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NONSYMMETRIC SOLUTIONS OF A NONLINEAR BOUNDARY VALUE PROBLEM

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Abstract. We study the existence and multiplicity of positive nonsymmetric and sign-changing nonantisymmetric solutions of a nonlinear second order ordinary differential equation with symmetric nonlinear boundary conditions, where both of the nonlinearities are of power type. The given problem has already been studied by other authors, but the number of its positive nonsymmetric and sign-changing nonantisymmetric solutions has been determined only under some technical conditions. It was a long-standing open question whether or not these conditions can be omitted. In this article we provide the answer. Our main tool is the shooting method.

Keywords: nonlinear second order ordinary differential equation; existence of solution; multiplicity of solution; nonlinear boundary condition; shooting method; time map

MSC 2010: 34B18, 34B15, 34B08

1. Introduction

In this paper we deal with positive nonsymmetric (i.e. noneven) solutions of the problem

$$\left\{\begin{aligned} u''(x) &= a|u(x)|^{p-1}u(x), \quad x \in (-l, l),\\ u'(\pm l) &= \pm|u(\pm l)|^{q-1}u(\pm l) \end{aligned}\right. \tag{1.1}$$

for $p \geqslant 1$, $q > (p+1)/2$, $a, l > 0$ and with its sign-changing nonantisymmetric (i.e. nonodd) solutions for $p \geqslant 1$, $0 < q < (p+1)/2$, $a, l > 0$. (The choice of these conditions will be explained a few paragraphs later.)

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The first study of positive solutions of (1.1) was done by M. Chipot, M. Fila and P. Quittner in [5]. They also studied the N-dimensional version of (1.1), but they were interested mainly in global existence and boundedness or blow-up of positive solutions of the corresponding N-dimensional parabolic problem

$$(1.2) \qquad \begin{cases} u_t = \Delta u - a|u|^{p-1}u & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial n} = |u|^{q-1}u & \text{in } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in } \overline{\Omega}, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, n is the unit outer normal vector to $\partial\Omega$, $u_0\colon \overline{\Omega} \to [0,\infty)$, considering $p, q > 1$ and $a > 0$. The same problem was independently studied in [12] for $N = 1$.

The results from [5] have been generalized in many directions: In [15] the behaviour of positive solutions of (1.2) was examined for all $p, q > 1$, while sign-changing solutions were considered in [6] for $p \geqslant 1$, $q > 1$. Positive solutions of the elliptic problem with $-\lambda u + u^p$ on the right-hand side of the equation were dealt with in [13] for $\lambda \in \mathbb{R}$, $p, q > 1$, and later in [10] for $\lambda \in \mathbb{R}$, $p, q > 0$, $(p,q) \notin (0,1)^2$. In [11] and [16], positive and sign-changing solutions of the parabolic problem with more general nonlinearities $f(u)$, $g(u)$ instead of $a|u|^{p-1}u$, $|u|^{q-1}u$ have been studied, while $f(x,u)$, $g(x,u)$ were considered in [2]. Many results concerning elliptic problems with nonlinear boundary conditions were summarised in [17]. In the recent paper [14], (1.1) was studied for $p, q \in \mathbb{R}$, and its solvability was examined for $p \geqslant -1$, $0 \leqslant q \leqslant (p+1)/2$ and $p > -1$, $q > (p+1)/2$, but only symmetric solutions were dealt with in the latter case. Further extensions of the results from [5] can be found in [1], [3], [4], [7], [8], [9].

It was shown in [5], Theorems 3.1 and 3.2, that assuming $p, q > 1$, (1.1) possesses positive nonsymmetric solutions only for $q > (p+1)/2$. (In general for $q > \max\{0, (p+1)/2\}$, if we consider $p, q \in \mathbb{R}$ (see [14], Theorem 2.6 (i)).) The existence of at most one pair of nonsymmetric solutions was proved under the condition

$$(1.3) \qquad p \leqslant 4 \quad \text{or} \quad p > 4,\ q \geqslant p - 1 - \frac{1}{p-2},$$

(see [5], Theorem 3.4). The first result of this paper—stated in Theorem 2.10—is that condition (1.3) is superfluous.

