mathrm{Im}(i)$ such that $\theta h_1(y) = f_1(y) = \theta(x') - x$. This means that $x = \theta(x') - \theta h_1(y) = \theta(x' - h_1(y))$. In addition, $h_2(x' - h_1(y)) = h_2(x') - h_2 h_1(y) = h_2(x') = p$. So $(p, x) = (h_2(x' - h_1(y)), \theta(x' - h_1(y))) \in \mathrm{Im}(\alpha)$. Now we get an exact sequence

$$0 \longrightarrow N \xrightarrow{\alpha} P \oplus N \xrightarrow{\beta} D \longrightarrow 0$$

with $N$, $D \in \mathscr{F}$. Hence $P \oplus N \in \mathscr{F}$ and so $P \in \mathscr{F}$. Since $G$ is injective, we get a commutative diagram

$$\begin{array}{ccccc} & & G & & \\ & & {\scriptstyle h_1 i}\uparrow & \nwarrow{\scriptstyle \phi} & \\ 0 & \longrightarrow & K & \xrightarrow[\iota]{} & E \end{array}$$

and hence the diagram

$$\begin{array}{ccccc} & & N & & \\ & & {\scriptstyle h_1 i}\uparrow & \nwarrow{\scriptstyle \phi} & \\ 0 & \longrightarrow & K & \xrightarrow[\iota]{} & E \end{array}$$

is also commutative. So the rows exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{\iota} & E & \xrightarrow{\pi} & L & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \phi} & & \downarrow & & \\ 0 & \longrightarrow & K & \xrightarrow{h_1 i} & N & \xrightarrow{h_2} & P & \longrightarrow & 0 \end{array}$$

can be completed to a commutative diagram. Similarly to the proof above, we obtain an exact sequence

$$0 \longrightarrow E \longrightarrow L \oplus N \longrightarrow P \longrightarrow 0.$$

Note that $E$, $P \in \mathscr{F}$. Hence $L \oplus N \in \mathscr{F}$ and so $L \in \mathscr{F}$. But then $\mathrm{Ext}^1(L, K) = 0$ again by [8], Corollary 7.2.3, page 156. Thus, the sequence $0 \to K \to E \to L \to 0$ is split exact, and hence $K$ is injective. Since $K$ is the kernel of the $\mathscr{F}$-cover $F \to C$, by [14], Corollary 1.2.8, page 13, $K$ is zero. Hence $C \cong F$, and so $C$ is $(n, 0)$-injective. It follows that $R$ is right $n$-coherent by Proposition 2.1. The proof is complete. □

When $n = 1$ in Theorem 3.13, we have

**Corollary 3.14.** *Let $R$ be a ring such that every absolutely pure right $R$-module is pure extending. If every right $R$-module has an absolutely pure cover, then $R$ is right coherent.*

**Proposition 3.15.** *The following assertions are equivalent for a ring $R$ and $n \geqslant 1$:*
(1) *$R$ is right $n$-coherent.*
(2) *Every $n$-pure submodule of an $(n,1)$-injective right $R$-module is $(n,1)$-injective.*

Proof. (1) $\Rightarrow$ (2) holds by Lemma 3.1.

(2) $\Rightarrow$ (1). Let $0 \to A \to B \to C \to 0$ be any exact sequence of right $R$-modules where $A$ and $B$ are $(n,0)$-injective. Then we get a long exact sequence

$$0 = \mathrm{Ext}^1(P,B) \to \mathrm{Ext}^1(P,C) \to \mathrm{Ext}^2(P,A)$$

for any $n$-presented right $R$-module $P$. Let $0 \to A \to E$ be exact with $E$ injective. Then $A$ is $(n,1)$-injective by (2). So $\mathrm{Ext}^2(P,A) = 0$ and hence $\mathrm{Ext}^1(P,C) = 0$. Thus, (1) follows. □

## 4. Applications to $(n,d)$-rings

In this section, we will say that $R$ is a *right $(n,d)$-FC ring* provided it is right $n$-coherent and $R_R$ is $(n,d)$-injective. We note that right $(1,0)$-$FC$ rings are also called right $FC$ rings [5], and $(0,0)$-$FC$ rings coincide with $QF$ rings.

