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THE WEAK MCSHANE INTEGRAL

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Abstract. We present a weaker version of the Fremlin generalized McShane integral (1995) for functions defined on a σ-finite outer regular quasi Radon measure space $(S, \Sigma, \mathcal{T}, \mu)$ into a Banach space X and study its relation with the Pettis integral. In accordance with this new method of integration, the resulting integral can be expressed as a limit of McShane sums with respect to the weak topology. It is shown that a function f from S into X is weakly McShane integrable on each measurable subset of S if and only if it is Pettis and weakly McShane integrable on S. On the other hand, we prove that if an X-valued function is weakly McShane integrable on S, then it is Pettis integrable on each member of an increasing sequence $(S_l)_{l\geqslant 1}$ of measurable sets of finite measure with union S. For weakly sequentially complete spaces or for spaces that do not contain a copy of c_0, a weakly McShane integrable function on S is always Pettis integrable. A class of functions that are weakly McShane integrable on S but not Pettis integrable is included.

Keywords: Pettis integral; McShane integral; weak McShane integral; uniform integrability

MSC 2010: 28B05, 46G10, 26A39

1. Introduction

In [7], Fremlin generalized the classical McShane integral to the case of an arbitrary σ-finite outer regular quasi Radon measure space $(S, \Sigma, \mathcal{T}, \mu)$. It turns out that for any McShane integrable function taking values in a Banach space $(X, \|\cdot\|)$, the McShane integral on S can be approximated with respect to the norm $\|\cdot\|$ by a sequence consisting of McShane sums. In this paper, we will consider a weaker method of integrability, namely *weak McShane integrability.* Roughly speaking, a function f from S into X is weakly McShane integrable on S if all sequences consisting of McShane sums of f corresponding to some class of generalized McShane partitions of S converge to the same limit with respect to the weak topology. Unlike the McShane integral, the resulting integral does not include automatically the Pettis integral, in general: a function which is weakly McShane integrable on S may fail to be Pettis integrable (Theorem 4.6 and Corollary 4.3), therefore not weak McShane integrable on a measurable subset of S (Corollary 3.1). By contrast with the scalar McShane integrability introduced in [12] and [17] (only for functions defined on compact intervals in $\mathbb{R}^m$ into a Banach space X, see also [15]), the generalized McShane partitions considered here does not depend on the choice of the vectors $x^* \in X^*$. As a starting point of this study, we need to extend some basic properties of the McShane integral developed by Fremlin in [7] into the context of weak McShane integrals. It is also shown that a function is weakly McShane integrable on Σ (i.e. weakly McShane integrable on each member of Σ) if and only if it is weakly McShane integrable on S and Pettis integrable. We then proceed to describe the relationship between the weak McShane integral (on S) and the Pettis integral. The Vitali convergence theorem for the Pettis integral of Musial (Theorem 1, [13], see also [11]) will play a central role in our investigation. With help of this theorem, it is shown that the concept of weak McShane integrability on S does not stray too far from the weak McShane integrability on Σ: if a function is weakly McShane integrable on S, then there exists a sequence $(S_l)_{l\geqslant 1}$ in Σ of finite measure with union S such that $1_{S_l} f$ is Pettis integrable and weakly McShane integrable on Σ for each $l \geqslant 1$ (Theorem 4.1). Consequently, for functions $f\colon S \to X$ satisfying the condition that $\{\langle x^*, f\rangle\colon x^* \in \overline{B}_{X^*}\}$ is uniformly integrable, the weak McShane integrability pass from S to Σ (Theorem 4.1 and Corollary 4.1). We also study the special case of functions taking values in weakly sequentially complete Banach spaces (for instance $L^1_{\mathbb{R}}(\Omega, \mathcal{F}, \nu)$, where $(\Omega, \mathcal{F}, \nu)$ is a σ-finite measure space) or in Banach spaces that do not contain a copy of c_0. It will be shown that for such Banach spaces it is possible to pass from weak McShane integrability on S to Pettis integrability; equivalently, a function is weakly McShane integrable on S if and only if it is weakly McShane integrable on Σ (Theorems 4.3 and 4.4).

The paper is organized as follows. Section 2 contains preliminaries and known facts on the Pettis integral. In Section 3 the notion of weak McShane integrability for functions defined on a σ-finite outer regular quasi Radon measure space $(S, \Sigma, \mathcal{T}, \mu)$ into a Banach space is presented and discussed. Further, some basic properties of the (strong) McShane integral are extended into the context of weak McShane integrals. It is also shown that a function is weakly McShane integrable on Σ if and only if it is weakly McShane integrable on S and Pettis integrable. Section 4 deals with the relationship between the weak McShane and the Pettis integrals. The crucial fact (Theorem 4.1) is that if a function is weakly McShane integrable on S, then it is Pettis integrable on each member of an increasing sequence $(S_l)_{l\geqslant 1}$ of measurable sets of finite measure with union S. The special case of weakly sequentially complete Banach spaces as well as the case of Banach spaces that do not contain a copy of

c_0 are discussed (Theorems 4.3 and 4.4). A class of functions which are weakly McShane integrable on S but not Pettis integrable is presented (Theorem 4.6 and Corollary 4.3).

2. Preliminaries and known facts on the Pettis integral

In the sequel, X stands for a Banach space, whose norm is denoted by $\|\cdot\|$, and X^* for the topological dual of X. The closed unit ball of X^* is denoted by $\overline{B}_{X^*}$. By w we denote the weak topology of X. Let (S,Σ,μ) be a positive measure space. By Σ_f we denote the collection of all measurable sets of finite measure. By $L^1_{\mathbb{R}}(\mu)$ we denote the Banach space of all (equivalence classes of) Σ-measurable and μ-integrable real-valued functions on Ω, equipped with the classical norm $\|f\|_1 := \int_S |f|\,\mathrm{d}\mu$. A function $f\colon S\to X$ is said to be *scalarly integrable* if for every $x^*\in X^*$, the real-valued function $\langle x^*, f\rangle$ is a member of $L^1_{\mathbb{R}}(\mu)$. We say also that f is *Dunford integrable*. If $f\colon S\to X$ is a scalarly integrable function, then for each $E\in\Sigma$ there is $x^{**}_E\in X^{**}$ such that

$$\langle x^*, x^{**}_E\rangle = \int_E \langle x^*, f\rangle\,\mathrm{d}\mu.$$

The vector x^{**}_E is called the *Dunford integral* of f over E. In the case that $x^{**}_E\in X$ for all $E\in\Sigma$, then f is called *Pettis integrable* and we write $(\mathcal{P}e)\int_E f\,\mathrm{d}\mu$ instead of x^{**}_E to denote the *Pettis integral* of f over E. If $f\colon S\to X$ is a Pettis integrable function, then the set $\{\langle x^*, f\rangle\colon x^*\in\overline{B}_{X^*}\}$ is relatively weakly compact in $L^1_{\mathbb{R}}(\mu)$ (see [13], page 162).

Definition 2.1 (Definition 246A, [8]). A subset H of $L^1_{\mathbb{R}}(\mu)$ is *uniformly integrable* if for every $\varepsilon>0$ we can find a set $E\in\Sigma_f$ and an $M\geqslant 0$ such that

$$\int_S (|h|-M1_E)^+\,\mathrm{d}\mu \leqslant \varepsilon \quad \text{for every } h\in H,$$

where $(|h|-M1_E)^+ := \max(|h|-M1_E, 0)$.

▷ Every uniformly integrable subset H of $L^1_{\mathbb{R}}(\mu)$ is $L^1_{\mathbb{R}}(\mu)$-bounded. Indeed, taking $\varepsilon=1$, there must be $E\in\Sigma_f$, $M\geqslant 0$ such that $\int_S(|h|-M1_E)^+\,\mathrm{d}\mu\leqslant 1$ for every $h\in H$. Hence

$$\int_S |h|\,\mathrm{d}\mu \leqslant \int_S (|h|-M1_E)^+\,\mathrm{d}\mu + \int_S M1_E\,\mathrm{d}\mu \leqslant 1+M\mu(E)$$

for every $h\in H$, so H is $L^1_{\mathbb{R}}(\mu)$-bounded.

▷ Let $\varphi\in L^1_{\mathbb{R}^+}(\mu)$. Then $\{h\in L^1_{\mathbb{R}}(\mu)\colon |h|\leqslant\varphi\}$ is uniformly integrable.

Theorem 2.1 ([8], Theorem 246G). *A subset H of $L^1_{\mathbb{R}}(\mu)$ is uniformly integrable if and only if*

(1) $|\int_A h\,\mathrm{d}\mu| < \infty$ *for every μ-atom (in the measure space sense (see* [8], *211I)) $A \in \Sigma$ and*

(2) *for every $\varepsilon > 0$ there are $E \in \Sigma_f$ and an $\eta > 0$ such that $|\int_F h\,\mathrm{d}\mu| \leqslant \varepsilon$ for every $h \in H$ and for every $F \in \Sigma$ with $\mu(F \cap E) \leqslant \eta$.*

Remark 2.1. Condition (2) is equivalent to

(2′) $\lim\limits_{n\to\infty} \sup\limits_{h\in H} |\int_{F_n} h\,\mathrm{d}\mu| = 0$ for every non-increasing sequence $(F_n)_{n\geqslant 1}$ in Σ with empty intersection.

Indeed, the proof of implication (2) $\Rightarrow$ (2′) is easy while the implication (2′) $\Rightarrow$ (2) follows from the proof of Theorem 246G in [8].

Note ([8], Corollary 246I) that in case (S, Σ, μ) is a probability space, (1) and (2) may be replaced by

$$\lim_{\lambda\to\infty} \sup_{h\in H} \int_{\{t\in S\colon |h(t)|\geqslant\lambda\}} |h|\,\mathrm{d}\mu = 0.$$

Theorem 2.2 ([8], Theorem 247C). *A subset H of $L^1_{\mathbb{R}}(\mu)$ is uniformly integrable if and only if it is relatively weakly compact in $L^1_{\mathbb{R}}(\mu)$.*

The following well known result ([11], [13]), which is the Pettis analogue of the classical Vitali convergence theorem, will play a key role in this work. An alternative proof based on the Eberlein-Smulyan-Grothendieck theorem can be found in [2].

Theorem 2.3. *Let $f\colon S \to X$ be a scalarly integrable function satisfying the following two conditions:*

(1) $\{\langle x^*, f\rangle\colon x^* \in \overline{B}_{X^*}\}$ *is relatively weakly compact in $L^1_{\mathbb{R}}(\mu)$.*

(2) *There exists a sequence (f_n) of Pettis integrable functions from S into X such that*

$$\lim_{n\to\infty} \int_E \langle x^*, f_n\rangle\,\mathrm{d}\mu = \int_E \langle x^*, f\rangle\,\mathrm{d}\mu$$

for each $x^* \in X^*$ and each $E \in \Sigma$.

Then f is Pettis integrable.

3. The weak McShane integral

In this section, we introduce the concept of the weak McShane integral and investigate some of its properties. For this purpose, we need to introduce some terminology. Assume that (S, Σ, μ) is a σ-finite positive measure space and $\mathcal{T} \subset \Sigma$ a topology on S making $(S, \mathcal{T}, \Sigma, \mu)$ a *quasi-Radon* measure space which is *outer regular*, that is, such that

$$\mu(E) = \inf\{\mu(G)\colon\ E \subset G,\ G \in \mathcal{T}\}\ (E \in \Sigma).$$

For an extensive study of quasi-Radon measure spaces, the reader is referred to [9], Chapter 41. A *partial McShane partition* is a countable (maybe finite) collection $\{(E_i, t_i)\}_{i\in I}$, where the E_i's are pairwise disjoint measurable subsets of S with finite measure and t_i is a point of S for each $i \in I$. A *generalized McShane partition* of S is an infinite partial McShane partition $\{(E_i, t_i)\}_{i\geqslant 1}$ such that $\mu\Big(S \setminus \bigcup\limits_{i=1}^{\infty} E_i\Big) = 0$. A *gauge* on S is a function $\Delta\colon\ S \to \mathcal{T}$ such that $t \in \Delta(t)$ for every $t \in S$. For a given Δ on S, we say that a partial McShane partition $\{(E_i, t_i)\}_{i\in I}$ is *subordinate* to Δ if $E_i \subset \Delta(t_i)$ for every $i \in I$. Let $f\colon\ S \to X$ be a function. We set

$$\sigma_n(f, \mathcal{P}_\infty) := \sum_{i=1}^{n} \mu(E_i) f(t_i)$$

for each infinite partial McShane partition $\mathcal{P}_\infty = \{(E_i, t_i)\}_{i\geqslant 1}$.