On the other hand, sign-changing nonantisymmetric solutions of (1.1) for $p \geqslant 1$, $q > 1$ exist only in the case of $q < (p+1)/2$ (see [6], Theorem 1.3 (i)). According to [6], Theorem 1.3 (iii), if

$$(1.4) \qquad (p-q)(2q+1-p)(p+1) \geqslant 2q(p-1),$$

then either four or no sign-changing nonantisymmetric solutions exist. As our second result, we prove this property in Theorem 3.10 without assuming (1.4), including also some $q \leqslant 1$.

2. Positive nonsymmetric solutions

We start this section with recalling the shooting method as it was used in [5].

Let $p, q \in \mathbb{R}$, $a, l > 0$. If u is a positive solution of (1.1), then $u'(-l) < 0 < u'(l)$, therefore u has a stationary point $x_0 \in (-l, l)$. So the function $u(\cdot + x_0)$ solves

$$\begin{cases} u'' = au^p, \\ u(0) = m, \\ u'(0) = 0 \end{cases} \tag{2.1}$$

for some $m > 0$. Since $u \mapsto au^p$ is locally Lipschitz continuous on $(0, \infty)$, (2.1) has a unique maximal solution, which is apparently even and strictly convex. We will denote it by $u_{m,p,a}$ and its domain by $(-\Lambda_{m,p,a}, \Lambda_{m,p,a})$.

Let us also introduce the notation $\mathcal{N}^+(l) = \mathcal{N}^+(l; p, q, a)$ for the set of all positive nonsymmetric (i.e. noneven) solutions of (1.1). Obviously, $\mathcal{N}^+(l)$ consists of all such functions $u_{m,p,a}(\cdot - (l_1 - l_2)/2)|_{[-l,l]}$ that $l_1 + l_2 = 2l$, $l_1 \neq l_2$ and $0 < l_i < \Lambda_{m,p,a}$, $u'_{m,p,a}(l_i) = u^q_{m,p,a}(l_i)$ for $i = 1, 2$.

Lemma 2.1 ([5], pages 53–55, for $p, q > 1$, or [14], Lemma 2.4, for $p, q \in \mathbb{R}$). *Let $p \neq -1$, $q \in \mathbb{R}$, $a > 0$. Then the following statements are equivalent for arbitrary $m, l > 0$:*

(i) $l < \Lambda_{m,p,a}$ *and* $u'_{m,p,a}(l) = u^q_{m,p,a}(l)$,

(ii) *the equation*

$$0 = \mathcal{F}(m, x) := \mathcal{F}_{p,q,a}(m, x) := \frac{x^{2q}}{2a} - \frac{x^{p+1}}{p+1} + \frac{m^{p+1}}{p+1}$$

with the unknown $x > 0$ has a solution $R > m$, and

$$l = \frac{m^{(1-p)/2}}{\sqrt{2a}} I_p\Big(\frac{R}{m}\Big),$$

where

$$I_p(y) := \int_1^y \sqrt{\frac{p+1}{V^{p+1} - 1}}\, \mathrm{d}V, \quad y \geqslant 1.$$

Let us remark that an assertion analogous to Lemma 2.1 holds for $p=-1$, in which $\mathcal{F}_{p,q,a}$ and I_p are replaced by their limits for $p \to -1$.

One can see that $\mathcal{F}(m,\cdot)$ has different behaviour for $p>-1$, $p=-1$ and $p<-1$ as well as for $q>0$, $q=0$ and $q<0$. It also matters which of the exponents $2q$, $p+1$ is greater. However, from now on we will consider only the case of $p>-1$, $q>(p+1)/2$.

Lemma 2.2 ([5], pages 57–58, for $p>1$, or [14], Lemma 2.5 (iv), for $p>-1$). *Let $p>-1$, $q>(p+1)/2$, $a,m>0$ and let us introduce*

$$M := M_{p,q,a} := \Big(\frac{2q-p-1}{2q}\Big)^{1/(p+1)} \Big(\frac{a}{q}\Big)^{1/(2q-p-1)}.$$

