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ON THE SIGNLESS LAPLACIAN SPECTRAL CHARACTERIZATION OF THE LINE GRAPHS OF T-SHAPE TREES

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Abstract. A graph is determined by its signless Laplacian spectrum if no other non-isomorphic graph has the same signless Laplacian spectrum (simply G is DQS). Let $T(a,b,c)$ denote the T-shape tree obtained by identifying the end vertices of three paths P_{a+2}, P_{b+2} and P_{c+2}. We prove that its all line graphs $\mathcal{L}(T(a,b,c))$ except $\mathcal{L}(T(t,t,2t+1))$ $(t \geqslant 1)$ are DQS, and determine the graphs which have the same signless Laplacian spectrum as $\mathcal{L}(T(t,t,2t+1))$. Let $\mu_1(G)$ be the maximum signless Laplacian eigenvalue of the graph G. We give the limit of $\mu_1(\mathcal{L}(T(a,b,c)))$, too.

Keywords: signless Laplacian spectrum; cospectral graphs; T-shape tree

MSC 2010: 05C50, 15A18

1. Introduction

All graphs considered here are undirected and simple. Suppose that G is a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and let $d_G(v_i)$ be the degree of the vertex v_i. Then $D(G) = \operatorname{diag}(d_G(v_1), \ldots, d_G(v_n))$ is a diagonal matrix of the vertex degrees of G. If $A(G)$ is the adjacency matrix of G, then the matrix $Q(G) = D(G) + A(G)$ is the *signless Laplacian matrix* of G. Since matrices $A(G)$ and $Q(G)$ are real and symmetric, all their eigenvalues are real numbers. Assume that $\varrho_1(G) \geqslant \varrho_2(G) \geqslant \ldots \geqslant \varrho_n(G)$ and $\mu_1(G) \geqslant \mu_2(G) \geqslant \ldots \geqslant \mu_n(G)$ are, respectively, the *adjacent eigenvalues* and the *signless Laplacian eigenvalues* of the graph G. The *A-spectrum* (or *Q-spectrum*) of the graph G consists of the adjacency eigenvalues (or signless Laplacian eigenvalues). Two graphs are said to be *A-cospectral* (or *Q-cospectral*) if they have the same A-spectrum (or Q-spectrum). A graph is said to

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be *determined by its A-spectrum* (or Q-spectrum) (simply G is DAS or DQS) if no other non-isomorphic graph is A-cospectral (or Q-cospectral) to it.

Finding new families of DS graphs is an interesting problem. For the background and some known results about this problem, we refer the reader to [9], [10] and the references therein. Let $T(a,b,c)$ denote the T-shape tree on n vertices obtained by identifying the end vertices of three paths P_{a+2}, P_{b+2} and P_{c+2} (see Figure 1). G. R. Omidi [7] showed that $T(a,b,c)$ is DQS. W. Wang and C. X. Xu in [13] and [12] proved respectively that $T(a,b,c)$ is DLS and that $T(a,b,c)$ is DAS if and only if $(a+1,b+1,c+1) \neq (l,l,2l-2)$ for any integer $l \geqslant 2$. Let $\mathcal{L}(T(a,b,c))$ be the line graph of $T(a,b,c)$. D. Cvetković, P. Rowlinson and S. K. Simić [2] verified that if two graphs are Q-cospectral, then their line graphs are A-cospectral. So from [7] we know that $\mathcal{L}(T(a,b,c))$ is DAS.

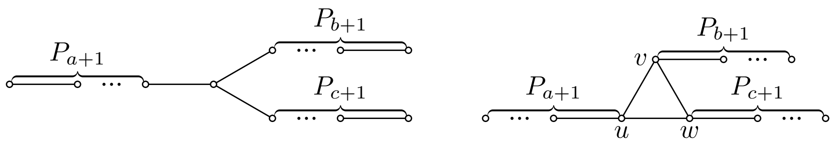

Figure 1. The T-shape tree $T(a,b,c)$ and its line graph $\mathcal{L}(T(a,b,c))$.

In this paper we mainly show that all $\mathcal{L}(T(a,b,c))$ except $\mathcal{L}(T(t,t,2t+1))$ $(t \geqslant 1)$ are DQS, and determine that $Q(2t+3;t+1,t)$ (see Figure 2) is the unique graph which is Q-cospectral to $\mathcal{L}(T(t,t,2t+1))$. We give the limit of $\mu_1(\mathcal{L}(T(a,b,c)))$, too.

2. Some lemmas on Q-spectrum

In this section we give some lemmas which are used in the next section to prove our main results.

Lemma 2.1 ([9]). *For the adjacent matrix of a graph, the following data can be obtained from the spectrum:*

(i) *the number of vertices;*

(ii) *the number of edges;*

(iii) *the number of closed walks of any length.*

Lemma 2.2 ([2]). *Let G be a connected graph of order $n \geqslant 2$. Then*

(i) $\mu_1(G) \leqslant \max\{d_G(v_i)+d_G(v_j);\ v_iv_j \in E(G)\}$, *with equality if and only if G is a regular or semi-regular bipartite graph;*

(ii) $\mu_1(G) \geqslant \Delta(G)+1$, *with equality if and only if G is the star $K_{1,n-1}$, where $\Delta(G)$ is the maximum degree of the graph G.*

Let $N_G(H)$ be the number of subgraphs of a graph G which are isomorphic to H.

Lemma 2.3 ([2])**.** *Let G be a graph with n vertices and m edges, and $T_k(G) = \sum_{i=1}^{n} \mu_i(G)^k$ $(k = 0, 1, \ldots)$. Then $T_0(G) = n$, $T_1(G) = \sum_{i=1}^{n} d_G(v_i) = 2m$, $T_2(G) = 2m + \sum_{i=1}^{n} d_G(v_i)^2$, $T_3(G) = 6N_G(C_3) + 3\sum_{i=1}^{n} d_G(v_i)^2 + \sum_{i=1}^{n} d_G(v_i)^3$.*

From the above lemma, we easily obtain

Lemma 2.4. *If G and H are Q-cospectral and have the same degree sequences, then $N_G(C_3) = N_H(C_3)$.*

Recall that the polynomial $\phi(G, \lambda) = \det(\lambda I - A(G)) = a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_n$ is the *characteristic polynomial* of G, where I is the identity matrix.