Recall that an $(n,d)$-injective envelope $\phi\colon M \to E$ of $M$ has the *unique mapping property* [6] if for any homomorphism $f\colon M \to A$ with $A$ $(n,d)$-injective, there is a unique homomorphism $g\colon E \to A$ such that $g\phi = f$. The concept of an $(n,d)$-injective cover with the unique mapping property can be defined similarly.

**Theorem 4.1.** *Let $R$ be a ring. Then the following assertions are equivalent:*
(1) *$R$ is a right $(n,d)$-ring;*
(2) *$R$ is a right $(n,d)$-$FC$ ring, and the kernel of any $(n,d)$-injective cover of a right $R$-module is $(n,d)$-injective;*
(3) *$R$ is a right $(n,d)$-$FC$ ring, and the cokernel of any $(n,d)$-injective preenvelope of a right $R$-module is $(n,d)$-injective;*
(4) *$R$ is a right $(n,d)$-$FC$ ring, and every factor module of a right $(n,d)$-injective $R$-module is $(n,d)$-injective;*
(5) *$R_R$ is $(n,d)$-injective, and every right $R$-module has a monomorphic $(n,d)$-injective cover;*
(6) *every right $R$-module has an epimorphic $(n,d)$-injective cover with the unique mapping property;*

(7) *every right $R$-module has an $(n, d)$-injective envelope with the unique mapping property;*

(8) *$R$ is a right $(n, d)$-$FC$ ring, and $R$ is a right $(n, d+m)$-ring for some $m \geqslant 0$;*

(9) *$R$ is a right $(n, d)$-$FC$ ring, and every right $R$-module has an $(n, d+m)$-injective cover with the unique mapping property for some $m \geqslant 0$;*

(10) *$R$ is a right $(n, d)$-$FC$ ring, and every right $R$-module has an $(n, d+m)$-injective envelope with the unique mapping property for some $m \geqslant 0$.*

Proof. (1) $\Rightarrow$ (3), (1) $\Rightarrow$ (4) and (1) $\Rightarrow$ (8) are clear.

(3) $\Rightarrow$ (2). Let $M$ be any right $R$-module. Then, by Corollary 3.10, $M$ has an epimorphic $(n, d)$-injective cover $E \to M$ with $E$ $(n, d)$-injective. On the other hand, by [10], Theorem 3.9, $K = \ker(E \to M)$ has a monomorphic $(n, d)$-injective preenvelope $g\colon K \to E^1$. So we get the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E^1 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & L & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

with exact rows and columns. Since $K \to E^1$ is an $(n, d)$-injective preenvelope, the homomorphism $K \to E$ can be extended to a homomorphism $E^1 \to E$ and so the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \longrightarrow & E \\ & & \downarrow & \nearrow & \\ & & E^1 & & \end{array}$$

is commutative. It induces a homomorphism $L \to N$ so that the diagram

$$\begin{array}{ccccccc} & & & & N & & \\ & & & \nearrow & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

is also commutative. This implies that the sequence $0 \to E \to N \to L \to 0$ is split exact. Now applying $\mathrm{Hom}(A, -)$ to the first commutative diagram with $A$ $(n, d)$-

injective, we obtain the commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(A,K) & \longrightarrow & \mathrm{Hom}(A,E) & \longrightarrow & \mathrm{Hom}(A,M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(A,E^1) & \longrightarrow & \mathrm{Hom}(A,N) & \longrightarrow & \mathrm{Hom}(A,M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(A,L) & \longrightarrow & \mathrm{Hom}(A,L) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Clearly, the bottom row and the right two columns are exact. Since $E \to M$ is an $(n,d)$-injective cover, the top row is also exact. So the middle row is exact. It follows that the left column is also exact by [12], Lemma 6.31, page 354. Note that $L$ is $(n,d)$-injective by (3). So by setting $E = L$, we see that $0 \to K \to E^1 \to L \to 0$ is split exact, and hence $K$ is $(n,d)$-injective, as desired.

(2) $\Rightarrow$ (1). Let $M$ be any right $R$-module. Then, by Corollary 3.10, $M$ has an epimorphic $(n,d)$-injective cover $E \to M$ with $E$ $(n,d)$-injective. So we get an exact sequence $0 \to K \to E \to M \to 0$. By (2), $K$ is $(n,d)$-injective. It follows that $M$ is also $(n,d)$-injective by Proposition 2.5, as desired.