If $m>M$, then $\mathcal{F}(m,\cdot)$ has no zero. If $m=M$, then the only zero of $\mathcal{F}(m,\cdot)$ is

$$\Big(\frac{a}{q}\Big)^{1/(2q-p-1)} =: R_{p,q,a}(M) =: R(M) > M.$$

If $m<M$, then $\mathcal{F}(m,\cdot)$ has two zeros, which will be denoted by $R_{i;p,q,a}(m) =: R_i(m)$, $i=1,2$, satisfying

$$m < R_1(m) < R(M) < R_2(m). \tag{2.2}$$

Definition 2.3. Let $p>-1$, $q>(p+1)/2$, $a>0$ and put

$$L_i(m) := L_{i;p,q,a}(m) := \frac{m^{(1-p)/2}}{\sqrt{2a}} I_p\Big(\frac{R_{i;p,q,a}(m)}{m}\Big)$$

for $i=1,2$ and $m\in(0,M)$. We introduce $L_{p,q,a}(M) =: L(M)$ analogously. Functions L, L_1 and L_2 will be called *time maps* (associated with (2.1)).

Using Lemmata 2.1 and 2.2, we can describe $\mathcal{N}^+(l)$ by means of the time maps:

Lemma 2.4. *For all $p>-1$, $q>(p+1)/2$ and $a,l>0$:*

$$\mathcal{N}^+(l) = \Big\{u_{m,p,a}\Big(\cdot \pm \frac{L_2(m)-L_1(m)}{2}\Big)\Big|_{[-l,l]} :\ L_1(m)+L_2(m)=2l\Big\}.$$

Thus, to determine the number of positive nonsymmetric solutions of (1.1) for given p, q, a, l, we need to calculate the limits of L_1+L_2 at 0 and M, to examine its monotonicity and to estimate its possible relative extrema. Therefore, the following two lemmata will be needed.

Lemma 2.5 ([5], Lemmata 3.1 and 3.3, for $p, q > 1$, or [14], Lemmata 8.3 and 8.4, for $p > -1$). *Let $p > -1$, $q > (p+1)/2$ and $a > 0$. Then*

$$\lim_{m\to M} L_i(m) = L(M), \quad i = 1, 2$$

and if, in addition, $p \geqslant 1$, then

$$\lim_{m\to 0} L_2(m) = \infty.$$

Lemma 2.6 ([5], Proof of Theorem 3.1, for $p > 1$, or [14], Lemma 2.9, for $p > -1$). *If $p > -1$, $q > (p+1)/2$, $a > 0$, $i \in \{1, 2\}$, then L_i is differentiable on $(0, M)$, fulfilling*

$$L_i'(m) = \frac{1-p}{2m} L_i(m) + \frac{2q-p-1}{2am} \frac{R_i^{q-p}(m)}{1 - qa^{-1} R_i^{2q-p-1}(m)}.$$

The next lemma will be used in the proof of Lemma 2.8.

Lemma 2.7. *If $p > -1$, $q > (p+1)/2$ and $a > 0$, then $R_1 R_2 < R^2(M)$.*

Proof. Choose $p > -1$, $q > (p+1)/2$, $a > 0$, $m \in (0, M)$, and set $\alpha := R_2(m)/R(M)$. Evidently, $\alpha > 1$ (see (2.2)). Our aim is to prove that

$$R_1(m) < \frac{R(M)}{\alpha}. \tag{2.3}$$

Since $\mathcal{F}(m, \cdot)$ is decreasing on $(0, R(M)]$, (2.3) is equivalent to

$$\mathcal{F}(m, R_1(m)) > \mathcal{F}\Big(m, \frac{R(M)}{\alpha}\Big),$$

which can be rewritten in the form

$$\mathcal{F}(m, \alpha R(M)) - \mathcal{F}\Big(m, \frac{R(M)}{\alpha}\Big) > 0,$$

using the definition of $R_1(m)$ and $R_2(m)$. One can derive that

$$\mathcal{F}(m, \alpha R(M)) - \mathcal{F}\Big(m, \frac{R(M)}{\alpha}\Big) = \underbrace{R^{p+1}(M)}_{>0}(F_\alpha(2q) - F_\alpha(p+1)),$$

where

$$F_\alpha(x) := \frac{\alpha^x - \alpha^{-x}}{x}, \quad x > 0, \tag{2.4}$$

therefore, the verification of the increase of F_α on $(0,\infty)$ will make the proof complete. Defining

$$G(z) := (z^2+1)\ln z - z^2 + 1, \quad z > 1,$$

we have that

$$F'_\alpha(x) = \frac{G(\alpha^x)}{x^2\alpha^x}.$$

Thus, it suffices to prove that $G(z) > 0$ for $z > 1$. And this holds indeed because $G(1) = 0$, $G'(1) = 0$ and

$$G''(z) = 2\ln z + \frac{z^2-1}{z^2} > 0, \quad z > 1.$$

□

Lemma 2.8. *If $p \geqslant 1$, $q > (p+1)/2$ and $a > 0$, then $(L_1 + L_2)' < 0$.*

Proof. Let $p \geqslant 1$, $q > (p+1)/2$, $a > 0$ and $m \in (0, M)$.