Lemma 2.5 ([1])**.** *Let v be a vertex of a graph G and let $\mathscr{C}(v)$ denote the collection of cycles containing v. Then the characteristic polynomial of G satisfies $\phi(G, \lambda) = \lambda\phi(G - v, \lambda) - \sum_{u \sim v} \phi(G - u - v, \lambda) - 2 \sum_{C \in \mathscr{C}(v)} \phi(G - V(C), \lambda)$.*

Lemma 2.6 ([6])**.** *For $n \geqslant 1$ we have $\phi(P_n, 2) = n + 1$, and $\phi(C_n, 2) = 0$ for $n \geqslant 3$.*

Lemma 2.7 ([2])**.** *Let G be a graph. Then the following statements hold:*

(i) *$\mu_1(G) = 0$ if and only if G has no edges;*

(ii) *$0 < \mu_1(G) < 4$ if and only if all components of G are paths;*

(iii) *for a connected graph G, $\mu_1(G) = 4$ if and only if G is a cycle C_n or $K_{1,3}$.*

Lemma 2.8 ([1])**.** *Let H be a proper subgraph of a connected graph G. Then $\mu_1(H) < \mu_1(G)$.*

Lemma 2.9 (Edge-Interlacing [3])**.** *Let G be a graph with order n and $e \in E(G)$. Then $0 \leqslant \mu_n(G - e) \leqslant \mu_n(G) \leqslant \ldots \leqslant \mu_2(G - e) \leqslant \mu_2(G) \leqslant \mu_1(G - e) \leqslant \mu_1(G)$.*

Lemma 2.10 ([2])**.** *If two graphs are Q-cospectral, then their line graphs are A-cospectral.*

Let $N_G(i)$ be the number of closed walks of length i in G.

Lemma 2.11 ([7]). *$N_G(4) = 2m + 4N_G(P_3) + 8N_G(C_4)$ and $N_G(5) = 30N_G(K_3) + 10N_G(C_5) + 10N_G(H_1)$, where H_1 is the graph $K_{1,3}$ with two end vertices joined by an edge.*

3. The line graph of a T-shape tree is a DQS-graph

Suppose without loss of generality that $a \leqslant b \leqslant c$ in $\mathcal{L}(T(a,b,c))$. Note that $\mathcal{L}(T(0,0,c))$ is isomorphic to the lollipop graph which is obtained by identifying a vertex of a cycle and an end vertex of a path. In [14], Y. P. Zhang et al. showed that all lollipop graphs are DQS. So we assume that $c \geqslant b \geqslant 1$.

Lemma 3.1. *If G and $\mathcal{L}(T(a,b,c))$ $(0 \leqslant a \leqslant b \leqslant c)$ are Q-cospectral, then the following implications hold:*
(i) *If $a = 0$, then* $\deg(G) = (3^2, 2^{n-4}, 1^2)$.
(ii) *If $a \geqslant 1$, then* $\deg(G) = (3^3, 2^{n-6}, 1^3)$ *or* $(4, 2^{n-3}, 1^2)$.

Proof. Since G and $\mathcal{L}(T(a,b,c))$ are Q-cospectral, we know by Lemma 2.3 that G and $\mathcal{L}(T(a,b,c))$ have the same order and size and that $\sum\limits_{i=1}^{n} d_G(v_i)^2 = \sum\limits_{i=1}^{n} d_{\mathcal{L}(T(a,b,c))}(v_i)^2$. By Lemma 2.2, we have $4 < \mu_1(\mathcal{L}(T(a,b,c))) < 6$, which implies that $\Delta(G) \leqslant 4$. Let x_i be the number of vertices of degree i. Then we know that $0 \leqslant i \leqslant 4$. If $a \geqslant 1$, then we have

$$
\begin{aligned}
x_0 + x_1 + x_2 + x_3 + x_4 &= n,\\
x_1 + 2x_2 + 3x_3 + 4x_4 &= 2n,\\
x_1 + 4x_2 + 9x_3 + 16x_4 &= 4n + 6.
\end{aligned}
$$

From these equations we have $x_0 + x_3 + 3x_4 = 3$ and so $x_4 \in \{0, 1\}$. Next we distinguish two cases.

Case 1. Suppose that $x_4 = 0$. Then $x_0 + x_3 = 3$. The case that $x_0 \geqslant 1$ implies that 0 lies on the Q-spectrum of G. This contradicts the fact that 0 does not lie on the Q-spectrum of $\mathcal{L}(T(a,b,c))$, and so $x_0 = 0$. Thus we obtain that $x_3 = 3$, $x_1 = 3$ and $x_2 = n - 6$, that is, $\deg(G) = (3^3, 2^{n-6}, 1^3)$.

Case 2. Suppose that $x_4 = 1$. Then $x_0 = x_3 = 0$. From the first two equations, we obtain $x_1 = 2$, $x_2 = n - 3$. Thus, $\deg(G) = (4, 2^{n-3}, 1^2)$.

If $a = 0$, then similarly we get that $\deg(G) = (3^2, 2^{n-4}, 1^2)$. □

Recall that the subdivision graph $S(G)$ of G is obtained from G by replacing each edge of G with a path of length two. The following result can be found in [5], [11].

Lemma 3.2. *Let G and H be two graphs. Then G and H are Q-cospectral if and only if $S(G)$ and $S(H)$ are A-cospectral.*

From [8], we know that for any $n \geqslant -2$, $\phi(P_n,\lambda) = (x^{2n+2}-1)/(x^{n+2}-x^n)$, where x satisfies $x^2-\lambda x+1=0$. Let $Q(q;k_1,k_2)$ be the unicyclic graph of order n with the degree sequence $(4,2^{n-3},1^2)$ shown in Figure 2.

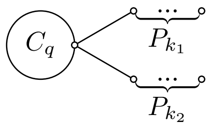

Figure 2. The unicyclic graph $Q(q;k_1,k_2)$.

