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SOME RESULTS ON THE LOCAL COHOMOLOGY OF MINIMAX MODULES

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Abstract. Let R be a commutative Noetherian ring with identity and I an ideal of R. It is shown that, if M is a non-zero minimax R-module such that $\dim \operatorname{Supp} H^i_I(M) \leqslant 1$ for all i, then the R-module $H^i_I(M)$ is I-cominimax for all i. In fact, $H^i_I(M)$ is I-cofinite for all $i \geqslant 1$. Also, we prove that for a weakly Laskerian R-module M, if R is local and t is a non-negative integer such that $\dim \operatorname{Supp} H^i_I(M) \leqslant 2$ for all $i < t$, then $\operatorname{Ext}^j_R(R/I, H^i_I(M))$ and $\operatorname{Hom}_R(R/I, H^t_I(M))$ are weakly Laskerian for all $i < t$ and all $j \geqslant 0$. As a consequence, the set of associated primes of $H^i_I(M)$ is finite for all $i \geqslant 0$, whenever $\dim R/I \leqslant 2$ and M is weakly Laskerian.

Keywords: local cohomology module; Krull dimension; minimax module; cofinite module; weakly Laskerian module; associated primes

MSC 2010: 13D45, 13E10, 13C05

1. Introduction

Let R be a commutative Noetherian ring with identity and I an ideal of R. For an R-module M, the ith local cohomology module of M with respect to I is defined as

$$H^i_I(M) \cong \varinjlim_{n \in \mathbb{N}} \operatorname{Ext}^i_R(R/I^n, M).$$

We refer the reader to [5] for more details about the local cohomology. In [12] Hartshorne defined an R-module M to be I-cofinite if $\operatorname{Supp} M \subseteq V(I)$ and $\operatorname{Ext}^j_R(R/I, M)$ is finite for all j and he asked:

For which rings R and ideals I are the modules $H^i_I(M)$ I-cofinite for all i and all finite R-modules M?

Concerning this question, Hartshorne in [12] and later Chiriacescu in [6] showed that if R is a complete regular local ring and I is a prime ideal such that $\dim R/I = 1$, then $H^i_I(M)$ is I-cofinite for any finite R-module M (see [12], Corollary 7.7). Huneke and Koh [13], Theorem 4.1, proved that if R is a complete Gorenstein local domain and I is an ideal of R such that $\dim R/I = 1$, then for all non-negative integers i and j, $\operatorname{Ext}^j_R(N, H^i_I(M))$ is finite for any finite R-modules M and N such that $\operatorname{Supp} N \subseteq V(I)$. Furthermore, Delfino [7] proved that if R is a complete local domain then under some mild conditions similar results hold. Also, Delfino and Marley [8], Theorem 1, and Yoshida [18], Theorem 1.1, have eliminated the completeness hypothesis entirely. Finally, Bahmanpour and Naghipour [4], Theorem 2.6, have removed the local assumption on R. They proved for a non-negative integer t and a finite R-module M such that $\dim \operatorname{Supp} H^i_I(M) \leqslant 1$ for all $i < t$, the R-modules $H^i_I(M)$ are I-cofinite for all $i < t$ and $\operatorname{Hom}_R(R/I, H^t_I(M))$ is a finite R-module. Azami, Naghipour and Vakili in [1] defined an R-module M to be I-cominimax, as a generalization of I-cofiniteness, if $\operatorname{Supp}(M) \subseteq V(I)$ and $\operatorname{Ext}^j_R(R/I, M)$ is minimax for all j. As one of the main results of this paper, we generalize the Bahmanpour and Naghipour's result to the class of minimax modules. More precisely, we show that if M is a minimax module over an arbitrary commutative Noetherian ring R such that $\dim \operatorname{Supp} H^i_I(M) \leqslant 1$ for all i, then $H^i_I(M)$ is I-cominimax for all i. As a consequence of this result, we prove that if M is minimax with $\dim \operatorname{Supp} H^i_I(M) \leqslant 1$ for all i, then the Bass numbers and Betti numbers of $H^i_I(M)$ are finite for all $i \geqslant 0$.