1. For arbitrary $y > 1$ we have

$$I_p(y) > \int_1^y \sqrt{\frac{p+1}{V^{p+1}-1}}\Big(\frac{V}{y}\Big)^p \,\mathrm{d}V = \frac{2}{y^p}\sqrt{\frac{y^{p+1}-1}{p+1}}.$$

Consequently,

$$L_i(m) > \frac{\sqrt{2}}{\sqrt{a}R_i^p(m)}\sqrt{\frac{R_i^{p+1}(m) - m^{p+1}}{p+1}} = \frac{R_i^{q-p}(m)}{a}, \quad i = 1, 2.$$

(Recall that $\mathcal{F}(m, R_i(m)) = 0$.) Using Lemma 2.6 and the last inequality, we obtain that

$$(L_1+L_2)'(m) \leqslant \frac{R^{q-p}(M)}{2am}\Big(F\Big(\frac{R_1(m)}{R(M)}\Big) + F\Big(\frac{R_2(m)}{R(M)}\Big)\Big),$$

where

$$F(x) := F_{p,q}(x) := (1-p)x^{q-p} + \frac{(2q-p-1)x^{q-p}}{1-x^{2q-p-1}}, \quad x \in (0,1)\cup(1,\infty).$$

Thus,

$$F\Big(\frac{R_1(m)}{R(M)}\Big) + F\Big(\frac{R_2(m)}{R(M)}\Big) < 0 \tag{2.5}$$

is a sufficient condition for $(L_1+L_2)'(m) < 0$.

2. Let us prove that F is increasing on $(0,1)$ for all $p \geqslant 1$, $q > (p+1)/2$.

For this purpose, it is useful to introduce parameters

$$\alpha := p - 1, \quad \beta := 2(p - q).$$

Thus, we consider $\alpha \geqslant 0$, $\beta < \alpha$. One can derive that

$$F(x) = -\alpha x^{-\beta/2} + \frac{(\alpha-\beta)x^{-\beta/2}}{1 - x^{\alpha-\beta}},$$
$$F'(x) = \underbrace{\frac{x^{-\beta/2-1}}{2(1-x^{\alpha-\beta})^2}}_{>0} g(x^{\alpha-\beta}),$$

where

$$g(z) := g_{\alpha,\beta}(z) := \alpha\beta z^2 + (2\alpha^2 - 5\alpha\beta + \beta^2)z + \beta^2.$$

So it suffices to prove that $g > 0$ on $(0,1)$.

If $\beta \leqslant 0$, then the statement follows from the facts that $g(0) = \beta^2 \geqslant 0$, $g(1) = 2(\alpha-\beta)^2 > 0$, and g is concave. Therefore, assume $\beta > 0$. In that case, g is strictly convex, attaining its minimum at

$$\frac{-2\alpha^2 + 5\alpha\beta - \beta^2}{2\alpha\beta} =: z_{0;\alpha,\beta} =: z_0.$$

If $z_0 \leqslant 0$, then $g(z) > g(0) > 0$ for $z \in (0,1)$. If $z_0 > 0$, then

$$g(z_0) = \frac{(\alpha-\beta)^2(-4\alpha^2 + 12\alpha\beta - \beta^2)}{4\alpha\beta} = (\alpha-\beta)^2\Big(z_0 + \frac{1}{2} + \frac{\beta}{4\alpha}\Big) > 0,$$

yielding again that $g > 0$ on $(0,1)$.

So F is indeed increasing on $(0,1)$.