From now on $(S, \mathcal{T}, \Sigma, \mu)$ is a σ-finite outer regular quasi-Radon measure space.

Definition 3.1 ([7]). A function $f\colon\ S \to X$ is *McShane integrable* (*$\mathcal{M}$-integrable* for short), with integral ϖ, if for every $\varepsilon > 0$ there is a gauge $\Delta\colon\ S \to \mathcal{T}$ such that

$$\limsup_{n\to\infty} \|\sigma_n(f, \mathcal{P}_\infty) - \varpi\| \leqslant \varepsilon$$

for every generalized McShane partition $\mathcal{P}_\infty$ of S subordinate to Δ. We set $\varpi := (\mathcal{M}) \int_S f \,\mathrm{d}\mu$.

Recall that for a compact Radon measure space $(S, \mathcal{T}, \Sigma, \mu)$, generalized McShane partitions can be replaced by *finite strict generalized McShane partitions* of S (that is, finite partial McShane partitions $\{(E_i, t_i)\}_{1\leqslant i\leqslant p}$ such that $\bigcup\limits_{i=1}^{p} E_i = S$) (see [7]).

Remark 3.1. For the sake of comparison with the weak McShane integral, it is interesting to observe the following sequential formulation of the preceding definition.

A function $f\colon\ S \to X$ is $\mathcal{M}$-integrable, with integral ϖ, if and only if there is a sequence of gauges (Δ_m) from S into $\mathcal{T}$ such that

$$\lim_{m\to\infty} \sup_{\mathcal{P}_\infty\in\Pi_\infty(\Delta_m)} \limsup_{n\to\infty} \|\sigma_n(f, \mathcal{P}_\infty) - \varpi\| = 0,$$

where $\Pi_\infty(\Delta_m)$ denotes the collection of all generalized McShane partitions of S subordinate to Δ_m.

Every McShane integrable function is Pettis integrable and the respective integrals coincide whenever defined ([7], Theorem 1Q) but the converse does not hold in general (see [3], [10], and [14]). Nevertheless, for some classes of Banach spaces these two notions coincide: this happens for separable spaces ([7], [10], [12]), super-reflexive spaces, $c_0(I)$ (for any nonempty set I) [5] and $L^1_{\mathbb{R}}(\nu)$ (for any probability measure ν) [14]. More recently, R. Deville and J. Rodríguez [3] have proved the coincidence of the McShane and Pettis integrals for functions taking values in a subspace of a *Hilbert generated Banach space*, thus generalizing all previously mentioned results on such coincidence. Recall that a Banach space is *Hilbert generated* if there exists a Hilbert space Y and an operator $T\colon Y \to X$ such that its range $T(Y)$ is dense in X. For an extensive study of this class of spaces, see [6] and the references therein.

Before proceeding further, we list below some basic properties of the McShane integral that will be needed in this work. They are borrowed from [7].

Theorem 3.1. *Let $f\colon S \to X$ be a function.*

(1) *If f is $\mathcal{M}$-integrable, then the restriction $f_{|A}$ is $\mathcal{M}$-integrable (with respect to the σ-finite outer regular quasi-Radon measure space $(A, A\cap\mathcal{T}, A\cap\Sigma, \mu_{|A})$) for every $A \subset S$.*

(2) *Let $E \in \Sigma$. Then f is $\mathcal{M}$-integrable on E if and only if $f_{|E}$ is $\mathcal{M}$-integrable, and in this case the integrals are equal.*

(3) *Suppose $X = \mathbb{R}$. Then f is $\mathcal{M}$-integrable, if and only if it is integrable in the ordinary sense, and the two integrals are equal.*

We now introduce the concept of the weak McShane integral. For this purpose, we need an extra definition. A sequence $(\mathcal{P}^m_\infty)$ of generalized McShane partitions of S is said to be *adapted* to a sequence of gauges (Δ_m) from S into $\mathcal{T}$ if $\mathcal{P}^m_\infty$ is subordinate to Δ_m for each $m \geqslant 1$.

Definition 3.2. A function $f\colon S \to X$ is said to be *weakly McShane integrable* ($\mathcal{WM}$-integrable for short) on S, with weak McShane integral ϖ, if there is a sequence of gauges (Δ_m) from S into $\mathcal{T}$ such that the following condition (†) holds:

$$(\dagger)\qquad \lim_{m\to\infty}\limsup_{n\to\infty}|\langle x^*, \sigma_n(f, \mathcal{P}^m_\infty)\rangle - \langle x^*, \varpi\rangle| = 0,$$

for every $x^* \in X^*$ and for every sequence $(\mathcal{P}^m_\infty)$ of generalized McShane partitions of S adapted to (Δ_m).

We set $\varpi = (\mathcal{WM})\int_S f\,\mathrm{d}\mu$.

- ▷ f is $\mathcal{WM}$-integrable on a measurable subset E of S, if the function $1_E f$ is $\mathcal{WM}$-integrable on S. We set $(\mathcal{WM})\int_E f\,\mathrm{d}\mu := (\mathcal{WM})\int_S 1_E f\,\mathrm{d}\mu$.
- ▷ f is $\mathcal{WM}$-integrable on Σ, if it is $\mathcal{WM}$-integrable on every measurable subset of S.

Evidently, if f is $\mathcal{WM}$-integrable on Σ, then it is $\mathcal{WM}$-integrable on S. This implication cannot be reversed, see Section 4 to come.

Proposition 3.1. *If $f\colon S \to X$ is $\mathcal{M}$-integrable, then it is $\mathcal{WM}$-integrable on Σ.*

P r o o f. It is an immediate consequence of Theorem 3.1 (1)–(2), Remark 3.1 and Definition 3.2. □

The next theorem provides the linearity properties of the weak McShane integral.

Theorem 3.2. *Let f, $g\colon S \to X$ be two functions.*

(i) *If f and g are $\mathcal{WM}$-integrable on S, then $f+g$ is $\mathcal{WM}$-integrable on S and*

$$(\mathcal{WM})\int_S f+g\,\mathrm{d}\mu = (\mathcal{WM})\int_S f\,\mathrm{d}\mu + (\mathcal{WM})\int_S g\,\mathrm{d}\mu.$$

(ii) *If f is $\mathcal{WM}$-integrable on S and if α is a real number, then αf is $\mathcal{WM}$-integrable on S and*

$$(\mathcal{WM})\int_S \alpha f\,\mathrm{d}\mu = \alpha(\mathcal{WM})\int_S f\,\mathrm{d}\mu.$$

(iii) *If f is $\mathcal{WM}$-integrable on S and if $f = g$ μ-a.e., then the function g is $\mathcal{WM}$-integrable on S and*

$$(\mathcal{WM})\int_S g\,\mathrm{d}\mu = (\mathcal{WM})\int_S f\,\mathrm{d}\mu.$$

P r o o f. We will prove (iii) only; the rest of the proof is straightforward. Set $\theta := f - g$. Since $\theta := 0$ μ-a.e., by ([7], Corollary 2G), θ is $\mathcal{M}$-integrable, therefore $\mathcal{WM}$-integrable on Σ. In turn, by (i), $g = f + \theta$ is $\mathcal{WM}$-integrable on S. □

The following equivalent formulation of the weak McShane integral helps to translate Theorem 3.1 into the context of weak McShane integrals.

Proposition 3.2. *A function $f\colon S \to X$ is $\mathcal{WM}$-integrable on S with weak McShane integral ϖ if and only if there is a sequence (Δ_m) of gauges from S into $\mathcal{T}$ such that*

$$(\dagger') \qquad \lim_{m\to\infty} \sup_{\mathcal{P}_\infty \in \Pi_\infty(\Delta_m)} \limsup_{n\to\infty} |\langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \langle x^*, \varpi\rangle| = 0 \quad \textit{for all } x^* \in X^*.$$

P r o o f. Of course $(\dagger')$ implies $(\dagger)$. To see that $(\dagger)$ implies $(\dagger')$, set

$$M_m := \sup_{\mathcal{P}_\infty \in \Pi_\infty(\Delta_m)} \limsup_{n\to\infty} |\langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \langle x^*, \varpi\rangle|$$

and suppose that

$$\alpha := \limsup_{m\to\infty} M_m > 0.$$

For each $m \geqslant 1$, choose $\mathcal{P}_\infty^m \in \Pi_\infty(\Delta_m)$ such that

$$\begin{aligned} \limsup_{n\to\infty} |\langle x^*, \sigma_n(f, \mathcal{P}_\infty^m)\rangle - \langle x^*, \varpi\rangle| &\geqslant M_m - \frac{\alpha}{2} \quad \text{if } M_m < \infty, \\ &\geqslant \frac{\alpha}{2} \quad \text{if } M_m = \infty. \end{aligned}$$

Taking the $\limsup$ on m we get $0 \geqslant \frac{\alpha}{2}$! a contradiction. □

As a first consequence of this proposition, we obtain a version of Proposition 1E in [7] for the weak McShane integral dealing with compact Radon measure spaces.

Proposition 3.3. *Suppose that $(S, \mathcal{T}, \Sigma, \mu)$ is a compact Radon measure space and let $f\colon S \to X$ be a function. Then f is $\mathcal{WM}$-integrable on S, with weak McShane integral ϖ, if and only if there is a sequence (Δ_m) of gauges from S into $\mathcal{T}$ such that*

$$(\dagger``) \qquad \lim_{m\to\infty} \sup_{\{(E_i,t_i)\}_{1\leqslant i\leqslant p} \in \Pi_f(\Delta_m)} \left| \left\langle x^*, \sum_{i=1}^p \mu(E_i) f(t_i) \right\rangle - \langle x^*, \varpi\rangle \right| = 0 \quad \textit{for all } x^* \in X^*,$$

where $\Pi_f(\Delta_m)$ denotes the collection of all finite strict generalized McShane partitions of S subordinate to Δ_m.

P r o o f. The "only if" part follows easily from Proposition 3.2 applied to each function $\langle x^*, f\rangle$ (note that a finite partial McShane partition can be extended to an infinite one by adding empty sets). To prove the "if" part let $x^* \in X^*$, $\varepsilon > 0$ and choose a positive integer N (which may depend on x^*) such that

$$\sup_{\{(E_i,t_i)\}_{1\leqslant i\leqslant p} \in \Pi_f(\Delta_m)} \left| \left\langle x^*, \sum_{i=1}^p \mu(E_i) f(t_i) \right\rangle - \langle x^*, \varpi\rangle \right| \leqslant \varepsilon$$

for every $m \geqslant N$. According to Proposition 1E in [7] and its proof, we get

$$\sup_{\{(E_i,t_i)\}_{1\leqslant i\leqslant p}\in\Pi_f(\Delta_m)} \limsup_{n\to\infty} |\langle x^*, \sigma_n(f,\mathcal{P}_\infty)\rangle - \langle x^*, \varpi\rangle| \leqslant \varepsilon$$

for every $m \geqslant N$. Thus

$$\lim_{m\to\infty} \sup_{\mathcal{P}_\infty\in\Pi_\infty(\Delta_m)} \limsup_{n\to\infty} |\langle x^*, \sigma_n(f,\mathcal{P}_\infty)\rangle - \langle x^*, \varpi\rangle| = 0.$$

Since this holds for all $x^* \in X^*$, we conclude that f is $\mathcal{WM}$-integrable on S. $\square$

Remark 3.2. By repeating mutatis mutandis the arguments of the proof of Proposition 3.2, we see that $(\dagger'')$ is equivalent to

$$\lim_{m\to\infty} \left\langle x^*, \sum_{i=1}^{p} \mu(E_i^m) f(t_i^m) \right\rangle = \langle x^*, \varpi\rangle \quad \text{for all } x^* \in X^*$$

for every sequence $(\{(E_i^m, t_i^m)\}_{1\leqslant i\leqslant p})_{m\geqslant 1}$ of finite strict generalized McShane partitions of S adapted to (Δ_m).