Also, Bahmanpour and Naghipour in [4], Theorem 3.1, proved that if R is local, M is a finite R-module and t is a non-negative integer such that $\dim \operatorname{Supp} H^i_I(M) \leqslant 2$ for all $i < t$, then $\operatorname{Ext}^j_R(R/I, H^i_I(M))$ and $\operatorname{Hom}_R(R/I, H^t_I(M))$ are weakly Laskerian for all $i < t$ and all $j \geqslant 0$. As a generalization, we prove that this result holds under the more general assumption that M is weakly Laskerian. As a consequence, it follows that the set of associated primes of $H^i_I(M)$ is finite for all $i \geqslant 0$, whenever $\dim R/I \leqslant 2$ and M is weakly Laskerian.

Throughout the article, R denotes a commutative Noetherian ring, I is an ideal of R and $V(I)$ is the set of all prime ideals of R containing I.

2. Maim results

In [20], Zöschinger introduced the class of minimax modules and in [20] and [21] he gave many equivalent conditions for a module to be minimax. An R-module M is called *minimax* if there is a finite submodule N of M such that M/N is Artinian. It was shown by T. Zink [19] and by E. Enochs [11] that a module over a complete local ring is minimax if and only if it is Matlis reflexive. We first recall briefly the definitions and basic properties of minimax modules that we shall use.

Remark 2.1.

(i) The class of minimax modules contains all finite and all Artinian modules.

(ii) Let $0 \to L \to M \to N \to 0$, be an exact sequence of R-modules. Then M is minimax if and only if both L and N are minimax (see [3], Lemma 2.1). Thus any submodule and quotient of a minimax module is minimax. Moreover, if M and N are two R-modules such that N is finite and M is minimax, then $\operatorname{Ext}^j_R(N, M)$ and $\operatorname{Tor}^R_j(N, M)$ are minimax for all $j \geqslant 0$

(iii) The set of associated primes of any minimax R-module is finite.

(iv) If M is a minimax R-module and $\mathfrak{p}$ is a non-maximal prime ideal of R, then $M_{\mathfrak{p}}$ is a finite $R_{\mathfrak{p}}$-module.

Theorem 2.2. *Let M be a minimax R-module such that* $\dim \operatorname{Supp} H^i_I(M) \leqslant 1$ *for all i and let N be a finite R-module with* $\operatorname{Supp} N \subseteq V(I)$. *Then* $\operatorname{Ext}^j_R(N, H^i_I(M))$ *is a minimax R-module for all i and j. In fact,* $\operatorname{Ext}^j_R(N, H^i_I(M))$ *is finite for all j and $i \geqslant 1$.*

Proof. By Gruson's theorem one can reduce the problem to the case where $N = R/I$. Since $H^0_I(M)$ is a submodule of M, the assertion holds for $i = 0$ and so we prove the claim for $i \geqslant 1$. To do this, since M is a minimax R-module, there exists an exact sequence

$$0 \to S \to M \to T \to 0$$

where S is finite and T is Artinian. So, we obtain the exact sequence

$$0 \to H^0_I(S) \to H^0_I(M) \to H^0_I(T) \xrightarrow{f} H^1_I(S) \to H^1_I(M) \to 0,$$

and the isomorphism

$$H^i_I(S) \cong H^i_I(M)$$

for all $i \geqslant 2$ by the Artinianness of T. Put $L := \operatorname{Im} f$ and consider the exact sequence

$$0 \to L \to H^1_I(S) \to H^1_I(M) \to 0.$$

So, it is easy to see that $\dim \operatorname{Supp} H^i_I(S) \leqslant 1$ for all i. Hence, $H^i_I(S)$ is I-cofinite for all i by [4], Theorem 2.6. So, $H^i_I(M)$ is I-cofinite for all $i \geqslant 2$. Moreover, $(0 :_L I)$ is of finite length, since L is Artinian and $(0 :_{H^1_I(S)} I)$ is finite. Therefore, by [15] Proposition 4.1, L is I-cofinite. So, in view of the long exact sequence

$$\ldots \to \operatorname{Ext}^j_R(R/I, H^1_I(S)) \to \operatorname{Ext}^j_R(R/I, H^1_I(M)) \to \operatorname{Ext}^{j+1}_R(R/I, L) \to \ldots$$

we deduce that $H^1_I(M)$ is I-cofinite. Hence $H^i_I(M)$ is I-cofinite for all $i \geqslant 1$. □

Corollary 2.3. *Suppose that M is a minimax R-module and*

$$\dim \operatorname{Supp} H_I^i(M) \leqslant 1$$

for all i. Then for all $i \geqslant 0$ and any minimax submodule X of $H_I^i(M)$, the R-module $H_I^i(M)/X$ is I-cominimax. In particular, the Bass numbers of $H_I^i(M)$ are finite.