3. Lemmata 2.7 and 2.2 imply that

$$0 < \frac{R_1(m)}{R(M)} < \frac{R(M)}{R_2(m)} < 1.$$

Thus, due to 2.,

$$F\Big(\frac{R(M)}{R_2(m)}\Big) + F\Big(\frac{R_2(m)}{R(M)}\Big) \leqslant 0$$

is a sufficient condition for (2.5). And since the range of $R_2/R(M)$ is a subset of $(1,\infty)$ (actually, it equals $(1, R_2(0)/R(M))$, see [14], Lemma 8.1), the verification of

$$\forall\, p \geqslant 1,\ q > \frac{p+1}{2},\ x > 1: \quad F\Big(\frac{1}{x}\Big) + F(x) \leqslant 0 \tag{2.6}$$

will complete the proof.

Let us reformulate (2.6) by means of α and β, and let us multiply the inequality in it by $x^{\beta/2}(1 - x^{\alpha-\beta})$, to obtain the equivalent assertion

$$\forall\, \alpha \geqslant 0,\ \beta < \alpha,\ x > 1\colon\ u_{\alpha,\beta}(x) := \beta x^{\alpha} + \alpha x^{\alpha-\beta} - \alpha x^{\beta} - \beta \geqslant 0.$$

Trivially, $u_{0,\beta} \equiv 0$, so we will consider only $\alpha > 0$. Since $u_{\alpha,\beta}(1) = 0$, it suffices to prove that $u_{\alpha,\beta}$ is nondecreasing on $[1, \infty)$. However,

$$u'_{\alpha,\beta}(x) = \underbrace{\alpha x^{\beta-1}}_{>0}\underbrace{(\beta x^{\alpha-\beta} + (\alpha-\beta)x^{\alpha-2\beta} - \beta)}_{=:v_{\alpha,\beta}(x)}$$

with $v_{\alpha,\beta}(1) = \alpha - \beta > 0$, so it suffices to verity the nondecrease of $v_{\alpha,\beta}$ on $[1, \infty)$. And that is guaranteed by the equality

$$v'_{\alpha,\beta}(x) = \underbrace{(\alpha-\beta)x^{\alpha-2\beta-1}}_{>0}\underbrace{(\beta x^{\beta} + \alpha - 2\beta)}_{=:w_{\alpha,\beta}(x)},$$

$w_{\alpha,\beta}(1) = \alpha - \beta > 0$ and the nondecrease of $w_{\alpha,\beta}$. □

Remark 2.9. The proof of Lemma 2.8 was motivated by [6], Remark 5.3, where a sufficient condition of $(\overline{L}_1 + \overline{L}_2)' < 0$ ($\overline{L}_1$ and $\overline{L}_2$ being the time maps associated with (3.1), see Definition 3.3), looking similar to (2.5), had been derived. That condition is based on a different integral estimate, and will be verified in the proof of Lemma 3.8.

The properties of $L_1 + L_2$ stated in Lemmata 2.5 and 2.8, together with Lemma 2.4, lead to the main result of this section:

Theorem 2.10. *Let $p \geqslant 1$, $q > (p+1)/2$ and $a, l > 0$. There exists such a number $L(M) > 0$ (see Lemma 2.2 and Definition 2.3) that (1.1) has two positive nonsymmetric solutions for $l > L(M)$ and none for $l \leqslant L(M)$. (See Figure 1.)*

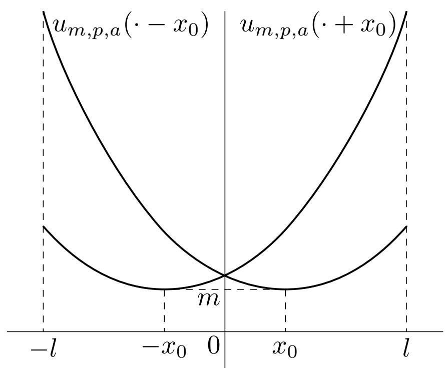

Figure 1. The two positive nonsymmetric solutions of (1.1) in the case dealt with in Theorem 2.10. (Here $x_0 = (L_2(m) - L_1(m))/2$, see Lemma 2.4).

3. Sign-changing nonantisymmetric solutions

This section will be again started with recalling the shooting method from [6]. Lemmata 3.1, 3.2, 3.5 and 3.6 will be stated under weaker assumptions on q than the corresponding assertions cited from [6], but we do not provide the proofs because they are unchanged.