Corollary 3.1. *Let $E \in \Sigma$ and let $f\colon S \to X$ be a function. If f is $\mathcal{WM}$-integrable on an E, then it is scalarly integrable on E (that is, $\langle x^*, f\rangle$ is Lebesgue integrable on E for all $x^* \in X^*$), and we have*

$$\int_E \langle x^*, f\rangle \,\mathrm{d}\mu = \left\langle x^*, (\mathcal{WM}) \int_E f \,\mathrm{d}\mu \right\rangle \quad \textit{for all } x^* \in X^*.$$

Consequently, if f is $\mathcal{WM}$-integrable on Σ, then it is Pettis integrable and

$$(\mathcal{P}e) \int_E f \,\mathrm{d}\mu = (\mathcal{WM}) \int_E f \,\mathrm{d}\mu \quad \textit{for all } E \in \Sigma.$$

P r o o f. Suppose that f is $\mathcal{WM}$-integrable on E and let $x^* \in X^*$. By virtue of Proposition 3.2 and Remark 3.1, we can easily see that $\langle x^*, 1_E f\rangle$ is $\mathcal{M}$-integrable as a function from S into $\mathbb{R}$, with integral $\langle x^*, (\mathcal{WM}) \int_E f \,\mathrm{d}\mu\rangle$. Therefore $\langle x^*, f\rangle$ is Lebesgue integrable on E and

$$\int_E \langle x^*, f\rangle \,\mathrm{d}\mu = \int_S \langle x^*, 1_E f\rangle \,\mathrm{d}\mu = \left\langle x^*, (\mathcal{WM}) \int_E f \,\mathrm{d}\mu \right\rangle \quad \text{for all } x^* \in X^*$$

in view of Theorem 3.1 (3). $\square$

Lemma 3.1. *Let $f\colon S \to X$ be a function. If f is $\mathcal{WM}$-integrable on S, then there is a sequence (Δ_m) of gauges from S into $\mathcal{T}$ such that*

$$\lim_{m\to\infty} \sup_{\mathcal{P}_\infty \in \Pi_{\infty|E}(\Delta_m)} \limsup_{n\to\infty} \left| \langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \int_E \langle x^*, f\rangle \,\mathrm{d}\mu \right| = 0$$

for all $E \in \Sigma$ and for all $x^* \in X^*$, where $\Pi_{\infty|E}(\Delta_m)$ denotes the collection of all generalized McShane partitions of E (with respect to the σ-finite outer regular quasi-Radon measure space $(E, E\cap\Sigma, E\cap\mathcal{T}, \mu_{|E})$) subordinate to Δ_m.

Proof. By Proposition 3.2, there is a sequence (Δ_m) of gauges from S into $\mathcal{T}$ such that

$$\lim_{m\to\infty} \sup_{\mathcal{P}_\infty \in \Pi_\infty(\Delta_m)} \limsup_{n\to\infty} \left| \langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \left\langle x^*, (\mathcal{WM}) \int_S f \,\mathrm{d}\mu \right\rangle \right| = 0 \quad \text{for all } x^* \in X^*.$$

Let $x^* \in X^*$ and $\varepsilon > 0$. Then there exists a positive integer N (possibly depending on x^*) such that

$$\sup_{\mathcal{P}_\infty \in \Pi_\infty(\Delta_m)} \limsup_{n\to\infty} \left| \left\langle x^*, \sigma_n(f, \mathcal{P}_\infty)\right\rangle - \left\langle x^*, (\mathcal{WM}) \int_S f \,\mathrm{d}\mu \right\rangle \right| \leqslant \frac{\varepsilon}{2}$$

for every $m \geqslant N$. Let $E \in \Sigma$. We can then repeat mutatis mutandis the arguments used in the proof of ([7], Theorem 1N) for the function $\langle x^*, f\rangle$ to obtain

$$\text{(3.2.1)} \qquad \sup_{\mathcal{P}_\infty, \mathcal{Q}_\infty \in \Pi_{\infty|E}(\Delta_m)} \limsup_{n\to\infty} |\langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \langle x^*, \sigma_n(f, \mathcal{Q}_\infty)\rangle| \leqslant \frac{\varepsilon}{2}$$

for every $m \geqslant N$. On the other hand, as $\langle x^*, f\rangle_{|E}$ is $\mathcal{M}$-integrable (by Corollary 3.1 and Theorem 3.1 (1)–(3)) we may select for each $m \geqslant 1$ a gauge $\Lambda_m\colon S \to \mathcal{T}$ (which may depend on x^*) with $\Lambda_m(t) \subset \Delta_m(t)$ for all $t \in S$ such that

$$\text{(3.2.2)} \qquad \sup_{\mathcal{P}_\infty \in \Pi_{\infty|E}(\Lambda_m)} \limsup_{n\to\infty} \left| \langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \int_E \langle x^*, f\rangle \,\mathrm{d}\mu \right| \leqslant \frac{\varepsilon}{2}.$$

Now, by the triangle inequality and the fact that $\Lambda_m(t) \subset \Delta_m(t)$ for all $t \in S$, we have

$$\begin{aligned}
&\sup_{\mathcal{P}_\infty \in \Pi_{\infty|E}(\Delta_m)} \limsup_{n\to\infty} \left| \langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \int_E \langle x^*, f\rangle \,\mathrm{d}\mu \right| \\
&\qquad \leqslant \sup_{\mathcal{P}_\infty \in \Pi_{\infty|E}(\Delta_m), \mathcal{Q}_\infty \in \Pi_{\infty|E}(\Lambda_m)} \limsup_{n\to\infty} |\langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \langle x^*, \sigma_n(f, \mathcal{Q}_\infty)\rangle| \\
&\qquad\quad + \sup_{\mathcal{Q}_\infty \in \Pi_{\infty|E}(\Lambda_m)} \limsup_{n\to\infty} \left| \langle x^*, \sigma_n(f, \mathcal{Q}_\infty)\rangle - \int_E \langle x^*, f\rangle \,\mathrm{d}\mu \right| \\
&\qquad \leqslant \sup_{\mathcal{P}_\infty, \mathcal{Q}_\infty \in \Pi_{\infty|E}(\Delta_m)} \limsup_{n\to\infty} |\langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \langle x^*, \sigma_n(f, \mathcal{Q}_\infty)\rangle| \\
&\qquad\quad + \sup_{\mathcal{Q}_\infty \in \Pi_{\infty|E}(\Lambda_m)} \limsup_{n\to\infty} \left| \langle x^*, \sigma_n(f, \mathcal{Q}_\infty)\rangle - \int_E \langle x^*, f\rangle \,\mathrm{d}\mu \right|
\end{aligned}$$

for every $m \geqslant N$. Hence, by (3.2.1) and (3.2.2)

$$\sup_{\mathcal{P}_\infty \in \Pi_{\infty|E}(\Delta_m)} \limsup_{n\to\infty} \left| \langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \int_E \langle x^*, f\rangle \,\mathrm{d}\mu \right| \leqslant \varepsilon$$

for every $m \geqslant N$. Thus

$$\lim_{m\to\infty} \sup_{\mathcal{P}_\infty \in \Pi_{\infty|E}(\Delta_m)} \limsup_{n\to\infty} \left| \langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \int_E \langle x^*, f\rangle \,\mathrm{d}\mu \right| = 0.$$

□

As a consequence of Lemma 3.1, we have

Corollary 3.2. *Let $f\colon S \to X$ be a function and let $F \in \Sigma$. If f is $\mathcal{WM}$-integrable on S and Pettis integrable on F (that is, $1_F f$ is Pettis integrable), then $f_{|E\cap F}$ is $\mathcal{WM}$-integrable on $E \cap F$ for every $E \in \Sigma$, and we have*

$$(\mathcal{WM}) \int_{E\cap F} f_{|E\cap F} \,\mathrm{d}\mu = (\mathcal{P}e) \int_E 1_F f \,\mathrm{d}\mu.$$

Lemma 3.2 (The weak Saks-Henstock lemma)**.** *Let $f\colon S \to X$ be a scalarly integrable function. Suppose that (Δ_m) is a sequence of gauges from S into $\mathcal{T}$ such that*

$$\lim_{m\to\infty} \sup_{\mathcal{P}_\infty \in \Pi_\infty(\Delta_m)} \limsup_{n\to\infty} \left| \langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \int_S \langle x^*, f\rangle \,\mathrm{d}\mu \right| = 0 \quad \textit{for all } x^* \in X^*.$$

Then

$$\lim_{m\to\infty} \sup_{\{(E_i,t_i)\}_{1\leqslant i\leqslant p}\in P\Pi_f(\Delta_m)} \left|\left\langle x^*, \sum_{i=1}^{p} \mu(E_i)f(t_i)\right\rangle - \int_{\bigcup\limits_{i=1}^{p} E_i} \langle x^*, f\rangle \,\mathrm{d}\mu\right| = 0$$

for all $x^* \in X^*$*, where* $P\Pi_f(\Delta_m)$ *denotes the collection of all finite partial McShane partitions of* S *subordinate to* Δ_m*.*

P r o o f. We will follow the same line of reasoning as in the proof of [7], Lemma 2B with suitable modifications. Let $x^* \in X^*$ and $\varepsilon > 0$. By the hypothesis, there exists a positive integer N such that

$$\sup_{\mathcal{P}_\infty\in\Pi_\infty(\Delta_m)} \limsup_{n\to\infty} \left|\langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \int_S \langle x^*, f\rangle \,\mathrm{d}\mu\right| \leqslant \frac{\varepsilon}{2} \quad \text{for all } m \geqslant N.$$

Now fix for the moment $m \geqslant N$ and let $\{(E_i, t_i)\}_{1\leqslant i\leqslant p}$ be a member of $P\Pi_f(\Delta_m)$. Let $E := \bigcup\limits_{i=1}^{p} E_i$. As $\langle x^*, f\rangle_{|S\setminus E}$ is $\mathcal{M}$-integrable (by Theorem 3.1 (3)), we may select a generalized McShane partition $\{(F_i, u_i)\}_{i\geqslant 1}$ of $S \setminus E$ (which may depend on x^*) subordinate to Δ_m such that

$$\limsup_{n\to\infty} \left|\left\langle x^*, \sum_{i=1}^{n} \mu(F_i)f(u_i)\right\rangle - \int_{S\setminus E} \langle x^*, f\rangle \,\mathrm{d}\mu\right| \leqslant \frac{\varepsilon}{2}.$$

Set

$$E_{p+i} := F_i \quad \text{and} \quad t_{p+i} := u_i, \quad i \geqslant 1.$$

Then $\{(E_i, t_i)\}_{i\geqslant 1}$ is a generalized McShane partition of S that is subordinate to Δ_m and

$$\begin{aligned}
&\left|\left\langle x^*, \sum_{i=1}^{p} \mu(E_i)f(t_i) - \int_E \langle x^*, f\rangle \,\mathrm{d}\mu\right|\right. \\
&= \left|\left\langle x^*, \sum_{i=1}^{p+n} \mu(E_i)f(t_i)\right\rangle - \int_S \langle x^*, f\rangle \,\mathrm{d}\mu - \left\langle x^*, \sum_{i=1}^{n} \mu(F_i)f(u_i)\right\rangle + \int_{S\setminus E} \langle x^*, f\rangle \,\mathrm{d}\mu\right| \\
&\leqslant \left|\left\langle x^*, \sum_{i=1}^{p+n} \mu(E_i)f(t_i)\right\rangle - \int_S \langle x^*, f\rangle \,\mathrm{d}\mu\right| + \left|\left\langle x^*, \sum_{i=1}^{n} \mu(F_i)f(u_i)\right\rangle - \int_{S\setminus E} \langle x^*, f\rangle \,\mathrm{d}\mu\right|.
\end{aligned}$$

Letting $n \to \infty$, we get

$$\left|\left\langle x^*, \sum_{i=1}^{p} \mu(E_i)f(t_i)\right\rangle - \int_E \langle x^*, f\rangle \,\mathrm{d}\mu\right| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Taking the supremum over $\{(E_i, t_i)\}_{1\leqslant i\leqslant p} \in P\Pi_f(\Delta_m)$ in this inequality yields

$$\sup_{\{(E_i,t_i)\}_{1\leqslant i\leqslant p}\in P\Pi_f(\Delta_m)} \left|\left\langle x^*, \sum_{i=1}^{p} \mu(E_i)f(t_i)\right\rangle - \int_{\bigcup\limits_{i=1}^{p} E_i} \langle x^*, f\rangle \,\mathrm{d}\mu\right| \leqslant \varepsilon.$$

This holds for all $m \geqslant N$. Thus

$$\lim_{m\to\infty} \sup_{\{(E_i,t_i)\}_{1\leqslant i\leqslant p}\in P\Pi_f(\Delta_m)} \left|\left\langle x^*, \sum_{i=1}^{p} \mu(E_i)f(t_i)\right\rangle - \int_{\bigcup\limits_{i=1}^{p} E_i} \langle x^*, f\rangle \,\mathrm{d}\mu\right| = 0.$$

□

The next result is a reformulation of Proposition 2E of [7] for the weak McShane integral. Its proof is a modified version of the proof of Fremlin.