Lemma 3.3. *Let x satisfy $x^2-\lambda x+1=0$. Then we have*

(1) $x^{2n}(x^2-1)^3\phi(S(\mathcal{L}(T(a,b,c))),\lambda) = x^{4a+4b+4c+18}-3x^{4a+4b+4c+16}+x^{4b+4c+14}+x^{4a+4c+14}+x^{4a+4b+14}-x^{4b+4c+12}-x^{4a+4c+12}-x^{4a+4b+12}+2x^{4b+4c+10}+2x^{4a+4c+10}+2x^{4a+4b+10}-2x^{4c+8}-2x^{4b+8}-2x^{4a+8}+x^{4c+6}+x^{4b+6}+x^{4a+6}-x^{4c+4}-x^{4b+4}-x^{4a+4}+3x^2-1$;

(2) $x^{2n}(x^2-1)^3\phi(S(Q(q;k_1,k_2)),\lambda) = x^{4k_1+4q+4k_2+6}-3x^{4k_1+4q+4k_2+4}+2x^{4k_1+4q+2}-2x^{4k_1+4k_2+2q+6}+2x^{4q+4k_2+2}+2x^{4k_1+4k_2+2q+4}+x^{4k_1+4k_2+6}+2x^{4k_1+2q+4}+2x^{4k_2+2q+4}+x^{4k_1+4k_2+4}-2x^{4k_1+2q+2}-2x^{4k_2+2q+2}-x^{4q+2}-x^{4q}-2x^{2q+2}-2x^{4k_1+4}+2x^{2q}-2x^{4k_2+4}+3x^2-1$.

P r o o f. By applying Lemma 2.5 to the subdivision graph $S(\mathcal{L}(T(a,b,c)))$, we obtain

$$\begin{aligned}\phi(S(\mathcal{L}(T(a,b,c)))) = \lambda^3\phi(P_{2c})\phi(P_{2a+2b+3}) - \lambda^2(&\phi(P_{2c})\phi(P_{2b+2})\phi(P_{2a})\\ &+\phi(P_{2c})\phi(P_{2b})\phi(P_{2a+2}) - \phi(P_{2c})\phi(P_{2b})\phi(P_{2a})\\ &+\phi(P_{2c-1})\phi(P_{2a+2b+3})) + \lambda(\phi(P_{2c-1})\phi(P_{2b+2})\phi(P_{2a})\\ &+\phi(P_{2c-1})\phi(P_{2b})\phi(P_{2a+2}) - \phi(P_{2c-1})\phi(P_{2b})\phi(P_{2a})\\ &-2\phi(P_{2c})\phi(P_{2a+2b+3})) + \phi(P_{2c})\phi(P_{2b+2})\phi(P_{2a})\\ &+\phi(P_{2c})\phi(P_{2b})\phi(P_{2a+2}) - 2\phi(P_{2c})\phi(P_{2b})\phi(P_{2a}).\end{aligned}$$

By substituting $\phi(P_n,\lambda) = (x^{2n+2}-1)/(x^{n+2}-x^n)$ with $\lambda = (x^2+1)/x$ in the above equation, we get the first assertion. The second assertion can be obtained similarly. □

Lemma 3.4. *Graphs $\mathcal{L}(T(a,b,c))$ and $Q(q;k_1,k_2)$ are Q-cospectral if and only if $a=b=k_2=t, c=2t+1, k_1=t+1, q=2t+3$, where $t\geqslant 1$.*

P r o o f. From [2] we know that for the Q-spectrum the multiplicity of 0 gives the number of bipartite components, and so zero does not lie in the Q-spectrum of $\mathcal{L}(T(a,b,c))$. Therefore, if $Q(q;k_1,k_2)$ and $\mathcal{L}(T(a,b,c))$ are Q-cospectral then q is odd. By Lemma 3.2 we also know that $S(\mathcal{L}(T(a,b,c))$ and $S(Q(q;k_1,k_2))$ are A-cospectral. By Lemma 3.3 we obtain that $x^{2n}(x^2-1)^3\phi(S(\mathcal{L}(T(a,b,c))),\lambda) = x^{2n}(x^2-1)^3\phi(S(Q(q;k_1,k_2)),\lambda)$. We assume without loss of generality that $a\leqslant b\leqslant c$ and $k_1\geqslant k_2$. Since the coefficients of the third, fourth and fifth terms of $x^{2n}(x^2-1)^3\phi(S(Q(q;k_1,k_2)),\lambda)$ are all even, the third and fourth terms of $x^{2n}\times(x^2-1)^3\phi(S(\mathcal{L}(T(a,b,c))),\lambda)$ are equal, that is $a=b$. If the third, fourth and fifth terms of $x^{2n}(x^2-1)^3\phi(S(Q(q;k_1,k_2)),\lambda)$ are equal, then we have $q=2k_1+2$, a contradiction. This implies that $k_1>k_2$ and that $4k_1+4q+2=4b+4c+14$. Thus we obtain $a=k_2$.

Note that $2x^{4a+6}$ and $-4x^{4a+8}$ are, respectively, the last fourth and fifth terms of the polynomial obtained by simplifying $x^{2n}(x^2-1)^3\phi(S(\mathcal{L}(T(a,b,c))),\lambda)$ with $a=b$. We have that $x^{4a+6}=x^{2q}$ and $-4x^{4a+8}=-2x^{4k_1+4}-2x^{2q+2}$. Thus we obtain that $a=b=k_2=t$, $c=2t+1$, $k_1=t+1$ and $q=2t+3$, where $t\geqslant 1$.

Conversely, if $a=b=k_2=t$, $c=2t+1$, $k_1=t+1$ and $q=2t+3$, then we can easily verify that $\phi(S(\mathcal{L}(T(a,b,c))),\lambda)=\phi(S(Q(q;k_1,k_2)),\lambda)$. □

In order to state the following lemma we need to add some further notation. The *odd-unicyclic graph* is a unicyclic graph which contains an odd cycle. A spanning subgraph H of G is its *TU-subgraph* if the components of H are trees or odd-unicyclic graphs. Suppose that a TU-subgraph H of G contains c unicyclic graphs and trees $T_1,T_2,\ldots,T_s$. Then the weight $W(H)$ of H is defined by $W(H)=4^c\prod\limits_{i=1}^{s}(1+|E(T_i)|)$. Note that isolated vertices in H do not contribute to $W(H)$ and may be ignored.