Let $p \geqslant 1$, $q \in \mathbb{R}$, $a, l > 0$. If u is a sign-changing solution of (1.1) and x_0 is its zero, then $u(\cdot + x_0)$ solves

$$\begin{cases} u'' = a|u|^{p-1}u, \\ u(0) = 0, \\ u'(0) = \theta \end{cases} \tag{3.1}$$

for some $\theta \in \mathbb{R}$. Since $u \mapsto a|u|^{p-1}u$ is locally Lipschitz continuous on $\mathbb{R}$, (3.1) has a unique maximal solution, which is obviously odd. It will be denoted by $\overline{u}_{\theta,p,a}$ and its domain by $(-\overline{\Lambda}_{\theta,p,a}, \overline{\Lambda}_{\theta,p,a})$. Clearly, $\overline{u}_{0,p,a} \equiv 0$ on $\mathbb{R}$ and thus, $x_0 \in (-l, l)$ and $\theta \neq 0$. One can also see that u is strictly convex on the intervals where it has positive values, and strictly concave on the intervals where it has negative values. As a consequence, $\overline{u}'_{\theta,p,a} > 0$ if $\theta > 0$, and $\overline{u}'_{\theta,p,a} < 0$ if $\theta < 0$. In addition, $\overline{u}_{-\theta,p,a} = -\overline{u}_{\theta,p,a}$, therefore we will restrict our further considerations to $\theta > 0$.

Let us also introduce the notation $\mathcal{N}^{\pm}(l) = \mathcal{N}^{\pm}(l; p, q, a)$ for the set of all sign-changing nonantisymmetric (i.e. nonodd) solutions of (1.1). Obviously, $\mathcal{N}^{\pm}(l)$ consists of all such functions $\pm\overline{u}_{\theta,p,a}(\cdot - (l_1 - l_2)/2)|_{[-l,l]}$ that $\theta > 0$, $l_1 + l_2 = 2l$, $l_1 \neq l_2$ and $0 < l_i < \overline{\Lambda}_{\theta,p,a}$, $\overline{u}'_{\theta,p,a}(l_i) = \overline{u}^q_{\theta,p,a}(l_i)$ for $i = 1, 2$.

Lemma 3.1 (for $q > 1$ see [6], pages 114–116)**.** *Let $p \geqslant 1$, $q \in \mathbb{R}$, $a > 0$, and set $b := 2a/(p+1)$. Then the following statements are equivalent for arbitrary θ, $l > 0$:*

(i) *$l < \overline{\Lambda}_{\theta,p,a}$ and $\overline{u}'_{\theta,p,a}(l) = \overline{u}^q_{\theta,p,a}(l)$,*

(ii) *the equation*

$$0 = \overline{\mathcal{F}}(\theta, x) := \overline{\mathcal{F}}_{p,q,a}(\theta, x) := x^{2q} - bx^{p+1} - \theta^2$$

with the unknown $x > 0$ has some solution $\overline{R}$, and

$$l = \theta^{-(p-1)/(p+1)} I_{p,b}(\theta^{-2/(p+1)}\overline{R}),$$

where

$$I_{p,b}(y) := \int_0^y \frac{\mathrm{d}s}{\sqrt{bs^{p+1} + 1}}, \quad y \geqslant 0.$$

Clearly, $\overline{\mathcal{F}}(\theta,\cdot)$ has different behaviour for $q \in (-\infty,0)$, $\{0\}$, $(0,(p+1)/2)$, $\{(p+1)/2\}$, $((p+1)/2,\infty)$. In the rest of this article, we will deal only with the third case.

Lemma 3.2 (for $q > 1$ see [6], page 115)**.** *Let $p \geqslant 1$, $0 < q < (p+1)/2$, $a, \theta > 0$, and let us introduce*

$$\Theta := \Theta_{p,q,a} := \sqrt{\frac{p+1-2q}{p+1}}\Big(\frac{q}{a}\Big)^{q/(p+1-2q)}.$$

If $\theta > \Theta$, then $\overline{\mathcal{F}}(\theta,\cdot)$ has no zero. If $\theta = \Theta$, then the only zero of $\overline{\mathcal{F}}(\theta,\cdot)$ is

$$\Big(\frac{q}{a}\Big)^{1/(p+1-2q)} =: \overline{R}_{p,q,a}(\Theta) =: \overline{R}(\Theta).$$