Proposition 3.4. *Let $f\colon S \to X$ be a function and let $E \in \Sigma$. Then $1_E f$ is $\mathcal{WM}$-integrable on S if and only if the restriction $f_{|E}$ is $\mathcal{WM}$-integrable on E, and the two integrals are equal.*

Proof. Set $g := 1_E f$. If g is $\mathcal{WM}$-integrable on S, then by Lemma 3.1 there exists a sequence (Δ_m) of gauges from S into $\mathcal{T}$ such that

$$\lim_{m\to\infty} \sup_{\mathcal{P}_\infty\in\Pi_{\infty|E}(\Delta_m)} \limsup_{n\to\infty} \left|\langle x^*, \sigma_n(g, \mathcal{P}_\infty)\rangle - \int_E \langle x^*, g\rangle \,\mathrm{d}\mu\right| = 0$$

for all $x^* \in X^*$ with

$$\int_E \langle x^*, g\rangle \,\mathrm{d}\mu = \int_S 1_E \langle x^*, g\rangle \,\mathrm{d}\mu = \int_S \langle x^*, g\rangle \,\mathrm{d}\mu = \left\langle x^*, (\mathcal{WM})\int_S g \,\mathrm{d}\mu\right\rangle,$$

where the last equality follows from Corollary 3.1. As $g_{|E} = f_{|E}$, we obtain

$$\lim_{m\to\infty} \sup_{\mathcal{P}_\infty\in\Pi_{\infty|E}(\Delta_m)} \limsup_{n\to\infty} \left|\langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \left\langle x^*, (\mathcal{WM})\int_S g \,\mathrm{d}\mu\right\rangle\right| = 0$$

for all $x^* \in X^*$. Thus $f_{|E}$ is $\mathcal{WM}$-integrable on E, with integral $(\mathcal{WM})\int_E f_{|E} \,\mathrm{d}\mu = (\mathcal{WM})\int_S g \,\mathrm{d}\mu$. Conversely, suppose that $f_{|E}$ is $\mathcal{WM}$-integrable on E and set $\varpi_E := (\mathcal{WM})\int_E f_{|E} \,\mathrm{d}\mu$. To prove that g is $\mathcal{WM}$-integrable on S, we will use several arguments of the proof of Proposition 2E, [7] with appropriate modifications. Let $(\Delta_{m,E})$ be a sequence of gauges from E into $\mathcal{T}$ such that

$$\lim_{m\to\infty} \sup_{\mathcal{P}_\infty\in\Pi_{\infty|E}(\Delta_{m,E})} \limsup_{n\to\infty} |\langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \langle x^*, \varpi_E\rangle| = 0 \quad \text{for all } x^* \in X^*.$$

Noting that

$$\langle x^*, \varpi_E\rangle = \int_E \langle x^*, f_{|E}\rangle \,\mathrm{d}\mu = \int_E \langle x^*, f\rangle \,\mathrm{d}\mu$$

and applying the Weak Saks-Henstock Lemma to $f_{|E}$, we get

$$\lim_{m\to\infty} \sup_{\{(E_i,t_i)\}_{1\leqslant i\leqslant p}\in P\Pi_f(\Delta_{m,E})} \left| \left\langle x^*, \sum_{i=1}^{p} \mu(E_i) f(t_i) \right\rangle - \int_{\bigcup_{i=1}^{p} E_i} \langle x^*, f\rangle \,\mathrm{d}\mu \right| = 0$$

for all $x^* \in X^*$. Then for any fixed $\varepsilon > 0$ and $x^* \in X^*$ there exists a positive integer N such that

$$\sup_{\{(E_i,t_i)\}_{1\leqslant i\leqslant p}\in P\Pi_f(\Delta_{m,E})} \left| \left\langle x^*, \sum_{i=1}^{p} \mu(E_i) f(t_i) \right\rangle - \int_{\bigcup_{i=1}^{p} E_i} \langle x^*, f\rangle \,\mathrm{d}\mu \right| \leqslant \varepsilon \tag{3.4.1}$$

for every $m \geqslant N$. Now for each $n \geqslant 1$, choose a closed set F_n and an open set O_n with $F_n \subset E \subset O_n$ such that

$$\mu(E \setminus F_n) \leqslant \frac{1}{n} \tag{3.4.2}$$

and

$$\mu(O_n \setminus E) \leqslant \frac{2^{-n}}{n+1}\varepsilon \tag{3.4.3}$$

and define the sequence (Δ_m) of gauges from S into $\mathcal{T}$ by

$$\Delta_m(t) := \begin{cases} \Delta_{m,E}(t) \cap O_n & \text{if } t \in E \text{ and } n \leqslant \|f(t)\| < n+1, \\ S \setminus F_m & \text{if } t \in S \setminus E. \end{cases}$$

Let $(\{(E_i^m, t_i^m)\}_{i\geqslant 1})_{m\geqslant 1}$ be a sequence of generalized McShane partitions of S adapted to (Δ_m) and for each $i \geqslant 1$ set

$$H_i^m := \begin{cases} E_i^m \cap E & \text{if } t_i^m \in E, \\ \emptyset & \text{otherwise.} \end{cases}$$

Since $(\{(H_i^m, t_i^m)\}_{i\geqslant 1})_{m\geqslant 1}$ is a sequence of partial McShane partitions of E adapted to $(\Delta_{m,E})$, (3.4.1) gives

$$\left| \left\langle x^*, \sum_{i=1}^{n} \mu(H_i^m) f(t_i^m) \right\rangle - \int_{\bigcup_{i=1}^{n} H_i^m} \langle x^*, f\rangle \,\mathrm{d}\mu \right| \leqslant \varepsilon$$

for every $n \geqslant 1$ and for every $m \geqslant N$. Therefore, by the triangle inequality and the definition of H_i^m, we find that

$$
\begin{aligned}
&\left|\left\langle x^*, \sum_{i=1}^{n} \mu(E_i^m) g(t_i^m)\right\rangle - \int_{\bigcup_{i=1}^{n} H_i^m} \langle x^*, f\rangle \, \mathrm{d}\mu\right| \\
&\qquad \leqslant \left|\left\langle x^*, \sum_{i=1}^{n} \mu(E_i^m) g(t_i^m)\right\rangle - \left\langle x^*, \sum_{i=1}^{n} \mu(H_i^m) f(t_i^m)\right\rangle\right| \\
&\qquad\quad + \left|\left\langle x^*, \sum_{i=1}^{n} \mu(H_i^m) f(t_i^m)\right\rangle - \int_{\bigcup_{i=1}^{n} H_i^m} \langle x^*, f\rangle \, \mathrm{d}\mu\right| \\
&\qquad \leqslant \sum_{\{i=1,\ldots,n,\ t_i^m \in E\}} \mu(E_i^m \setminus E)\|f(t_i^m)\| + \varepsilon \\
&\qquad = \sum_{k=1}^{\infty} \sum_{\{i=1,\ldots,n,\ t_i^m \in E,\, k \leqslant \|f(t_i^m)\| < k+1\}} \mu(E_i^m \setminus E)\|f(t_i^m)\| + \varepsilon
\end{aligned}
$$

for every $n \geqslant 1$ and for every $m \geqslant N$. As $E_i^m \subset \Delta(t_i^m) \subset O_k$ for all $i \geqslant 1$ such that $t_i \in E$ and $k \leqslant \|f(t_i^m)\| < k+1$, we obtain

$$
\begin{aligned}
(3.4.4) \qquad &\left|\left\langle x^*, \sum_{i=1}^{n} \mu(E_i^m) g(t_i^m)\right\rangle - \int_{\bigcup_{i=1}^{n} H_i^m} \langle x^*, f\rangle \, \mathrm{d}\mu\right| \\
&\qquad \leqslant \sum_{k=1}^{\infty} (k+1)\mu(O_k \setminus E) + \varepsilon \leqslant \sum_{k=1}^{\infty} 2^{-k}\varepsilon + \varepsilon = 2\varepsilon
\end{aligned}
$$

for every $n \geqslant 1$ and for every $m \geqslant N$. On the other hand, we have

$$
(3.4.5) \qquad \left|\int_E \langle x^*, f\rangle \, \mathrm{d}\mu - \int_{\bigcup_{i=1}^{\infty} H_i^m} \langle x^*, f\rangle \, \mathrm{d}\mu\right| \leqslant \int_{E \setminus F_m} |\langle x^*, f\rangle| \, \mathrm{d}\mu
$$

because, for every $m \geqslant 1$,

$$
\begin{aligned}
E &= \left[\bigcup_{i \geqslant 1, t_i^m \in E} (E \cap E_i^m)\right] \cup \left[\bigcup_{i \geqslant 1, t_i \in S \setminus E} (E \cap E_i^m)\right] \\
&\subset \left(\bigcup_{i=1}^{\infty} H_i^m\right) \cup \left[\bigcup_{i \geqslant 1, t_i \in S \setminus E} (E \cap \Delta_m(t_i^m))\right] = \left(\bigcup_{i=1}^{\infty} H_i^m\right) \cup (E \setminus F_m) \subset E,
\end{aligned}
$$

in view of the definition of H_i^m and Δ_m. Putting (3.4.4) and (3.4.5) together, we get

$$\left|\left\langle x^*, \sum_{i=1}^{n} \mu(E_i^m) g(t_i^m)\right\rangle - \langle x^*, \varpi_E\rangle\right| \leqslant \left|\left\langle x^*, \sum_{i=1}^{n} \mu(E_i^m) g(t_i^m)\right\rangle - \int_{\bigcup\limits_{i=1}^{n} H_i^m} \langle x^*, f\rangle \, \mathrm{d}\mu\right|$$
$$+ \left|\int_E \langle x^*, f\rangle \, \mathrm{d}\mu - \int_{\bigcup\limits_{i=1}^{\infty} H_i^m} \langle x^*, f\rangle \, \mathrm{d}\mu\right| \leqslant 2\varepsilon + \int_{E\setminus F_m} |\langle x^*, f\rangle| \, \mathrm{d}\mu$$

for every $n \geqslant 1$ and for every $m \geqslant N$. As (3.4.1) and the integrability of $\langle x^*, f\rangle$ on E ensure

$$\lim_{m\to\infty} \int_{E\setminus F_m} |\langle x^*, f\rangle| \, \mathrm{d}\mu = 0,$$

we obtain

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \left|\left\langle x^*, \sum_{i=1}^{n} \mu(E_i^m) g(t_i^m)\right\rangle - \langle x^*, \varpi_E\rangle\right| \leqslant 2\varepsilon,$$

and hence

$$\lim_{m\to\infty} \limsup_{n\to\infty} \left|\left\langle x^*, \sum_{i=1}^{n} \mu(E_i^m) g(t_i^m)\right\rangle - \langle x^*, \varpi_E\rangle\right| = 0,$$

by the arbitrariness of ε. Thus g is $\mathcal{WM}$-integrable on S with integral ϖ_E. □

Corollary 3.3. *A function $f\colon S \to X$ is $\mathcal{WM}$-integrable on Σ if and only if it is $\mathcal{WM}$-integrable on S and Pettis integrable, and the corresponding integrals are equal.*

P r o o f. The only "if part" is proved by Corollary 3.1, whereas the "if part" is a direct consequence of Proposition 3.4 and Corollary 3.2. □

We conclude this section by providing the following lemma which, together with Theorem 2.3, will play a crucial role in Section 4.