Recall that the polynomial

$$\varphi(G)=\varphi(G,\mu)=\det(\mu I-Q(G))=q_0\mu^n+q_1\mu^{n-1}+\ldots+q_n$$

is the *signless Laplacian characteristic polynomial* of G. The lemma below shows the relation between the coefficients of $\varphi(G,\mu)$ and the weights of a TU-subgraph of G.

Lemma 3.5 ([2]). *Numbers $q_0=1$ and $q_j=\sum\limits_{H_j}(-1)^jW(H_j)$ $(j=1,2,\ldots,n)$, where the summation runs over all TU-subgraphs H_j of G with j edges.*

Lemma 3.6. *No two non-isomorphic line graphs of T-shape trees are Q-cospectral.*

P r o o f. Suppose that $\mathcal{L}(T(a,b,c))$ and $\mathcal{L}(T(a_1,b_1,c_1))$ are Q-cospectral, where $a \leqslant b \leqslant c$ and $a_1 \leqslant b_1 \leqslant c_1$. Then we know by Lemma 2.3 that

$$a + b + c = a_1 + b_1 + c_1 \tag{1}$$

and by Lemma 3.2 that $S(\mathcal{L}(T(a,b,c)))$ and $S(\mathcal{L}(T(a_1,b_1,c_1)))$ are A-cospectral. By Lemma 3.3, we know that the third smallest exponents of x in $x^{2n}(x^2-1)^3 \times \phi(S(\mathcal{L}(T(a,b,c))),\lambda)$ and $x^{2n}(x^2-1)^3\phi(S(\mathcal{L}(T(a_1,b_1,c_1))),\lambda)$ are equal to $4a+4$ and $4a_1+4$, respectively, and so $4a+4 = 4a_1+4$, that is,

$$a = a_1. \tag{2}$$

Using Lemma 3.5 we easily obtain that

$$q_{n-1}(\mathcal{L}(T(a,b,c))) = (-1)^{n-1}(2(a^2+b^2+c^2)+5n-6)$$

and

$$q_{n-1}(\mathcal{L}(T(a_1,b_1,c_1))) = (-1)^{n-1}(2(a_1^2+b_1^2+c_1^2)+5n-6),$$

from which we obtain that $a^2+b^2+c^2 = a_1^2+b_1^2+c_1^2$. The assertion follows from (1) and (2). □

Lemma 3.7 ([3])**.** *Let G be a graph of order n and size m. Then $\phi(S(G),\mu) = \mu^{m-n}\varphi(G,\mu^2)$.*

From Lemmas 3.3 and 3.7 we easily obtain

Lemma 3.8. $\varphi(\mathcal{L}(T(a,b,c)),4) \neq 0$.

Lemma 3.9. *If G and $\mathcal{L}(T(a,b,c))$ are Q-cospectral, then G does not contain a cycle as its component.*

P r o o f. Since G and $\mathcal{L}(T(a,b,c))$ are Q-cospectral, by Lemma 3.8 we have $\varphi(G,4) \neq 0$. If $G = G' \cup C_l$, then $\varphi(G,\mu) = \varphi(G',\mu)\cdot\varphi(C_l,\mu)$. By Lemma 2.7 (iii) we get $\varphi(G,4) = 0$. This is a contradiction. □

Lemma 3.10. *For any graph $\mathcal{L}(T(a,b,c))$, we have the following assertions:*
(i) *If $a=0$, then $\mu_2(\mathcal{L}(T(a,b,c)))<4$.*
(ii) *If $a\geqslant 1$, then $\mu_3(\mathcal{L}(T(a,b,c)))<4$.*

P r o o f. Let uv and uw be the edges of $\mathcal{L}(T(a,b,c))$ shown in Figure 1. If $a=0$, then by Lemma 2.9 we have $\mu_2(\mathcal{L}(T(0,b,c)))\leqslant\mu_1(\mathcal{L}(T(0,b,c))-vw)=\mu_1(P_n)<4$. If $a\geqslant 1$, then we know by Lemma 2.9 that $\mu_3(\mathcal{L}(T(a,b,c)))\leqslant\mu_2(\mathcal{L}(T(a,b,c))-uv)$ and that $\mu_2(\mathcal{L}(T(a,b,c))-uv)\leqslant\mu_1(\mathcal{L}(T(a,b,c))-uv-uw)=\mu_1(P_{n-a-1}\cup P_{a+1})$. By Lemma 2.7 (ii) we have $\mu_3(\mathcal{L}(T(a,b,c)))<4$. □

Lemma 3.11. *If G and $\mathcal{L}(T(0,b,c))$ are Q-cospectral, then G is a connected graph.*

P r o o f. Suppose for a contradiction that $G=G_1\cup G_2\cup\ldots\cup G_k$, where $k>1$ and G_i is a connected component of G. Without loss of generality, set $\mu_1(G)=\mu_1(G_1)$. Since G and $\mathcal{L}(T(0,b,c))$ are Q-cospectral, it follows from Lemma 3.10 (i) that $\mu_2(G)=\max\{\mu_2(G_1),\mu_1(G_i);\ 2\leqslant i\leqslant k\}<4$, and so by Lemma 2.7 we know that each G_i $(2\leqslant i\leqslant k)$ is a path or an isolated vertex. This implies that zero lies on the Q-spectrum of G, a contradiction. □

Theorem 3.12. *$\mathcal{L}(T(0,b,c))$ is DQS.*

P r o o f. Suppose that G and $\mathcal{L}(T(0,b,c))$ are Q-cospectral. Then we know by Lemma 3.1 (i) that the degree sequence of G is $(3^2,2^{n-4},1^2)$ and by Lemma 3.11 that G is a connected unicyclic graph. By Lemma 2.4, we have $N_G(C_3)=N_{\mathcal{L}(T(0,b,c))}(C_3)=1$. All connected unicyclic graphs U_i $(1\leqslant i\leqslant 2)$ containing C_3 on n vertices with the degree sequence $(3^2,2^{n-4},1^2)$ are shown in Figure 3.

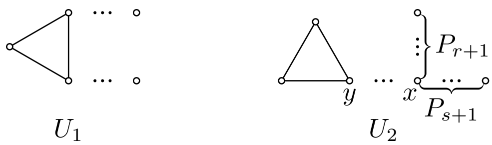

Figure 3.