Proof. The first assertion follows from the short exact sequence $0 \to X \to H_I^i(M) \to H_I^i(M)/X \to 0$ and Theorem 2.2. For the last assertion, let $\mathfrak{p} \in \operatorname{Spec}(R)$ and let $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ be the residue field of $R_{\mathfrak{p}}$. If $I \nsubseteq \mathfrak{p}$, then $\mathfrak{p} \notin \operatorname{Supp}_R(H_I^i(M))$. So, there is nothing to prove in this case. Otherwise, $\operatorname{Ext}_R^j(R/\mathfrak{p}, H_I^i(M))$ is finite for all j and $i \geqslant 1$ and $\operatorname{Ext}_R^j(R/\mathfrak{p}, H_I^0(M))$ is minimax for all j by Theorem 2.2. If $\mathfrak{p}$ is a non-maximal prime ideal of R, it follows from Remark 2.1 (iv) that $(\operatorname{Ext}_R^j(R/\mathfrak{p}, H_I^0(M)))_{\mathfrak{p}}$ is finite over $R_{\mathfrak{p}}$ for all j. Also, if $\mathfrak{p}$ is a maximal ideal of R, since $(\operatorname{Ext}_R^j(R/\mathfrak{p}, H_I^0(M)))_{\mathfrak{p}}$ is also a $k(\mathfrak{p})$-vector space, the $R_{\mathfrak{p}}$-module must be a finite length for all j by minimaxness of $H_I^0(M)$. Thus, in either case, $(\operatorname{Ext}_R^j(R/\mathfrak{p}, H_I^i(M)))_{\mathfrak{p}}$ is finite for all j and $i \geqslant 0$ and the claim is true. □

Theorem 2.4. *Under the hypotheses of Corollary 2.3, $\operatorname{Tor}_j^R(R/I, H_I^i(M))$ is minimax for all i and j. In fact, the R-modules $\operatorname{Tor}_j^R(R/I, H_I^i(M))$ are finite for all j and $i \geqslant 1$.*

Proof. The result follows from Remark 2.1, Theorem 2.2 and [15], Theorem 2.1. □

Corollary 2.5. *Under the hypotheses of Corollary 2.3, the Betti numbers of $H_I^i(M)$ are finite for all i.*

An R-module M is said to be *weakly Laskerian* if the set of associated primes of any quotient module of M is finite. Note that in some texts the weakly Laskerian modules are called *skinny* modules. For example see [17]. Recently, in [16], Quy has introduced the class of FSF modules, i.e., modules containing some finite submodules such that the support of the quotient module is finite. Also, more recently in [2], Theorem 3.3, it has been shown by Bahmanpour that over a Noetherian ring R, an R-module M is weakly Laskerian if and only if it is FSF.

Here, we prove that [4], Theorem 3.1, holds for the larger class of weakly Laskerian modules instead of the class of finite modules.

Theorem 2.6. *Let R be a local ring, M a weakly Laskerian R-module and N a finite R-module with $\operatorname{Supp} N \subseteq V(I)$. Let t be a non-negative integer such that $\dim \operatorname{Supp} H_I^i(M) \leqslant 2$ for all $i < t$. Then the R-modules $\operatorname{Ext}_R^j(N, H_I^i(M))$ for all j and $i < t$, $\operatorname{Hom}_R(N, H_I^t(M))$ and $\operatorname{Ext}_R^1(N, H_I^t(M))$ are weakly Laskerian. In particular, the set of associated primes of $H_I^i(M)$ is finite for all $i \leqslant t$.*

P r o o f. By Gruson's theorem one can reduce the problem to the case where $N = R/I$. In view of [14], Theorem 3.7, it is enough to show that if $\dim \operatorname{Supp} H^i_I(M) \leqslant 2$ for all i, then $\operatorname{Ext}^j_R(N, H^i_I(M))$ are weakly Laskerian for all i and j. Since $H^0_I(M)$ is a submodule of M, the assertion holds for $i = 0$ and so we prove the claim for $i \geqslant 1$. As we mentioned in the paragraph after Corollary 2.5, since M is a weakly Laskerian R-module, there exists an exact sequence

$$0 \to S \to M \to T \to 0$$

where S is finite and T has finite support. Since any module with finite support over a Noetherian ring has dimension at most one, we obtain the exact sequence