Lemma 3.3. *Suppose that $f\colon S \to X$ is $\mathcal{WM}$-integrable on S, (Δ_m) is a sequence of gauges from S into $\mathcal{T}$ as given in Definition 3.2 (or in Proposition 3.2) and R is a measurable set of finite measure. Then given any sequence $(\{(E_i^m, t_i^m)\}_{i\geqslant 1})_{m\geqslant 1}$ of generalized McShane partitions of S adapted to (Δ_m), there exists a strictly increasing sequence $(p_m)_{m\geqslant 1}$ of positive integers such that*

$$\lim_{m\to\infty} \left\langle x^*, \sum_{i=1}^{p_m} \mu(R\cap E\cap E_i^m) f(t_i^m)\right\rangle = \int_{R\cap E} \langle x^*, f\rangle \, \mathrm{d}\mu$$

for every $x^* \in X^*$ and for every $E \in \Sigma$.

Proof. By Proposition 3.2

$$\text{(a)}\qquad \lim_{m\to\infty}\sup_{\mathcal{P}_\infty\in\Pi_\infty(\Delta_m)}\limsup_{n\to\infty}\left|\langle x^*,\sigma_n(f,\mathcal{P}_\infty)\rangle-\left\langle x^*,(\mathcal{WM})\int_S f\,\mathrm{d}\mu\right\rangle\right|=0$$

for every $x^*\in X^*$. Let $(\{(E_i^m,t_i^m)\}_{i\geqslant 1})_{m\geqslant 1}$ be a sequence of generalized McShane partitions of S adapted to (Δ_m) and let $(p_m)_{m\geqslant 1}$ be a strictly increasing sequence of positive integers such that

$$\text{(b)}\qquad \lim_{m\to\infty}\mu\biggl(R\setminus\bigcup_{i=1}^{p_m}E_i^m\biggr)=0.$$

Let $E\in\Sigma$ and let $x^*\in X^*$. Since $\{(R\cap E\cap E_i^m,t_i^m)\}_{i\geqslant 1}$ is an infinite partial McShane partition of S that is subordinate to Δ_m for each $m\geqslant 1$, equality (a) and Lemma 3.2 yield

$$\begin{aligned}\limsup_{m\to\infty}&\left|\left\langle x^*,\sum_{i=1}^{p_m}\mu(R\cap E\cap E_i^m)f(t_i^m)\right\rangle-\int_{\bigcup\limits_{i=1}^{p_m}R\cap E\cap E_i^m}\langle x^*,f\rangle\,\mathrm{d}\mu\right|\\&\leqslant\lim_{m\to\infty}\sup_{\{(F_i,u_i)\}_{1\leqslant i\leqslant p}\in P\Pi_f(\Delta_m)}\left|\left\langle x^*,\sum_{i=1}^{p}\mu(F_i)f(u_i)\right\rangle-\int_{\bigcup\limits_{i=1}^{p}F_i}\langle x^*,f\rangle\,\mathrm{d}\mu\right|=0.\end{aligned}$$

Moreover, one has

$$\lim_{m\to\infty}\int_{\bigcup\limits_{i=1}^{p_m}R\cap E\cap E_i^m}\langle x^*,f\rangle\,\mathrm{d}\mu=\int_{R\cap E}\langle x^*,f\rangle\,\mathrm{d}\mu,$$

because $\langle x^*,f\rangle$ is integrable and $\lim\limits_{m\to\infty}\mu\Bigl(\bigcup\limits_{i=1}^{p_m}R\cap E\cap E_i^m\Bigr)=\mu(R\cap E)$ (by (b)). Hence

$$\lim_{m\to\infty}\left\langle x^*,\sum_{i=1}^{p_m}\mu(R\cap E\cap E_i^m)f(t_i^m)\right\rangle=\int_{R\cap E}\langle x^*,f\rangle\,\mathrm{d}\mu.$$

□

4. Weak McShane integrability from S to Σ

In this section we attempt to determine when weak McShane integrability passes from S to Σ.

The following theorem due to the first author will play a crucial role in this section. It represents the combined efforts of Theorem 2.3, Lemma 3.2 and Lemma 4.1 below.

Theorem 4.1 (M. Saadoune). *If a function $f\colon S \to X$ is $\mathcal{WM}$-integrable on S, then there exists an increasing sequence $(S_l)_{l\geqslant 1}$ in Σ_f with union S such that $1_{S_l} f$ is Pettis integrable and $\mathcal{WM}$-integrable on Σ for each $l \geqslant 1$.*

Lemma 4.1. *If $f\colon S \to X$ is a scalarly integrable function, then there exists an increasing sequence $(S_l)_{l\geqslant 1}$ in Σ_f with union S such that $\{\langle x^*, 1_{S_l} f\rangle\colon x^* \in \overline{B}_{X^*}\}$ is uniformly integrable for each $l \geqslant 1$.*

Proof. Since μ is σ-finite, there is an increasing sequence $(R_k)_{k\geqslant 1}$ in Σ_f such that $S = \bigcup\limits_{k\geqslant 1} R_k$. For each $k \geqslant 1$, set

$$C_k := \{t \in R_k\colon \ \|f(t)\| \leqslant k\}.$$

Then $(C_k)_{k\geqslant 1}$ is an increasing sequence with union S and $\mu^*(C_k) < \infty$ for all $k \geqslant 1$, where μ^* stands for the outer measure induced by μ. Let $D_k \in \Sigma_f$ be such that $C_k \subset D_k$ and $\mu(D_k) = \mu^*(C_k)$. Since f is uniformly bounded on C_k and $\mu^*(C_k) = \mu(D_k)$, $\langle x^*, f\rangle$ is uniformly bounded almost everywhere on D_k for each $x^* \in X^*$. Set

$$S_l := \bigcup_{k=1}^{l} D_k, \quad l \geqslant 1.$$

Clearly, $(S_l)_{l\geqslant 1}$ is a non-decreasing sequence in Σ_f with union S. Further, the function $\{\langle x^*, 1_{S_l} f\rangle$ is uniformly bounded almost everywhere for each $x^* \in X^*$ and each $l \geqslant 1$, in turn $\{\langle x^*, 1_{S_l} f\rangle\colon x^* \in \overline{B}_{X^*}\}$ is uniformly integrable. $\square$

Remark 4.1. Actually, repeating several arguments used in the proof of Theorem 4 of [13], Lemma 4.1 remains valid when dealing only with σ-finite positive measure spaces instead of σ-finite outer regular quasi-Radon measure spaces.

Proof of Theorem 4.1. Let $(S_l)_{l\geqslant 1}$ be the sequence given in Lemma 4.1. Let (Δ_m) be as mentioned in Definition 3.2 and let $(\{(E_i^m, t_i^m)\}_{i\geqslant 1})_{m\geqslant 1}$ be a fixed sequence of generalized McShane partitions of S adapted to (Δ_m). Then for any fixed $l \geqslant 1$, Lemma 3.3 provides a strictly increasing sequence $(p_m)_{m\geqslant 1}$ of positive integers (possibly depending on l) such that

$$\lim_{m\to\infty} \left\langle x^*, \sum_{i=1}^{p_m} \mu(S_l \cap E_i^m \cap E) f(t_i^m)\right\rangle = \int_{S_l\cap E} \langle x^*, f\rangle \,\mathrm{d}\mu$$

for all $x^* \in X^*$ and for all $E \in \Sigma$. In other words, this equality becomes

$$\lim_{m\to\infty} \int_{S_l\cap E} \left\langle x^*, \sum_{i=1}^{p_m} 1_{E_i^m} f(t_i^m)\right\rangle \mathrm{d}\mu = \int_{S_l\cap E} \langle x^*, f\rangle \,\mathrm{d}\mu$$

for all $x^* \in X^*$ and for all $E \in \Sigma$. As the functions $\sum\limits_{i=1}^{p_m} 1_{E_i^m} f(t_i^m)$ $(m \geqslant 1)$ are obviously Pettis integrable and, by Lemma 4.1, the set $\{\langle x^*, 1_{S_l} f\rangle \colon x^* \in \overline{B}_{X^*}\}$ is uniformly integrable, it follows from Theorem 2.3 that $1_{S_l} f$ is Pettis integrable. Therefore, by Corollary 3.2, $f_{|S_l}$ is $\mathcal{WM}$-integrable on S_l. Equivalently, $1_{S_l} f$ is $\mathcal{WM}$-integrable on S, in view of Proposition 3.4. The desired conclusion then follows from Corollary 3.3. $\square$

In certain interesting situations, the weak McShane integrability passes from S to Σ as the following results (Theorems 4.2–4.5) show.

Theorem 4.2. *Let $f\colon S \to X$ be a function. If the following two conditions hold:*

(j) *f is $\mathcal{WM}$-integrable on S and*

(jj)
$$\lim_{n\to\infty} \sup_{x^*\in \overline{B}_{X^*}} \left| \int_{F_n} \langle x^*, f\rangle \,\mathrm{d}\mu \right| = 0$$

whenever (F_n) is a non-increasing sequence in Σ with empty intersection, then f is Pettis integrable. Consequently, f is $\mathcal{WM}$-integrable on Σ.

P r o o f. Fix $\varepsilon > 0$. Then, by condition (jj) and Remark 2.1, there are $E_0 \in \Sigma_f$ and $\eta > 0$ such that
$$\left| \int_E \langle x^*, f\rangle \,\mathrm{d}\mu \right| \leqslant \frac{\varepsilon}{2}$$
for every $x^* \in \overline{B}_{X^*}$ and for every $E \in \Sigma$ with $\mu(E \cap E_0) \leqslant \eta$. Now let $(S_l)_{l\geqslant 1}$ be given as in Theorem 4.1. As $\bigcup\limits_{l\geqslant 1} S_l = S$, we may choose an integer $l_0 \geqslant 1$ such that $\mu(E_0 \setminus S_l) \leqslant \eta$ for every $l \geqslant l_0$; therefore
$$\left| \int_{E\setminus S_l} \langle x^*, f\rangle \,\mathrm{d}\mu \right| \leqslant \frac{\varepsilon}{2}$$
for every $x^* \in \overline{B}_{X^*}$, for every $E \in \Sigma$ and for every $l \geqslant l_0$. Together with the Pettis integrability of each function $1_{S_l} f$, we get
$$\begin{aligned}\left| \left\langle x^*, (\mathcal{P}e)\int_{E\cap S_l} f \,\mathrm{d}\mu - (\mathcal{P}e)\int_{E\cap S_{l'}} f \,\mathrm{d}\mu \right\rangle \right| &= \left| \int_{E\cap S_l} \langle x^*, f\rangle \,\mathrm{d}\mu - \int_{E\cap S_{l'}} \langle x^*, f\rangle \,\mathrm{d}\mu \right| \\ &= \left| \int_{E\setminus S_l} \langle x^*, f\rangle \,\mathrm{d}\mu - \int_{E\setminus S_{l'}} \langle x^*, f\rangle \,\mathrm{d}\mu \right| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$
for all $E \in \Sigma$, $l \geqslant l' \geqslant l_0$ and $x^* \in \overline{B}_{X^*}$. Taking the supremum over $\overline{B}_{X^*}$ in the above estimation yields
$$\left\| (\mathcal{P}e)\int_{E\cap S_l} f \,\mathrm{d}\mu - (\mathcal{P}e)\int_{E\cap S_{l'}} f \,\mathrm{d}\mu \right\| \leqslant \varepsilon$$

for all $E \in \Sigma$ and for all $l \geqslant l' \geqslant l_0$. This shows that $((\mathcal{P}e)\int_{E\cap S_l} f\,\mathrm{d}\mu)_{l\geqslant 1}$ is a Cauchy sequence in X, so it necessarily converges to an element $x_E \in X$ with respect to the norm topology. In particular, we have

$$\lim_{l\to\infty}\left\langle x^*, (\mathcal{P}e)\int_{E\cap S_l} f\,\mathrm{d}\mu\right\rangle = \langle x^*, x_E\rangle \quad \text{for all } x^* \in \overline{B}_{X^*}.$$

As

$$\lim_{l\to\infty}\left\langle x^*, (\mathcal{P}e)\int_{E\cap S_l} f\,\mathrm{d}\mu\right\rangle = \lim_{l\to\infty}\int_{E\cap S_l}\langle x^*, f\rangle\,\mathrm{d}\mu = \int_E \langle x^*, f\rangle\,\mathrm{d}\mu$$

for all $x^* \in \overline{B}_{X^*}$ and all $E \in \Sigma$, we obtain

$$\int_E \langle x^*, f\rangle\,\mathrm{d}\mu = \langle x^*, x_E\rangle.$$

Thus f is Pettis integrable, and hence $\mathcal{WM}$-integrable on Σ, in view of Corollary 3.2. $\square$

Corollary 4.1. *A function $f\colon S \to X$ is $\mathcal{WM}$-integrable on Σ if and only if*

(j) *f is $\mathcal{WM}$-integrable on S and*

(jj)
$$\lim_{n\to\infty}\sup_{x^*\in\overline{B}_{X^*}}\left|\int_{F_n}\langle x^*, f\rangle\,\mathrm{d}\mu\right| = 0$$

whenever (F_n) is a non-increasing sequence in Σ with empty intersection.