So $G\cong U_1$ or U_2. If $G\cong U_1$, then by Lemma 3.6 we have $G\cong\mathcal{L}(T(0,b,c))$. If $G\cong U_2$, then we know by Lemma 2.10 that the line graphs $\mathcal{L}(G)$ and $\mathcal{L}(\mathcal{L}(T(a,b,c)))$ are A-cospectral, and so it follows from Lemma 2.1 that $N_{\mathcal{L}(\mathcal{L}(T(a,b,c)))}(4)=N_{\mathcal{L}(G)}(4)$. Using Lemma 2.11, we get

$$N_{\mathcal{L}(\mathcal{L}(T(0,b,c)))}(4)=\begin{cases}6n+56, & \text{if } b=c=1;\\ 6n+60, & \text{if } b=1,\ c\geqslant 2;\\ 6n+64, & \text{if } 2\leqslant b\leqslant c.\end{cases}$$

If $d_{U_2}(x,y) \geqslant 2$, then U_2 contains one cycle and one $K_{1,3}$ and so $\mu_2(U_2) > 4$, which contradicts Lemma 3.10 (i). Hence we assume that $d_{U_2}(x,y) = 1$; then we have

$$N_{\mathcal{L}(U_2)}(4) = \begin{cases} 6n+48, & \text{if } r = s = 1; \\ 6n+52, & \text{if } r = 1,\ s \geqslant 2; \\ 6n+56, & \text{if } 2 \leqslant r \leqslant s. \end{cases}$$

From $N_{\mathcal{L}(\mathcal{L}(T(0,b,c)))}(4) = N_{\mathcal{L}(U_2)}(4) = 6n+56$ we know that $b = c = 1$ and $s \geqslant r \geqslant 2$. But $n(\mathcal{L}(T(0,1,1))) = 5 < 8 \leqslant n(U_2)$, a contradiction. □

Lemma 3.13. *Suppose that the graph G is Q-cospectral to $\mathcal{L}(T(a,b,c))$ $(a \geqslant 1)$. Then we have*

(i) *G does not contain a subgraph isomorphic to the disjoint union of two cycles and one $K_{1,3}$;*

(ii) *G does not contain a subgraph isomorphic to the disjoint union of two $K_{1,3}$ and one cycle;*

(iii) *G does not contain a subgraph isomorphic to the disjoint union of three cycles.*

Proof. Since G and $\mathcal{L}(T(a,b,c))$ are Q-cospectral, we know by Lemma 3.10 (ii) that $\mu_3(G) < 4$. Suppose on the contrary that G contains a subgraph isomorphic to the disjoint union of two cycles C_{l_1} and C_{l_2} and one $K_{1,3}$. Then we know by Lemma 2.9 that $\mu_3(G) \geqslant \mu_3(C_{l_1} \cup C_{l_2} \cup K_{1,3})$. Since, by Lemma 2.7 (iii), $\mu_1(C_{l_1}) = \mu_1(C_{l_2}) = \mu_1(K_{1,3}) = 4$, we have $\mu_3(G) \geqslant 4$, a contradiction. Similarly, we can verify that (ii) and (iii) are also true. □

Theorem 3.14. *Let $a \geqslant 1$. Then all $\mathcal{L}(T(a,b,c))$ except $\mathcal{L}(T(a,a,2a+1))$ are DQS, and $Q(2a+3; a+1, a)$ is the unique graph which is Q-cospectral to $\mathcal{L}(T(a,a,2a+1))$.*

Proof. Suppose that G and $\mathcal{L}(T(a,b,c))$ are Q-cospectral. Then we know by Lemma 3.1 (ii) that the degree sequence of G is $(4, 2^{n-3}, 1^2)$ or $(3^3, 2^{n-6}, 1^3)$.

If $\deg(G) = (4, 2^{n-3}, 1^2)$, then by Lemma 3.9, $G \cong Q(q; k_1, k_2)$. We know from Lemma 3.4 that no $\mathcal{L}(T(a,b,c))$ except $\mathcal{L}(T(a,a,2a+1))$ can be Q-cospectral to $Q(q; k_1, k_2)$.

Now we suppose that $\deg(G) = (3^3, 2^{n-6}, 1^3)$. If G is connected, then we know by Lemma 2.4 that G contains one C_3. All connected unicyclic graphs A_i $(1 \leqslant i \leqslant 3)$ containing C_3 on n vertices with the degree sequence $(3^3, 2^{n-6}, 1^3)$ are shown in Figure 4.

If $G \cong A_3$, then by Lemma 3.3 we have $A_3 \cong \mathcal{L}(T(a,b,c))$. Next we will discuss two cases.

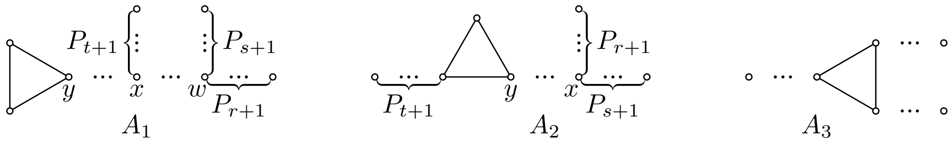

Figure 4.

Case 1. $G \cong A_1$.

If $d_{A_1}(x,y) \geqslant 2$ and $d_{A_1}(x,w) \geqslant 3$, then A_1 always has a subgraph isomorphic to the disjoint union of two $K_{1,3}$ and one cycle, which contradicts Lemma 3.13 (i). Thus we consider the following two subcases.

Subcase 1.1. $d_{A_1}(x,y) = 1$ and $d_{A_1}(x,w) = 1$.