If $\theta < \Theta$, then $\overline{\mathcal{F}}(\theta,\cdot)$ has two zeros, which will be denoted by $\overline{R}_{i;p,q,a}(\theta) =: \overline{R}_i(\theta)$, $i = 1,2$, being

$$\overline{R}_1(\theta) < \overline{R}(\Theta) < \overline{R}_2(\theta).$$

Definition 3.3. Let $p \geqslant 1$, $0 < q < (p+1)/2$, $a > 0$, $b := 2a/(p+1)$, and put

$$\overline{L}_i(\theta) := \overline{L}_{i;p,q,a}(\theta) := \theta^{-(p-1)/(p+1)} I_{p,b}(\theta^{-2/(p+1)}\overline{R}_{p,q,a}(\theta))$$

for $i = 1,2$ and $\theta \in (0,\Theta)$. We introduce $\overline{L}_{p,q,a}(\Theta) =: \overline{L}(\Theta)$ analogously. Functions $\overline{L}$, $\overline{L}_1$ and $\overline{L}_2$ will be called *time maps* (associated with (3.1)).

Using Lemmata 3.1 and 3.2, we can describe $\mathcal{N}^{\pm}(l)$ by means of the time maps:

Lemma 3.4. *For all $p \geqslant 1$, $q \in (0,(p+1)/2)$ and $a, l > 0$:*

$$\mathcal{N}^{\pm}(l) = \Big\{ \pm\,\overline{u}_{\theta,p,a}\Big(\,\cdot \pm \frac{\overline{L}_2(\theta) - \overline{L}_1(\theta)}{2}\Big)\Big|_{[-l,l]} :\ \overline{L}_1(\theta) + \overline{L}_2(\theta) = 2l\Big\},$$

where the two $\pm$ symbols on the right-hand side are independent (i.e., there are four sign-changing nonantisymmetric solutions corresponding to any $\theta > 0$ satisfying $\overline{L}_1(\theta) + \overline{L}_2(\theta) = 2l$).

We need to know the limits of $\overline{L}_1 + \overline{L}_2$ at 0 and Θ, and whether $\overline{L}_1 + \overline{L}_2$ is monotone. Therefore, we now cite the following two lemmata and afterwards state the new results.

Lemma 3.5 (for $q > 1$ see [6], Lemma 5.2). *If $p \geqslant 1$, $0 < q < (p+1)/2$ and $a > 0$, then*

$$\lim_{\theta\to\Theta} \overline{L}_i(\theta) = \overline{L}(\Theta), \quad i = 1, 2,$$
$$\lim_{\theta\to 0} \overline{L}_2(\theta) = \infty.$$

Lemma 3.6 (for $q > 1$ see [6], Proof of Lemma 5.1). *If $p \geqslant 1$, $0 < q < (p+1)/2$, $a > 0$, $i \in \{1, 2\}$, then $\overline{L}_i$ is differentiable on $(0, \Theta)$, fulfilling*

$$\overline{L}_i'(\theta) = -\frac{p-1}{(p+1)\theta}\overline{L}_i(\theta) + \frac{p+1-2q}{(p+1)q\theta}\frac{\overline{R}_i^{1-q}(\theta)}{1 - \frac{a}{q}\overline{R}_i^{p+1-2q}(\theta)}.$$

Lemma 3.7. *If $p \geqslant 1$, $0 < q < (p+1)/2$ and $a > 0$, then $\overline{R}_1\overline{R}_2 < \overline{R}^2(\Theta)$.*

Proof. It is much the same as the proof of Lemma 2.7. So let $p \geqslant 1$, $0 < q < (p+1)/2$, $a > 0$, $\theta \in (0, \Theta)$, and set $\alpha := \overline{R}_2(\theta)/\overline{R}(\Theta) > 1$. Using the increase of $\overline{\mathcal{F}}(\theta, \cdot)$ on $(0, \overline{R}(\Theta))$ and the definition of $\overline{R}_1(\theta)$ and $\overline{R}_2(\theta)$, one can see that it suffices to prove that

$$0 > \overline{\mathcal{F}}(\theta, \alpha\overline{R}(\Theta)) - \overline{\mathcal{F}}\Big(\theta, \frac{\overline{R}(\Theta)}{\alpha}\Big) = 2q\overline{R}^{2q}(\Theta)(F_\alpha(2q) - F_\alpha(p+1))$$