(5) $\Rightarrow$ (1). Let $M$ be any right $R$-module. By hypothesis, $M$ has a monomorphic $(n,d)$-injective cover $F \to M$. Since $R_R$ is $(n,d)$-injective, it is easy to see that $F \to M$ is an epimorphism. So $M$ is $(n,d)$-injective and (1) follows.

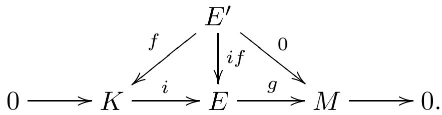
(4) $\Rightarrow$ (1). Let $M$ be any right $R$-module. Then by Corollary 3.10, $M$ has an epimorphic $(n,d)$-injective cover $g\colon E \to M$ with $E$ $(n,d)$-injective. So $M \cong \mathrm{coker}(g)$, and (4) implies that $M$ is $(n,d)$-injective, as desired.

(8) $\Rightarrow$ (1). If $m = 0$ then we are done. So assume $m \geqslant 1$. Let $M$ be any right $R$-module. Since $R$ is right $n$-coherent and $R_R$ is $(n,d)$-injective, by Corollary 3.10, $M$ has an epimorphic $(n,d)$-injective cover $f\colon E \to M$ with $E$ $(n,d)$-injective, which yields the exactness of the sequence $0 \to K \to E \to M \to 0$. So we get a long exact sequence

$$\mathrm{Ext}^{d+m}(P,E) \longrightarrow \mathrm{Ext}^{d+m}(P,M) \longrightarrow \mathrm{Ext}^{d+m+1}(P,K)$$

for any $n$-presented right $R$-module $P$. By (8), $K$ is $(n, d+m)$-injective, and so $\mathrm{Ext}^{d+m+1}(P,K) = 0$. But $\mathrm{Ext}^{d+m}(P,E) = 0$ since $R$ is right $n$-coherent and $E$ is $(n,d)$-injective. Hence $\mathrm{Ext}^{d+m}(P,M) = 0$, and so $M$ is $(n, d+m-1)$-injective. Thus $R$ is a right $(n, d+m-1)$-ring. Repeat this procedure to obtain $R$ is a right $(n,d)$-ring.

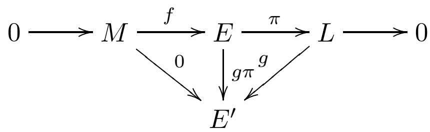
(6) $\Rightarrow$ (1). For any right $R$-module $M$, let $g\colon E \to M$ be an epimorphic $(n,d)$-injective cover of $M$ with the unique mapping property, where $E$ is $(n,d)$-injective. By (6), $K = \ker(g)$ has an epimorphic $(n,d)$-injective cover $f\colon E' \to K$. So, we obtain the following row exact commutative diagram:



Since $g(if) = 0$, we have $if = 0$ by uniqueness. Note that $f$ is an epimorphism. Hence $K = \mathrm{Im}(f) \subseteq \ker(i) = 0$. Hence, $M$ is $(n,d)$-injective. So (1) follows.

(1) $\Rightarrow$ (5), (1) $\Rightarrow$ (6) and (1) $\Rightarrow$ (7). Let $M$ be any right $R$-module. Then $M$ is $(n,d)$-injective by (1). Now it is easy to verify that the identity homomorphism on $M$ is an $(n,d)$-injective cover with the unique mapping property. It is also an $(n,d)$-injective envelope of $M$ which has the unique mapping property. Thus (5), (6) and (7) hold.

(7) $\Rightarrow$ (1). For any right $R$-module $M$, let $f\colon M \to E$ be an $(n,d)$-injective envelope of $M$ with the unique mapping property, where $E$ is $(n,d)$-injective. By (7), $L = \mathrm{coker}(f)$ has an $(n,d)$-injective envelope $g\colon L \to E'$. Therefore we get the commutative diagram



with exact row. Since $(g\pi)f = 0$, we have $g\pi = 0$ by uniqueness. Note that $g$ is a monomorphism. Hence, $L = \mathrm{Im}(\pi) \subseteq \ker(g) = 0$. So $M$ is $(n,d)$-injective, and (1) follows.