By Lemma 2.10, we know that the line graphs $\mathcal{L}(G)$ and $\mathcal{L}(\mathcal{L}(T(a,b,c)))$ are A-cospectral, and so it follows from Lemma 2.1 that $N_{\mathcal{L}(\mathcal{L}(T(a,b,c)))}(4) = N_{\mathcal{L}(G)}(4)$. Using Lemma 2.11, we obtain:

$$N_{\mathcal{L}(\mathcal{L}(T(a,b,c)))}(4) = \begin{cases} 6n+90, & \text{if } a=b=c=1; \\ 6n+94, & \text{if } a=b=1,\ c\geqslant 2; \\ 6n+98, & \text{if } a=1,\ 2\leqslant b\leqslant c; \\ 6n+102, & \text{if } 2\leqslant a\leqslant b\leqslant c. \end{cases}$$

$$N_{\mathcal{L}(A_1)}(4) = \begin{cases} 6n+70, & \text{if } t=r=s=1; \\ 6n+74, & \text{if } t=1,\ r=1,\ s\geqslant 2 \text{ or } t\geqslant 2,\ r=s=1; \\ 6n+78, & \text{if } t=1,\ 2\leqslant r\leqslant s \text{ or } t\geqslant 2,\ r=1,\ s\geqslant 2; \\ 6n+82, & \text{if } t\geqslant 2,\ 2\leqslant r\leqslant s. \end{cases}$$

Clearly, $N_{\mathcal{L}(\mathcal{L}(T(a,b,c)))}(4) \neq N_{\mathcal{L}(A_1)}(4)$, a contradiction.

Subcase 1.2. $d_{A_1}(x,y) = 1$ and $d_{A_1}(x,w) \geqslant 2$.

$$N_{\mathcal{L}(A_1)}(4) = \begin{cases} 6n+66, & \text{if } t=r=s=1; \\ 6n+70, & \text{if } t=1,\ r=1,\ s\geqslant 2 \text{ or } t\geqslant 2,\ r=s=1; \\ 6n+74, & \text{if } t=1,\ 2\leqslant r\leqslant s \text{ or } t\geqslant 2,\ r=1,\ s\geqslant 2; \\ 6n+78, & \text{if } t\geqslant 2,\ 2\leqslant r\leqslant s. \end{cases}$$

Clearly, $N_{\mathcal{L}(\mathcal{L}(H^n_{g,k}))}(4) \neq N_{\mathcal{L}(B_1)}(4)$, a contradiction.

Subcase 1.3. $d_{A_1}(x,y) \geqslant 2$ and $d_{A_1}(x,w) = 1$.

$$N_{\mathcal{L}(A_1)}(4) = \begin{cases} 6n+66, & \text{if } t=r=s=1; \\ 6n+70, & \text{if } t=1,\ r=1,\ s\geqslant 2 \text{ or } t\geqslant 2,\ r=s=1; \\ 6n+74, & \text{if } t=1,\ 2\leqslant r\leqslant s \text{ or } t\geqslant 2,\ r=1,\ s\geqslant 2; \\ 6n+78, & \text{if } t\geqslant 2,\ 2\leqslant r\leqslant s. \end{cases}$$

Clearly, $N_{\mathcal{L}(\mathcal{L}(H^n_{g,k}))}(4) \neq N_{\mathcal{L}(B_1)}(4)$, a contradiction.

Subcase 1.4. $d_{A_1}(x,y) \geqslant 2$ and $d_{A_1}(x,w) = 2$.

$$N_{\mathcal{L}(A_1)}(4) = \begin{cases} 6n+62, & \text{if } t=r=s=1; \\ 6n+66, & \text{if } t=1,\ r=1,\ s\geqslant 2 \text{ or } t\geqslant 2,\ r=s=1; \\ 6n+70, & \text{if } t=1,\ 2\leqslant r\leqslant s \text{ or } t\geqslant 2,\ r=1,\ s\geqslant 2; \\ 6n+74, & \text{if } t\geqslant 2,\ 2\leqslant r\leqslant s. \end{cases}$$

Clearly, $N_{\mathcal{L}(\mathcal{L}(H^n_{g,k}))}(4) \neq N_{\mathcal{L}(B_1)}(4)$, a contradiction.

Case 2. $G \cong A_2$.

Subcase 2.1. $d_{A_2}(x,y) = 1$.

$$N_{\mathcal{L}(A_2)}(4) = \begin{cases} 6n+78, & \text{if } t=r=s=1; \\ 6n+82, & \text{if } t=1,\ r=1,\ s\geqslant 2 \text{ or } t\geqslant 2,\ r=s=1; \\ 6n+86, & \text{if } t=1,\ 2\leqslant r\leqslant s \text{ or } t\geqslant 2,\ r=1,\ s\geqslant 2; \\ 6n+90, & \text{if } t\geqslant 2,\ 2\leqslant r\leqslant s. \end{cases}$$

Clearly, $N_{\mathcal{L}(\mathcal{L}(T(a,b,c)))}(4) \neq N_{\mathcal{L}(A_2)}(4)$, a contradiction.

Subcase 2.2. $d_{A_2}(x,y) \geqslant 2$.

$$N_{\mathcal{L}(A_2)}(4) = \begin{cases} 6n+74, & \text{if } t=r=s=1; \\ 6n+78, & \text{if } t=1,\ r=1,\ s\geqslant 2 \text{ or } t\geqslant 2,\ r=s=1; \\ 6n+82, & \text{if } t=1,\ 2\leqslant r\leqslant s \text{ or } t\geqslant 2,\ r=1,\ s\geqslant 2; \\ 6n+86, & \text{if } t\geqslant 2,\ 2\leqslant r\leqslant s. \end{cases}$$

Clearly, $N_{\mathcal{L}(\mathcal{L}(T(a,b,c)))}(4) \neq N_{\mathcal{L}(A_2)}(4)$, a contradiction.

Next we suppose that G is not connected. We have known from [2] that for the Q-spectrum the multiplicity of 0 gives the number of bipartite components. Thus, G does not contain a bipartite graph as its component. Let $U(p_1;s,t)$, $Z(p_2;s,t)$ and $H(p_3;k)$ be the three unicyclic graphs shown in Figure 5. Then we can determine by Lemmas 3.9 and 3.13 (iii) that $G \cong H(p_3;k) \cup U(p_1;s,t)$ or $G \cong H(p_3;k) \cup Z(p_2;s,t)$.

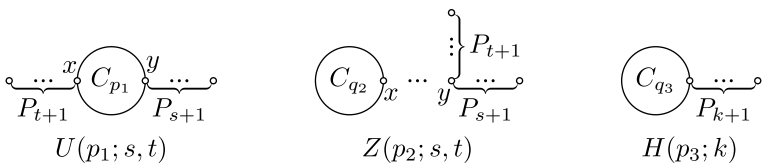

Figure 5. The graphs $U(p_1;s,t)$, $Z(p_2;s,t)$ and $H(p_3;k)$.