Condition (jj) *may be replaced by*

(jj)$'$ $\{\langle x^*, f\rangle\colon x^* \in \overline{B}_{X^*}\}$ *is uniformly integrable.*

P r o o f. Sufficiency is established by the preceding theorem. Necessity follows from Corollary 3.1, a lemma of Musial in ([13], page 162), Theorems 2.1–2.2 and Remark 2.1. $\square$

Theorem 4.3. *Suppose that X is weakly sequentially complete and let $f\colon S \to X$ be a function. If f is $\mathcal{WM}$-integrable on S, then it is Pettis integrable. Consequently, f is $\mathcal{WM}$-integrable on Σ if and only if it is $\mathcal{WM}$-integrable on S.*

P r o o f. Let $(S_l)_{l\geqslant 1}$ be given as in Theorem 4.1 and let $E \in \Sigma$. As each function $1_{S_l} f$ is Pettis integrable, we have

$$\left\langle x^*, (\mathcal{P}e)\int_{E\cap S_l} f\,\mathrm{d}\mu\right\rangle = \int_{E\cap S_l}\langle x^*, f\rangle\,\mathrm{d}\mu \quad \text{for all } x^* \in X^*.$$

Letting l go to ∞, we get

$$\lim_{l\to\infty}\left\langle x^*, (\mathcal{P}e)\int_{E\cap S_l} f\,\mathrm{d}\mu\right\rangle = \int_E \langle x^*, f\rangle\,\mathrm{d}\mu \quad \text{for all } x^* \in X^*$$

(because $S_l \uparrow S$ and $\langle x^*, f\rangle \in L^1_{\mathbb{R}}(\mu)$). So $((\mathcal{P}e)\int_{E\cap S_l} f\,\mathrm{d}\mu)_{l\geqslant 1}$ is a weak Cauchy sequence in X, therefore it w-converges to an element $\varpi_E \in X$, since, by hypothesis, X is weakly sequentially complete. By identifying the limits we get

$$\langle x^*, \varpi_E\rangle = \int_E \langle x^*, f\rangle\,\mathrm{d}\mu.$$

This happens for all $E \in \Sigma$, so f is Pettis integrable. The main implication of the second part of the theorem follows then from Corollary 3.2. $\square$

Corollary 4.2. *Suppose $X = L^1(\Omega, \mathcal{F}, \nu)$ (for any σ-finite measure space $(\Omega, \mathcal{F}, \nu)$) and let $f\colon S \to X$ be a function. Then the following conditions are equivalent:*
(1) *f is $\mathcal{M}$-integrable,*
(2) *f is $\mathcal{WM}$-integrable on Σ,*
(3) *f is $\mathcal{WM}$-integrable on S,*
(4) *f is Pettis integrable.*

Proof. As we have already mentioned in the previous section, every $\mathcal{M}$-integrable function, is $\mathcal{WM}$-integrable on Σ, by Proposition 3.1. Now the implication (2) $\Rightarrow$ (3) is obvious, implication (3) $\Rightarrow$ (4) follows from Theorem 4.3, because $L^1(\Omega, \mathcal{F}, \nu)$ is weakly sequentially complete. Further (4) $\Rightarrow$ (1) is due to R. Deville and J. Rodríguez ([3], Corollary 3.8). $\square$

Theorem 4.4. *Suppose that X does not contain any isomorphic copy of c_0 and let $f\colon S \to X$ be a function. If f is $\mathcal{WM}$-integrable on S, then it is Pettis integrable. Consequently, f is $\mathcal{WM}$-integrable on Σ if and only if it is $\mathcal{WM}$-integrable on S.*

Proof. Using Theorem 4.1 we obtain an increasing sequence $(S_l)_{l\geqslant 1}$ in Σ_f with $\bigcup\limits_{l\geqslant 1} S_l = S$ such that $1_{S_l} f$ is Pettis integrable for each $l \geqslant 1$. Define the sequence $(S'_l)_{l\geqslant 1}$ in Σ by

$$S'_1 := S_1 \quad \text{and} \quad S'_l := S_l \setminus S_{l-1} \quad \text{for } l > 1$$

and let $E \in \Sigma$. Then we have

$$\begin{aligned}\int_E |\langle x^*, f\rangle|\,\mathrm{d}\mu &= \int_{\bigcup\limits_{l\geqslant 1} E\cap S'_l} |\langle x^*, f\rangle|\,\mathrm{d}\mu = \sum_{l=1}^{\infty}\int_{E\cap S'_l} |\langle x^*, f\rangle|\,\mathrm{d}\mu \\ &\geqslant \sum_{l=1}^{\infty}\left|\int_{E\cap S'_l}\langle x^*, f\rangle\,\mathrm{d}\mu\right| = \sum_{l=1}^{\infty}\left|\left\langle x^*, (\mathcal{P}e)\int_{E\cap S'_l} f\,\mathrm{d}\mu\right\rangle\right|\end{aligned}$$

for all $x^* \in X^*$. Since f is scalarly integrable (by Corollary 3.1), it follows that the series $\sum\limits_{l\geqslant 1} |\langle x^*, (\mathcal{P}e)\int_{E\cap S'_l} f\,\mathrm{d}\mu\rangle|$ is convergent for all $x^* \in X^*$. Therefore, we can invoke the sequence characterization of Bessaga and Pełczyński of Banach spaces not containing c_0 ([4], Corollary I.4.5), which shows that the series $\sum\limits_{l\geqslant 1}(\mathcal{P}e)\int_{E\cap S'_l} f\,\mathrm{d}\mu$ is unconditionally convergent in the norm topology to an element $x_E \in X$, so

$$\lim_{n\to\infty}\left\|\sum_{l=1}^{n}(\mathcal{P}e)\int_{E\cap S'_l} f\,\mathrm{d}\mu - x_E\right\| = 0,$$

which implies

$$\lim_{n\to\infty}\sum_{l=1}^{n}\left\langle x^*, (\mathcal{P}e)\int_{E\cap S'_l} f\,\mathrm{d}\mu\right\rangle = \langle x^*, x_E\rangle \quad \text{for all } x^* \in X^*.$$

As

$$\begin{aligned}\lim_{n\to\infty}\sum_{l=1}^{n}\left\langle x^*, (\mathcal{P}e)\int_{E\cap S'_l} f\,\mathrm{d}\mu\right\rangle &= \lim_{n\to\infty}\sum_{l=1}^{n}\int_{E\cap S'_l}\langle x^*, f\rangle\,\mathrm{d}\mu \\ &= \lim_{n\to\infty}\int_{E\cap\bigcup\limits_{l=1}^{n} S'_l}\langle x^*, f\rangle\,\mathrm{d}\mu = \int_E \langle x^*, f\rangle\,\mathrm{d}\mu,\end{aligned}$$

we get

$$\int_E \langle x^*, f\rangle\,\mathrm{d}\mu = \langle x^*, x_E\rangle \quad \text{for all } x^* \in X^*.$$

Since this holds for all $E \in \Sigma$, we conclude that f is Pettis integrable. The main implication of the second part of the theorem follows then from Corollary 3.2. $\square$

Remark 4.2. Recalling that $L^1(\Omega, \mathcal{F}, \nu)$ does not contain any isomorphic copy of c_0 (because $L^1(\Omega, \mathcal{F}, \nu)$ is weakly sequentially complete and c_0 does not enjoy this property), implication (3) $\Rightarrow$ (4) of Corollary 4.2 can also be derived from the preceding theorem.

It is worth giving the following variant of Theorems 4.2–4.4

Theorem 4.5. *Suppose that $\mu(S) = 1$ and let $f\colon S \to X$ be a function. If f is $\mathcal{WM}$-integrable on S and if there exists a fixed closed convex and w-ball-compact subset Γ of X with $0 \in \Gamma$ (that is, Γ is w-ball-compact if the intersection of Γ with every closed ball is weakly compact) such that*

$$f(t) \in \Gamma \quad \text{a.e.},$$

then f is $\mathcal{WM}$-integrable on Σ.

P r o o f. Taking into account Theorem 3.2 (iii), we may suppose, without loss of generality, that $f(t) \in \Gamma$ for all $t \in S$. According to Lemma 3.4, we can choose a sequence $(\{(E_i^m, t_i^m)\}_{i\geqslant 1})_{m\geqslant 1}$ of generalized McShane partitions of S and a strictly increasing sequence (p_m) of positive integers such that

$$\lim_{m\to\infty} \left\langle x^*, \sum_{i=1}^{p_m} \mu(E\cap E_i^m) f(t_i^m) \right\rangle = \int_E \langle x^*, f\rangle \, \mathrm{d}\mu$$

for all $x^* \in X^*$ and for all $E \in \Sigma$. So $(\sum\limits_{i=1}^{m} \mu(E\cap E_i^m) f(t_i^m))$ is a weak Cauchy sequence in X, and $r := \sup\limits_{m\geqslant 1} \|\sum\limits_{i=1}^{m} \mu(E\cap E_i^m) f(t_i^m)\| < \infty$. Furthermore, since Γ is convex and contains 0, $\sum\limits_{i=1}^{p_m} \mu(E\cap E_i^m) f(t_i^m) \in \Gamma$ for all $m \geqslant 1$. So, the sum $\sum\limits_{i=1}^{p_m} \mu(E\cap E_i^m) f(t_i^m)$ belongs to the w-compact set $\overline{B}(0,r)\cap\Gamma$ for each $m \geqslant 1$. Consequently, $\Big(\sum\limits_{i=1}^{p_m} \mu(E\cap E_i^m) f(t_i^m)\Big)$ w-converges to an element $\varpi_E \in X$. By identifying the limits we get $\langle x^*, \varpi_E\rangle = \int_E \langle x^*, f\rangle \, \mathrm{d}\mu$ and so, by the arbitrariness of $E \in \Sigma$, f is Pettis integrable. $\square$

The next theorem is a weak McShane version of Theorem 15 of Gordon [12] dealing with the McShane integral on $[0,1]$. We first recall some definitions and known facts about series ([1], [4], [16]).

Given a series $\sum\limits_{n\geqslant 1} x_n$ with values in X, recall that $\sum\limits_{n\geqslant 1} x_n$ is called:

▷ *weakly convergent* or *convergent in norm* if the sequence of its partial sums $\sum\limits_{n=1}^{i} x_n$ w-converges or converges in norm, respectively.

▷ *unconditionally convergent in norm* if the series $\sum\limits_{k\geqslant 1} x_{\pi(k)}$ converges in norm for every sequence $(x_{\pi(k)})$ whenever π is a permutation of $\mathbb{N}$; equivalently, if all series of the form $\sum\limits_{k\geqslant 1} x_{\varphi(k)}$ where φ is a strictly increasing mapping from $\mathbb{N}$ onto $\mathbb{N}$ converge in norm.