(8) $\Leftrightarrow$ (9) $\Leftrightarrow$ (10). The proofs are analogous to those of (1) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7). $\square$

It is well-known that a ring $R$ is a right $(0,0)$-ring (or $(0,1)$-ring, $(1,0)$-ring, $(1,1)$-ring) if and only if $R$ is semisimple (or right hereditary, von Neumann regular, right semihereditary, respectively) (see [4], Theorem 1.3; or [16], Corollary 2.7). Specializing Theorem 4.1, we have

**Corollary 4.2.** *Let $R$ be a ring. Then the following assertions are equivalent:*

(1) *$R$ is von Neumann regular;*

(2) *$R$ is a right $FC$ ring, and the kernel of any $FP$-injective cover of a right $R$-module is $FP$-injective;*

(3) *$R$ is a right $FC$ ring, and the cokernel of any $FP$-injective preenvelope of a right $R$-module is $FP$-injective;*

(4) *$R_R$ is right $FP$-injective, and every factor module of a right $FP$-injective $R$-module is $FP$-injective;*

(5) *$R_R$ is $FP$-injective, and $R$ is right semihereditary;*

(6) *$R_R$ is $FP$-injective, and every right $R$-module has a monomorphic $FP$-injective cover;*

(7) *every right $R$-module has an epimorphic $FP$-injective cover with the unique mapping property;*

(8) *every right $R$-module has an $FP$-injective envelope with the unique mapping property;*

(9) *$R$ is a right $FC$ ring, and $R$ is a right $(1, m)$-ring for some $m \geqslant 0$;*

(10) *$R$ is a right $FC$ ring, and every right $R$-module has a $(1, m)$-injective cover with the unique mapping property for some $m \geqslant 0$;*

(11) *$R$ is a right $FC$ ring, and every right $R$-module has a $(1, m)$-injective envelope with the unique mapping property for some $m \geqslant 0$;*

(12) *$R$ is a right $FC$ ring and $wD(R) < \infty$.*

Proof. Due to Theorem 4.1, we need only to show that (4) $\Rightarrow$ (5) and (9) $\Leftrightarrow$ (12).

(4) $\Rightarrow$ (5). Let $M$ be any right $R$-module. There is an exact sequence $0 \to M \to E \to L \to 0$ with $E$ injective. So for any finitely presented right $R$-module $P$ we have $\mathrm{Ext}^2(P, M) \cong \mathrm{Ext}^1(P, L) = 0$ since $L$ is $FP$-injective by(4). Hence $M$ is $(1, 1)$-injective, and (5) follows.

(9) $\Leftrightarrow$ (12). This follows from [16], Proposition 2.6; and [16], Corollary 2.7. □

**Corollary 4.3.** *Let $R$ be a ring. Then the following assertions are equivalent:*

(1) *$R$ is semisimple;*

(2) *$R$ is a $QF$ ring, and the kernel of any injective cover of a right $R$-module is injective;*

(3) *$R$ is a $QF$ ring, and the cokernel of any injective envelope of a right $R$-module is injective;*

(4) *$R$ is a $QF$ ring, and $R$ is right hereditary;*

(5) *$R_R$ is injective, and every right $R$-module has a monomorphic injective cover;*

(6) *every right $R$-module has an epimorphic injective cover with the unique mapping property;*

(7) *every right $R$-module has an injective envelope with the unique mapping property;*

(8) *$R$ is a $QF$ ring, and $rD(R) < \infty$;*

(9) *$R$ is a $QF$ ring, and $R$ is a $(0,m)$-ring for some $m \geqslant 0$;*

(10) *$R$ is a $QF$ ring, and every right $R$-module has a $(0,m)$-injective cover with the unique mapping property for some $m \geqslant 0$;*

(11) *$R$ is a $QF$ ring, and every right $R$-module has a $(0,m)$-injective envelope with the unique mapping property for some $m \geqslant 0$.*

By Corollary 3.8, we see that $R$ is right Noetherian if and only if every right $R$-module has a $(0,d)$-injective cover, for any non-negative integer $d$. We end the paper with

**Remark 4.4.** Let $n \geqslant 1$. The question whether $R$ must necessarily be right $n$-coherent in order that every right $R$-module have an $(n,d)$-injective cover for any non-negative integer $d$, is open.

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