If $G \cong H(p_3;k) \cup U(p_1;s,t)$, then by Lemma 2.4 we know that $H(p_3;k) \cup U(p_1;s,t)$ contains only one C_3. Thus we have $p_1 = 3, p_3 \geqslant 5$ or $p_1 \geqslant 5, p_3 = 3$. Note that both p_1 and p_3 must be odd.

If $p_1 = 3, p_3 \geqslant 5$, then by Lemma 2.11 we get

$$N_{\mathcal{L}(H(p_3;k) \cup U(3;s,t))}(4) = \begin{cases} 6n+74, & \text{if } k=t=s=1; \\ 6n+78, & \text{if } k=1,\, t=1,\, s \geqslant 2 \quad \text{or} \quad k \geqslant 2,\, t=s=1; \\ 6n+82, & \text{if } k=1,\, 2 \leqslant t \leqslant s \quad \text{or} \quad k \geqslant 2,\, t=1,\, s \geqslant 2; \\ 6n+86, & \text{if } k \geqslant 2,\, 2 \leqslant t \leqslant s. \end{cases}$$

Clearly, $N_{\mathcal{L}(\mathcal{L}(T(a,b,c)))}(4) \neq N_{\mathcal{L}(H(p_3;k) \cup U(p_1;s,t))}(4)$, a contradiction.

If $p_1 \geqslant 5, p_3 = 3$ and $d_{U(p_1;s,t)}(x,y) \geqslant 3$, then G contains two $K_{1,3}$ and one cycle. Thus, we discuss two subcases.

If $d_{U(p_1;s,t)}(x,y) = 1$, then

$$N_{\mathcal{L}(H(3;k) \cup U(p_1;s,t))}(4) = \begin{cases} 6n+66, & \text{if } k=t=s=1; \\ 6n+70, & \text{if } k=1,\, t=1,\, s \geqslant 2 \quad \text{or} \quad k \geqslant 2,\, t=s=1; \\ 6n+74, & \text{if } k=1,\, 2 \leqslant t \leqslant s \quad \text{or} \quad k \geqslant 2,\, t=1,\, s \geqslant 2; \\ 6n+78, & \text{if } k \geqslant 2,\, 2 \leqslant t \leqslant s. \end{cases}$$

Clearly, $N_{\mathcal{L}(\mathcal{L}(T(a,b,c)))}(4) \neq N_{\mathcal{L}(H(3;k) \cup U(p_1;s,t))}(4)$, a contradiction.

If $d_{U(p_1;s,t)}(x,y) = 2$, then

$$N_{\mathcal{L}(H(3;k) \cup U(p_1;s,t))}(4) = \begin{cases} 6n+62, & \text{if } k=t=s=1; \\ 6n+66, & \text{if } k=1,\, t=1,\, s \geqslant 2 \quad \text{or} \quad k \geqslant 2,\, t=s=1; \\ 6n+70, & \text{if } k=1,\, 2 \leqslant t \leqslant s \quad \text{or} \quad k \geqslant 2,\, t=1,\, s \geqslant 2; \\ 6n+74, & \text{if } k \geqslant 2,\, 2 \leqslant t \leqslant s. \end{cases}$$

Clearly, $N_{\mathcal{L}(\mathcal{L}(T(a,b,c)))}(4) \neq N_{\mathcal{L}(H(3;k) \cup U(p_1;s,t))}(4)$, a contradiction.

If $G \cong H(p_3;k) \cup Z(p_2;s,t)$, then $p_2 = 3$, $p_3 \geqslant 5$ or $p_3 = 3$, $p_2 \geqslant 5$. Note that both p_2 and p_3 are odd.

If $d_{Z(p_2;s,t)}(x,y) \geqslant 2$, then G contains two cycles and one $K_{1,3}$. Thus, we only discuss the case that $d_{Z(p_2;s,t)}(x,y) = 1$.

If $p_2 = 3, p_3 \geqslant 5$, then

$$N_{\mathcal{L}(H(p_3;k) \cup Z(3;s,t))}(4) = \begin{cases} 6n+66, & \text{if } k=t=s=1; \\ 6n+70, & \text{if } k=1,\, t=1,\, s \geqslant 2 \quad \text{or} \quad k \geqslant 2,\, t=s=1; \\ 6n+74, & \text{if } k=1,\, 2 \leqslant t \leqslant s \quad \text{or} \quad k \geqslant 2,\, t=1,\, s \geqslant 2; \\ 6n+78, & \text{if } k \geqslant 2,\, 2 \leqslant t \leqslant s. \end{cases}$$

Clearly, $N_{\mathcal{L}(\mathcal{L}(T(a,b,c)))}(4) \neq N_{\mathcal{L}(H(p_3;k)\cup Z(3;s,t))}(4)$, a contradiction.

If $p_3 = 3, p_2 \geqslant 5$, then

$$N_{\mathcal{L}(H(3;k)\cup Z(p_2;s,t))}(4) = \begin{cases} 6n+66, & \text{if } k=t=s=1; \\ 6n+70, & \text{if } k=1,\ t=1,\ s\geqslant 2 \quad \text{or} \quad k\geqslant 2,\ t=s=1; \\ 6n+74, & \text{if } k=1,\ 2\leqslant t\leqslant s \quad \text{or} \quad k\geqslant 2,\ t=1,\ s\geqslant 2; \\ 6n+78, & \text{if } k\geqslant 2,\ 2\leqslant t\leqslant s. \end{cases}$$

Clearly, $N_{\mathcal{L}(\mathcal{L}(T(a,b,c)))}(4) \neq N_{\mathcal{L}(H(3;k)\cup Z(p_2;s,t))}(4)$, a contradiction.

So far we have verified that all $\mathcal{L}(T(a,b,c))$ but $\mathcal{L}(T(a,a,2a+1))$ $(a \geqslant 1)$ are *DQS*. Furthermore, by Lemma 3.4 we can determine that $Q(2a+3;a+1,a)$ is the unique graph which is Q-cospectral to $\mathcal{L}(T(a,a,2a+1))$. □

An *internal path* in a graph is a path joining two end vertices which are both of degree greater than two (not necessarily distinct), while all other vertices are of degree 2.