▷ *scalarly* (alias *weakly*) *absolutely convergent* if $\sum\limits_{n\geqslant 1} |\langle x^*, x_n\rangle| < \infty$ for all $x^* \in X^*$. We say also that $\sum\limits_{n\geqslant 1} x_n$ is *weakly unconditionally Cauchy.*

It is known that the following implications hold:

$\sum\limits_{n\geqslant 1} x_n$ is unconditionally convergent in norm $\Rightarrow$ $\sum\limits_{n\geqslant 1} x_n$ is scalarly absolutely convergent.

$\sum\limits_{n\geqslant 1} x_n$ is unconditionally convergent in norm $\Rightarrow$ $\sum\limits_{n\geqslant 1} x_n$ is convergent in norm $\Rightarrow$ $\sum\limits_{n\geqslant 1} x_n$ is weakly convergent.

Moreover, no one of these arrows can be reversed in general (see [1], Example 5, page 341). More precisely, the following conditions are equivalent (see [1], page 337):

(C_1) There exists a scalarly absolutely convergent series which is convergent in norm, but is not unconditionally convergent in norm.

(C_2) There exists a scalarly absolutely convergent series which is weakly convergent, but is not convergent in norm.

(C_3) There exists a scalarly absolutely convergent series which is not weakly convergent.

(C_4) The space X has a copy of c_0.

Theorem 4.6 (M. Saadoune). *Let $(E_n)_{n\geqslant 1}$ be a sequence of disjoint subsets of Σ, let $(x_n)_{n\geqslant 1}$ be a sequence in X, and let $f\colon S\to X$ be the function defined by*

$$f(t) := \sum_{n=1}^{\infty} x_n 1_{E_n}(t) \quad (t\in S).$$

If the series $\sum\limits_{n\geqslant 1} \mu(E_n)x_n$ is scalarly absolutely convergent, then f is scalarly integrable, and there is a sequence (Δ_m) of gauges from S into $\mathcal{T}$ such that

$$\lim_{m\to\infty} \sup_{\mathcal{P}_\infty\in\Pi_\infty(\Delta_m)} \limsup_{n\to\infty} \left|\langle x^*, \sigma_n(f,\mathcal{P}_\infty)\rangle - \int_S \langle x^*, f\rangle \,\mathrm{d}\mu\right| = 0$$
$$\text{for all } x^*\in X^* \text{ with } \int_S \langle x^*, f\rangle \,\mathrm{d}\mu = \sum_{n=1}^{\infty} \mu(E_n)\langle x^*, x_n\rangle.$$

Consequently, f is $\mathcal{WM}$-integrable on S if and only if the series $\sum\limits_{n\geqslant 1}\mu(E_n)x_n$ is weakly convergent and scalarly absolutely convergent. In this case we have

$$(\mathcal{WM})\int_S f\,\mathrm{d}\mu = w\text{-}\sum_{n=1}^{\infty}\mu(E_n)x_n.$$

P r o o f. For each $k\geqslant 1$ define

$$f_k(t) := \sum_{i=1}^{k} x_i 1_{E_i}(t).$$

As

$$\begin{aligned}(4.6.1)\qquad \int_S |\langle x^*, f\rangle| \,\mathrm{d}\mu &= \int_S \sum_{i=1}^{\infty} 1_{E_i}|\langle x^*, x_i\rangle| \,\mathrm{d}\mu \\ &= \sum_{i=1}^{\infty}\int_S 1_{E_i}|\langle x^*, x_i\rangle| \,\mathrm{d}\mu = \sum_{i=1}^{\infty}\mu(E_i)|\langle x^*, x_i\rangle|\end{aligned}$$

for all $x^* \in X^*$, and the series $\sum_{n\geqslant 1} \mu(E_n)x_n$ is scalarly absolutely convergent, f is scalarly integrable and hence

$$(4.6.2)\quad \int_S \langle x^*, f\rangle \,\mathrm{d}\mu = \int_S \lim_{k\to\infty} \langle x^*, f_k\rangle \,\mathrm{d}\mu = \lim_{k\to\infty} \int_S \langle x^*, f_k\rangle \,\mathrm{d}\mu = \sum_{i=1}^{\infty} \mu(E_i)\langle x^*, x_i\rangle,$$

by making use of the dominated convergence theorem, since

$$|\langle x^*, f_k\rangle| \leqslant |\langle x^*, f\rangle| \quad \text{for all } x^* \in X^* \text{ and for all } k \geqslant 1.$$

Next, by Proposition 1C (a) and Lemma 1I in [7], it is clear that each f_k is $\mathcal{M}$-integrable on S and

$$(\mathcal{M})\int_S f_k \,\mathrm{d}\mu = \sum_{i=1}^{k} \mu(E_i)x_i.$$

Therefore, we can invoke Lemma 2B of [7], which provides a gauge $\Lambda_k\colon\ S \to \mathcal{T}$ such that

$$(4.6.3)\quad \sup_{\{(F_j,t_j)\}_{1\leqslant j\leqslant n}\in P\Pi_f(\Lambda_k)} \left\| \sum_{j=1}^{n} \mu(F_j)f_k(t_j) - \sum_{i=1}^{k} \mu(E_i)x_i \right\| \leqslant \frac{1}{2^k}.$$

Next, for each $m \geqslant 1$ set

$$A_m := \bigcup_{i=1}^{m} E_i \cup \left(S \setminus \bigcup_{i=1}^{\infty} E_i \right)$$

and, for each $p \geqslant m$

$$B_p^m := A_m \quad \text{for } p = m \text{ and } B_p^m := E_p \text{ for } p > m.$$

Noting that for each $m \geqslant 1$ the B_p^m's are pairwise disjoint measurable sets and $\bigcup_{p=m}^{\infty} B_p^m = S$, we define a gauge $\Delta_m\colon\ S \to \mathcal{T}$ by

$$\Delta_m(t) := \bigcap_{i=1}^{k} \Lambda_i(t), \quad \text{for } t \in B_k^m \quad k \geqslant m.$$

Fix m for the moment and let $\mathcal{P}_\infty^m := \{(F_j^m, t_j^m)\}_{j\geqslant 1}$ be a generalized McShane partition of S subordinate to Δ_m. Fix x^* in X^*. We seek to estimate $|\langle x^*, \sigma_n(f_k, \mathcal{P}_\infty^m) -$

$\sum\limits_{i=1}^{k}\mu(E_i)x_i\rangle|$, where $k>m$ is an arbitrary fixed integer. To this end, define sets I_p $(p\geqslant m)$, J_k^1 and J_k^2 by

$$I_p := \{j\geqslant 1\colon\ t_j^m\in B_p^m\},\quad J_k^1 := \bigcup_{m\leqslant p<k} I_p \quad\text{and}\quad J_k^2 := \bigcup_{p\geqslant k} I_p$$

and collections

$$\mathcal{P}_\infty^{m,p} := \{(F_j^m,t_j^m)\}_{j\in I_p},\ \mathcal{Q}_\infty^m := \{(F_j^m,t_j^m)\}_{j\in J_k^1}\quad\text{and}\quad \mathcal{R}_\infty^m := \{(F_j^m,t_j^m)\}_{j\in J_k^2}.$$

It is clear that $J_k^1\cup J_k^2=\bigcup\limits_{p\geqslant m} I_p=\mathbb{N}^*$, since $\bigcup\limits_{p=m}^{\infty} B_p^m=S$. Further, observe that $\mathcal{P}_\infty^{m,p}$ is subordinate to Λ_p for each $m\leqslant p<k$, and that $\mathcal{R}_\infty^m$ is subordinate to Λ_k. Then by (4.6.3)

$$\left\|\sum_{j\in I_p,j\leqslant n}\mu(F_j^m)f_p(t_j^m)-\sum_{i=1}^{p}\mu\Big(E_i\cap\Big(\bigcup_{j\in I_p,j\leqslant n}F_j^m\Big)\Big)x_i\right\|\leqslant\frac{1}{2^p}\tag{4.6.4}$$

for each $m\leqslant p<k$ and

$$\left\|\sum_{j\in J_k^2,j\leqslant n}\mu(F_j^m)f_k(t_j^m)-\sum_{i=1}^{k}\mu\Big(E_i\cap\Big(\bigcup_{j\in J_k^2,j\leqslant n}F_j^m\Big)\Big)x_i\right\|\leqslant\frac{1}{2^k}.\tag{4.6.5}$$

Next, by the definition of J_k^1 and the triangle inequality we have

$$\begin{aligned}
&\left|\Big\langle x^*,\sum_{j\in J_k^1,j\leqslant n}\mu(F_j^m)f_k(t_j^m)\Big\rangle-\Big\langle x^*,\sum_{i=1}^{k}\mu\Big(E_i\cap\Big(\bigcup_{j\in J_k^1,j\leqslant n}F_j^m\Big)\Big)x_i\Big\rangle\right|\\
&\leqslant\sum_{p=m}^{k-1}\left|\Big\langle x^*,\sum_{j\in I_p,j\leqslant n}\mu(F_j^m)f_k(t_j^m)\Big\rangle-\Big\langle x^*,\sum_{j\in I_p,j\leqslant n}\mu(F_j^m)f_p(t_j^m)\Big\rangle\right|\\
&\quad+\sum_{p=m}^{k-1}\left|\Big\langle x^*,\sum_{j\in I_p,j\leqslant n}\mu(F_j^m)f_p(t_j^m)\Big\rangle-\Big\langle x^*,\sum_{i=1}^{p}\mu\Big(E_i\cap\Big(\bigcup_{j\in I_p,j\leqslant n}F_j^m\Big)\Big)x_i\Big\rangle\right|\\
&\quad+\sum_{p=m}^{k-1}\left|\Big\langle x^*,\sum_{i=1}^{p}\mu\Big(E_i\cap\Big(\bigcup_{j\in I_p,j\leqslant n}F_j^m\Big)\Big)x_i\Big\rangle\right.\\
&\quad\left.-\Big\langle x^*,\sum_{i=1}^{k}\mu\Big(E_i\cap\Big(\bigcup_{j\in I_p,j\leqslant n}F_j^m\Big)\Big)x_i\Big\rangle\right|.
\end{aligned}$$

The first term on the right hand side, is equal to 0, since $f_k = f_p$ on B_p^m for each $p \geqslant m$, whereas the second term is, by (4.6.4), smaller than $\sum\limits_{p=m}^{k-1} 1/2^p$ so that

$$\begin{aligned}
&\left|\left\langle x^*, \sum_{j\in J_k^1, j\leqslant n} \mu(F_j^m) f_k(t_j^m)\right\rangle - \left\langle x^* \sum_{i=1}^{k} \mu\Big(E_i \cap \Big(\bigcup_{j\in J_k^1, j\leqslant n} F_j^m\Big)\Big) x_i\right\rangle\right| \\
&\leqslant \sum_{p=m}^{k-1} \frac{1}{2^p} + \sum_{p=m}^{k-1} \sum_{i=p+1}^{k} |\langle x^*, x_i\rangle| \mu\Big(E_i \cap \Big(\bigcup_{j\in I_p, j\leqslant n} F_j^m\Big)\Big) \\
&= \sum_{p=m}^{k-1} \frac{1}{2^p} + \sum_{i=m+1}^{k} \sum_{p=m}^{i-1} |\langle x^*, x_i\rangle| \mu\Big(E_i \cap \Big(\bigcup_{j\in I_p, j\leqslant n} F_j^m\Big)\Big) \\
&= \sum_{p=m}^{k-1} \frac{1}{2^p} + \sum_{i=m+1}^{k} |\langle x^*, x_i\rangle| \mu\Big(E_i \cap \Big(\bigcup_{p=m}^{i-1} \bigcup_{j\in I_p, j\leqslant n} F_j^m\Big)\Big) \\
&\leqslant \sum_{p=m}^{\infty} \frac{1}{2^p} + \sum_{i=m+1}^{\infty} |\langle x^*, x_i\rangle| \mu(E_i).
\end{aligned}$$