$$\begin{aligned} 0 \to H^0_I(S) \to H^0_I(M) \to H^0_I(T) \xrightarrow{f} H^1_I(S) \xrightarrow{g} H^1_I(M) \xrightarrow{h} H^1_I(T) \\ \xrightarrow{k} H^2_I(S) \to H^2_I(M) \to 0, \end{aligned}$$

and the isomorphism

$$H^i_I(S) \cong H^i_I(M)$$

for all $i \geqslant 3$. Consider the following exact sequences:

$$\begin{aligned} &0 \to \operatorname{Im} f \to H^1_I(S) \to \operatorname{Im} g \to 0; \\ &0 \to \operatorname{Im} g \to H^1_I(M) \to \operatorname{Im} h \to 0; \\ &0 \to \operatorname{Im} h \to H^1_I(T) \to \operatorname{Im} k \to 0; \\ &0 \to \operatorname{Im} k \to H^2_I(S) \to H^2_I(M) \to 0. \end{aligned}$$

Hence, it is easy to see that $\dim \operatorname{Supp} H^i_I(S) \leqslant 2$ for all i. So, $\operatorname{Ext}^j_R(R/I, H^i_I(S))$ is weakly Laskerian for all i and j by [4], Theorem 3.1. Also, the fact that $\operatorname{Supp} H^i_I(T) \subseteq \operatorname{Supp} T$ for all i implies that $\operatorname{Im} f$, $\operatorname{Im} h$ and $\operatorname{Im} k$ have finite supports and so are weakly Laskerian. Therefore, in the light of the long exact sequences of Ext modules induced by the above short exact sequences we infer that $\operatorname{Ext}^j_R(R/I, H^i_I(M))$ are weakly Laskerian for all i and j. □

An R-module M is called *I-weakly cofinite* if $\operatorname{Supp}_R(M) \subseteq V(I)$ and $\operatorname{Ext}^j_R(R/I, M)$ is weakly Laskerian for all $j \geqslant 0$ (see [9] and [10]). Now, in view of Theorem 2.6, the proofs of Corollaries 3.2, 3.3 and 3.7 in [4] may be adapted. So, we may improve these results as follows.

Corollary 2.7. *Let I be an ideal of the local ring R such that $\dim R/I \leqslant 2$ and let M be a weakly Laskerian R-module. Then for any $i \geqslant 0$ and any weakly Laskerian submodule N of $H^i_I(M)$, the R-module $H^i_I(M)/N$ is I-weakly cofinite for all i. In particular, the set $\operatorname{Ass}_R H^i_I(M)$ is finite for all i.*

Corollary 2.8. *Let R be a local ring and M a weakly Laskerian R-module such that* $\dim \operatorname{Supp} H_I^i(M) \leqslant 2$ *for all* $i < \operatorname{cd}(I, M)$, *where* $\operatorname{cd}(I, M) := \max\{i \in \mathbb{Z}\colon H_I^i(M) \neq 0\}$ *is the cohomological dimension of M with respect to I. Then the modules* $\operatorname{Ext}_R^j(R/I, H_I^i(M))$ *are weakly Laskerian for all i and j.*

Proposition 2.9. *Let R be a commutative Noetherian (not necessarily local) ring and let $I \subseteq J$ be ideals of R such that* $\dim R/I = 2$. *Suppose that M is an I-weakly cofinite R-module such that $H_J^i(M)$ is I-weakly cofinite for $i = 0, 1$. Then $H_J^i(M)$ is J-weakly cofinite for all i.*

P r o o f. Since $\operatorname{Ext}_R^j(R/I, M)$ is weakly Laskerian and $I \subseteq J$, we conclude that $\operatorname{Ext}_R^j(R/J, M)$ is weakly Laskerian for all $j \geqslant 0$ by [10], Lemma 2.8. Also, as $\operatorname{Supp} M \subseteq V(I)$ and $\dim R/I = 2$ we have $\dim \operatorname{Supp} M \leqslant 2$. Thus $H_J^i(M) = 0$ for all $i \geqslant 3$. Now, the result follows from [4], Proposition 3.6. □

Corollary 2.10. *Let R be a local ring and let $I \subseteq J$ be ideals of R such that* $\dim R/I = 2$. *Suppose that M is a weakly Laskerian R-module such that $H_J^j(H_I^i(M))$ is I-weakly cofinite for all i and $j = 0, 1$. Then $H_J^j(H_I^i(M))$ is J-weakly cofinite for all i and j.*

P r o o f. It follows from Corollary 2.7 and Proposition 2.9. □

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