Lemma 3.15 ([4]). *Let uv be an edge of the connected graph G, and let G_{uv} be obtained from G by subdividing the edge uv of G.*

(i) *If uv is not in an internal path of $G \neq C_n$, then $\mu_1(G_{uv}) > \mu_1(G)$.*

(ii) *If uv belongs to an internal path of G, then $\mu_1(G_{uv}) < \mu_1(G)$.*

Theorem 3.16. $\mu_1(\mathcal{L}(T(a,b,c))) < 16/3$.

Proof. We know by Lemma 3.15 that $\mu_1(\mathcal{L}(T(r,r,r)))$ is an increasing function of r and by Lemma 2.2 that $\mu_1(\mathcal{L}(T(r,r,r))) < 6$. Thus, $\lim\limits_{r\to\infty} \mu_1(\mathcal{L}(T(r,r,r)))$ exists. Let $q = \lim\limits_{r\to\infty} \mu_1(\mathcal{L}(T(r,r,r)))$. Suppose that $P_{r+1} = v_1v_2\dots v_rv_{r+1}$ is a pendant path of $\mathcal{L}(T(r,r,r))$, where v_1 is the pendant vertex of $\mathcal{L}(T(r,r,r))$. Let $\mu = \mu_1(\mathcal{L}(T(r,r,r)))$ and let $x = (x_1, x_2, \dots, x_n)^{\mathrm{T}}$ be a Perron vector of $Q(\mathcal{L}(T(r,r,r)))$, where x_i corresponds to the vertex v_i. From $Q(\mathcal{L}(T(r,r,r)))x = \mu x$ we have $x_2 = (\mu-1)x_1, x_3 = (\mu-2)x_2 - x_1, \dots, x_{r+1} = (\mu-2)x_r - x_{r-1}$. Thus, we obtain

$$x_{r+1} = \frac{(1+\lambda_2)\lambda_1^{r+1} - (1+\lambda_1)\lambda_2^{r+1}}{\sqrt{\mu^2-4\mu}}x_1 \tag{1}$$

and

$$x_r = \frac{(1+\lambda_2)\lambda_1^{r} - (1+\lambda_1)\lambda_2^{r}}{\sqrt{\mu^2-4\mu}}x_1 \tag{2}$$

where $\lambda_1 = \frac{1}{2}\bigl(\mu - 2 + \sqrt{\mu^2 - 4\mu}\bigr)$ and $\lambda_2 = \frac{1}{2}\bigl(\mu - 2 - \sqrt{\mu^2 - 4\mu}\bigr)$. By the symmetry of the graph $\mathcal{L}(T(r,r,r))$ we have $(\mu - 3)x_{r+1} = 2x_{r+1} + x_r$ and so

$$\mu - 5 = \frac{x_r}{x_{r+1}}. \tag{3}$$

Substituting equations (1) and (2) in the equation (3), we get

$$\mu - 5 = \frac{(1+\lambda_2)\lambda_1^r - (1+\lambda_1)\lambda_2^r}{(1+\lambda_2)\lambda_1^{r+1} - (1+\lambda_1)\lambda_2^{r+1}}.$$

By taking $r \to \infty$ in the above equality, we have

$$q - 5 = \frac{q - \sqrt{q^2 - 4q}}{q + \sqrt{q^2 - 4q}}.$$

Thus, we have $q = 16/3$. By Lemma 2.8, we know that

$$\mu_1(\mathcal{L}(T(a,b,c))) < \mu_1(\mathcal{L}(T(r,r,r)))$$

for any positive integer $r > c$ and so $\mu_1(\mathcal{L}(T(a,b,c))) < 16/3$. □

References

[1] *D. M. Cvetković, M. Doob, H. Sachs*: Spectra of Graphs. Theory and Applications (3rd rev. a. enl., ed.). J. A. Barth, Leipzig, 1995.

[2] *D. Cvetković, P. Rowlinson, S. K. Simić*: Signless Laplacians of finite graphs. Linear Algebra Appl. *423* (2007), 155–171.

[3] *D. Cvetković, P. Rowlinson, S. K. Simić*: Eigenvalue bounds for the signless Laplacian. Publ. Inst. Math., Nouv. Sér. *81* (2007), 11–27.

[4] *D. Cvetković, S. K. Simić*: Towards a spectral theory of graphs based on signless Laplacian. I. Publ. Inst. Math., Nouv. Sér. *85* (2009), 19–33.

[5] *D. Cvetković, S. K. Simić*: Towards a spectral theory of graphs based on signless Laplacian. II. Linear Algebra Appl. *432* (2010), 2257–2272.

[6] *N. Ghareghani, G. R. Omidi, B. Tayfeh-Rezaie*: Spectral characterization of graphs with index at most $\sqrt{2+\sqrt{5}}$. Linear Algebra Appl. *420* (2007), 483–489.

[7] *G. R. Omidi*: On a signless Laplacian spectral characterizaiton of T-shape trees. Linear Algebra Appl. *431* (2009), 1607–1615.

[8] *F. Ramezani, N. Broojerdian, B. Tayfeh-Rezaie*: A note on the spectral characterization of θ-graphs. Linear Algebra Appl. *431* (2009), 626–632.

[9] *E. R. van Dam, W. H. Haemers*: Which graphs are determined by their spectrum? Linear Algebra Appl. Special issue on the Combinatorial Matrix Theory Conference (Pohang, 2002) *373* (2003), 241–272.

[10] *E. R. van Dam, W. H. Haemers*: Developments on spectral characterizations of graphs. Discrete Math. *309* (2009), 576–586.

[11] *J. F. Wang, Q. X. Huang, F. Belardo, E. M. L. Marzi*: On the spectral characterizations of ∞-graphs. Discrete Math. *310* (2010), 1845–1855.

[12] *W. Wang, C. X. Xu*: On the spectral characterization of T-shape trees. Linear Algebra Appl. *414* (2006), 492–501.
[13] *W. Wang, C. X. Xu*: Note: The T-shape tree is determined by its Laplacian spectrum. Linear Algebra Appl. *419* (2006), 78–81.
[14] *Y. P. Zhang, X. G. Liu, B. Y. Zhang, X. R. Yong*: The lollipop graph is determined by its Q-spectrum. Discrete Math. *309* (2009), 3364–3369.

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