By virtue of (4.6.5) and the decomposition

$$\begin{aligned}
&\left|\langle x^*, \sigma_n(f_k, \mathcal{P}_\infty^m)\rangle - \left\langle x^*, \sum_{i=1}^{k} \mu\Big(E_i \cap \Big(\bigcup_{j=1}^{n} F_j^m\Big)\Big) x_i\right\rangle\right| \\
&= \left\langle x^*, \sum_{j\in J_k^1, j\leqslant n} \mu(F_j^m) f_k(t_j^m)\right\rangle - \left\langle x^*, \sum_{i=1}^{k} \mu\Big(E_i \cap \Big(\bigcup_{j\in J_k^1, j\leqslant n} F_j^m\Big)\Big) x_i\right\rangle \\
&\quad + \left\langle x^*, \sum_{j\in J_k^2, j\leqslant n} \mu(F_j^m) f_k(t_j^m)\right\rangle - \left\langle x^*, \sum_{i=1}^{k} \mu\Big(E_i \cap \Big(\bigcup_{j\in J_k^2, j\leqslant n} F_j^m\Big)\Big) x_i\right\rangle
\end{aligned}$$

we get

$$\begin{aligned}
(4.6.6)\qquad &\left|\langle x^*, \sigma_n(f_k, \mathcal{P}_\infty^m)\rangle - \left\langle x^*, \sum_{i=1}^{k} \mu\Big(E_i \cap \Big(\bigcup_{j=1}^{n} F_j^m\Big)\Big) x_i\right\rangle\right| \\
&\leqslant \sum_{p=m}^{p=\infty} \frac{1}{2^p} + \sum_{i=m+1}^{\infty} \mu(E_i) |\langle x^*, x_i\rangle| + \frac{1}{2^k},
\end{aligned}$$

which is valid for all $k \geqslant 1$. Noting that (f_k) pointwise converges to f, we conclude that

$$
\begin{aligned}
&\left|\langle x^*, \sigma_n(f, \mathcal{P}_\infty^m)\rangle - \int_S \langle x^*, f\rangle \,\mathrm{d}\mu\right| \\
&\quad = \left|\langle x^*, \sigma_n(f, \mathcal{P}_\infty^m)\rangle - \sum_{i=1}^{\infty} \langle x^*, \mu(E_i)x_i\rangle\right| \quad \text{(by (4.6.2))} \\
&\quad = \lim_{k\to\infty} \left|\langle x^*, \sigma_n(f_k, \mathcal{P}_\infty^m)\rangle - \left\langle x^*, \sum_{i=1}^{k} \mu(E_i)x_i\right\rangle\right| \\
&\quad = \lim_{k\to\infty} \left[\left|\langle x^*, \sigma_n(f_k, \mathcal{P}_\infty^m)\rangle - \left\langle x^*, \sum_{i=1}^{k} \mu\left(E_i \cap \left(\bigcup_{j=1}^{n} F_j^m\right)\right)x_i\right\rangle\right.\right. \\
&\qquad \left.\left. + \left\langle x^*, \sum_{i=1}^{k} \mu\left(E_i \cap \left(\bigcup_{j=1}^{n} F_j^m\right)\right)x_i\right\rangle - \left\langle x^*, \sum_{i=1}^{k} \mu(E_i)x_i\right\rangle\right|\right] \\
&\quad \leqslant \limsup_{k\to\infty} \left|\langle x^*, \sigma_n(f_k, \mathcal{P}_\infty^m)\rangle - \left\langle x^*, \sum_{i=1}^{k} \mu\left(E_i \cap \left(\bigcup_{j=1}^{n} F_j^m\right)\right)x_i\right\rangle\right| \\
&\qquad + \limsup_{k\to\infty} \left|\left\langle x^*, \sum_{i=1}^{k} \mu\left(E_i \cap \left(\bigcup_{j=n+1}^{\infty} F_j^m\right)\right)x_i\right\rangle\right| \\
&\quad \leqslant \sum_{p=m}^{\infty} \frac{1}{2^p} + \sum_{i=m+1}^{\infty} \mu(E_i)|\langle x^*, x_i\rangle| \quad \text{(by (4.6.6))} \\
&\qquad + \sum_{i=1}^{\infty} \mu\left(E_i \cap \left(\bigcup_{j=n+1}^{\infty} F_j^m\right)\right)|\langle x^*, x_i\rangle|.
\end{aligned}
$$

This yields, by letting $n \to \infty$ and $m \to \infty$

$$
\begin{aligned}
&\limsup_{m\to\infty} \limsup_{n\to\infty} \left|\langle x^*, \sigma_n(f, \mathcal{P}_\infty^m)\rangle - \int_S \langle x^*, f\rangle \,\mathrm{d}\mu\right| \\
&\quad \leqslant \lim_{m\to\infty} \sum_{p=m}^{\infty} \frac{1}{2^p} + \lim_{m\to\infty} \sum_{i=m+1}^{\infty} \mu(E_i)|\langle x^*, x_i\rangle| \\
&\qquad + \limsup_{m\to\infty} \lim_{n\to\infty} \sum_{i=1}^{\infty} \mu\left(E_i \cap \bigcup_{j=n+1}^{\infty} F_j^m\right)|\langle x^*, x_i\rangle| \\
&\quad = \limsup_{m\to\infty} \lim_{n\to\infty} \sum_{i=1}^{\infty} \mu\left(E_i \cap \bigcup_{j=n+1}^{\infty} F_j^m\right)|\langle x^*, x_i\rangle|,
\end{aligned}
$$

because, by hypothesis, the series $\sum_{i\geqslant 1} \mu(E_i)|\langle x^*, x_i\rangle|$ is convergent. Since the series $\sum_{i\geqslant 1} \mu\left(E_i \cap \bigcup_{j=n+1}^{\infty} F_j^m\right)|\langle x^*, x_i\rangle$ is dominated term-by-term by the convergent series

$\sum_{i\geqslant 1} |\langle x^*, x_i\rangle|\mu(E_i)$, and $\lim_{n\to\infty} \mu\Bigl(E_i \cap \bigcup_{j=n+1}^{\infty} F_j^m\Bigr) = 0$, the dominated convergence theorem for series gives

$$\begin{aligned}\limsup_{m\to\infty} \lim_{n\to\infty} &\sum_{i=1}^{\infty} \mu\Bigl(E_i \cap \bigcup_{j=n+1}^{\infty} F_j^m\Bigr)|\langle x^*, x_i\rangle| \\ &= \limsup_{m\to\infty} \sum_{i=1}^{\infty} \lim_{n\to\infty} \mu\Bigl(E_i \cap \bigcup_{j=n+1}^{\infty} F_j^m\Bigr)|\langle x^*, x_i\rangle| = 0.\end{aligned}$$

Thus

$$\lim_{m\to\infty} \limsup_{n\to\infty} \Bigl|\langle x^*, \sigma_n(f, \mathcal{P}_\infty^m)\rangle - \int_S \langle x^*, f\rangle \,\mathrm{d}\mu\Bigr| = 0.$$

Now, if the series $\sum_{n\geqslant 1} \mu(E_n)x_n$ is weakly convergent, then

$$\int_S \langle x^*, f\rangle \,\mathrm{d}\mu = \sum_{i=1}^{\infty} \mu(E_i)\langle x^*, x_i\rangle = \Bigl\langle x^*, w\text{-}\sum_{i=1}^{\infty} \mu(E_i)x_i\Bigr\rangle \quad \text{for all } x^* \in X^*$$

and therefore f is $\mathcal{WM}$-integrable on S. Conversely, if f is $\mathcal{WM}$-integrable on S, then, by Corollary 3.1, it is scalarly integrable and

$$\Bigl\langle x^*, (\mathcal{WM}) \int_S f \,\mathrm{d}\mu\Bigr\rangle = \int_S \langle x^*, f\rangle \,\mathrm{d}\mu \quad \text{for all } x^* \in X^*.$$

Returning to the equalities (4.6.1) and (4.6.2)), we deduce that

$$\sum_{i=1}^{\infty} \mu(E_i)|\langle x^*, x_i\rangle| < \infty \quad \text{and} \quad \Bigl\langle x^*, (\mathcal{WM}) \int_S f \,\mathrm{d}\mu\Bigr\rangle = \sum_{i=1}^{\infty} \mu(E_i)\langle x^*, x_i\rangle$$

for every $x^* \in X^*$. Hence, the series $\sum_{n\geqslant 1} \mu(E_n)x_n$ is both scalarly absolutely and weakly convergent. This completes the proof. □

In the next corollary, we provide a class of functions which are weakly McShane integrable on S but not Pettis integrable.

Corollary 4.3. *If X contains a copy of c_0, then there is a function $f\colon S \to X$ which is $\mathcal{WM}$-integrable on S but not Pettis integrable.*

P r o o f. From what has been said above, we may choose a series $\sum_{n\geqslant 1} x_n$ in X which is both scalarly absolutely and weakly convergent, but not unconditionally convergent in the norm topology. Let $(E_n)_{n\geqslant 1}$ be any pairwise disjoint sequence of measurable sets of strictly positive measure and define the function $f\colon\ S\to X$ by

$$f(t):=\sum_{n=1}^{\infty}x_n\frac{1}{\mu(E_n)}1_{E_n}(t).$$

Then according to Theorem 4.6, f is $\mathcal{WM}$-integrable on S, in view of the choice of $\sum\limits_{n\geqslant 1}x_n$. On the other hand, as $\sum\limits_{n\geqslant 1}x_n$ is not unconditionally convergent, the Orlicz-Pettis Theorem ([4], Corollary I.4.4) ensures the existence of a strictly increasing sequence (k_n) of positive integers such that the series $\sum\limits_{n\geqslant 1}x_{k_n}$ is not weakly convergent. Since f is scalarly integrable and

$$\int_{E_{k_n}}\langle x^*,f\rangle\,\mathrm{d}\mu=\langle x^*,x_{k_n}\rangle,$$

we have

$$\int_{\bigcup\limits_{n=1}^{\infty}E_{k_n}}\langle x^*,f\rangle\,\mathrm{d}\mu=\sum_{n=1}^{\infty}\int_{E_{k_n}}\langle x^*,f\rangle\,\mathrm{d}\mu=\sum_{n=1}^{\infty}\langle x^*,x_{k_n}\rangle,$$

which shows that f cannot be Pettis integrable, since otherwise the series $\sum\limits_{n\geqslant 1}x_{k_n}$ would be weakly convergent. $\square$

To close this section we would like to mention some open problems in connection with the results of [3], [7], [5] and [14] dealing with McShane integrability.

Problem 1. Under the Continuum Hypothesis, Piazza-Preiss [5] and Rodríguez [14] provided examples of *scalarly null* functions defined on $[0,1]$ (endowed with the Lebesgue measure) which are not McShane integrable. In this connection, we ask whether their function is $\mathcal{WM}$-integrable on $[0,1]$ or not. The same question arises for the function constructed by Fremlin and Mendoza ([10], Example 3C). Unfortunately, we have been unable to find a function which is Pettis integrable but not $\mathcal{WM}$-integrable on a measurable subset of S. This leads us to put the following question: Is every Pettis integrable Banach space valued function $\mathcal{WM}$-integrable on Σ? If the answer to this question is negative, is there a class of Banach spaces including strictly the class of Hilbert generated Banach spaces for which these two integrals are equivalent?

Recall that a function $f\colon\ S\to X$ is said to be *scalarly null* if for each $x^*\in X^*$, the real-valued function $\langle x^*,f\rangle$ vanishes almost everywhere (the exceptional set depends on x^*).

Problem 2. If an X-valued function is $\mathcal{WM}$-integrable on Σ, does it have to be $\mathcal{M}$-integrable? If the answer is no, when do $\mathcal{WM}$-integrability on Σ and $\mathcal{M}$-integrability coincide? Let us mention at least that this is the case for example if X is a subspace of a Hilbert generated Banach space. What about the case of X being weakly compactly generated (WCG)? Recall that the Banach space X is (WCG) if there exists a weakly compact subset of X whose linear span is dense in X.

Problem 3. Suppose that $f\colon S \to X$ is $\mathcal{WM}$-integrable on Σ. Does it follow that the *indefinite Pettis integral* of f (that is, the set function $E \to (\mathcal{P}e)\int_E f\,\mathrm{d}\mu$ $(E \in \Sigma)$) has a totally bounded range?

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