(see (2.4) for the definition of F_α), which is a true inequality, due to the increase of F_α. □

Lemma 3.8. *If $a > 0$ and either $p = 1$, $q \in (0, 1)$ or $p > 1$, $q \in [1/2, (p+1)/2)$, then $(\overline{L}_1 + \overline{L}_2)' < 0$.*

Proof. Consider $p \geqslant 1$, $0 < q < (p+1)/2$, $a > 0$, $\theta \in (0, \Theta)$, and put $b := 2a/(p+1)$. We will proceed similarly to the proof of Lemma 2.8.

1. We start with the estimate suggested in [6], Remark 5.3:

$$I_{p,b}(y) > \frac{y}{\sqrt{by^{p+1} - 1}}, \quad y > 0,$$

which results in

$$\overline{L}_i(\theta) > \overline{R}_i^{1-q}(\theta), \quad i = 1, 2.$$

Applying this inequality to the formula included in Lemma 3.6, one can derive a sufficient condition for $(\overline{L}_1 + \overline{L}_2)'(\theta) < 0$ in the form of

$$\overline{F}\Big(\frac{\overline{R}_1(\theta)}{\overline{R}(\Theta)}\Big) + \overline{F}\Big(\frac{\overline{R}_2(\theta)}{\overline{R}(\Theta)}\Big) < 0,$$

where

$$\overline{F}(x) := \overline{F}_{p,q}(x) := (1-p)qx^{1-q} + \frac{(p+1-2q)x^{1-q}}{1-x^{p+1-2q}}, \quad x \in (0,1)\cup(1,\infty).$$

2. Now we prove the increase of $\overline{F}$ on $(0,1)$. Setting

$$\alpha := p-1 \geqslant 0, \quad \beta := 2(q-1) \in (-2,\alpha),$$

we obtain that

$$\overline{F}(x) = -\alpha\Big(\frac{\beta}{2}+1\Big)x^{-\beta/2} + \frac{(\alpha-\beta)x^{-\beta/2}}{1-x^{\alpha-\beta}} = F(x) - \frac{\alpha\beta}{2}x^{-\beta/2}.$$

Since F increases on $(0,1)$ due to Step 2. of the proof of Lemma 2.8, $\overline{F}$ increases on $(0,1)$ as well.

3. Using the same ideas as in Step 3. of the proof of Lemma 2.8, we can see that it suffices to verify the inequality

$$\overline{u}_{\alpha,\beta}(x) := \beta(\alpha+2)x^{\alpha} + \alpha(\beta+2)x^{\alpha-\beta} - \alpha(\beta+2)x^{\beta} - \beta(\alpha+2) \geqslant 0$$

for all $x>1$ and α, β fulfilling either $\alpha=0$, $\beta\in(-2,0)$ or $\alpha>0$, $\beta\in[-1,\alpha)$. The former case is clear. In the latter one we have that $\overline{u}_{\alpha,\beta}(1)=0$ and

$$\overline{u}'_{\alpha,\beta}(x) = \underbrace{\alpha x^{\beta-1}}_{>0}\underbrace{(\beta(\alpha+2)x^{\alpha-\beta} + (\alpha-\beta)(\beta+2)x^{\alpha-2\beta} - \beta(\beta+2))}_{=:\overline{v}_{\alpha,\beta}(x)},$$

so the verification ot the nonnegativity of $\overline{v}_{\alpha,\beta}$ on $(1,\infty)$ will complete the proof. And since $\overline{v}_{\alpha,\beta}(1) = 2(\alpha-\beta)(\beta+1) \geqslant 0$ and

$$\overline{v}'_{\alpha,\beta}(x) = \underbrace{(\alpha-\beta)x^{\alpha-2\beta-1}}_{>0}\underbrace{(\beta(\alpha+2)x^{\beta} + (\beta+2)(\alpha-2\beta))}_{=:\overline{w}_{\alpha,\beta}(x)},$$

we just need to observe that $\overline{w}_{\alpha,\beta}\geqslant 0$ on $(1,\infty)$ because $\overline{w}_{\alpha,\beta}(1) = 2(\alpha-\beta)(\beta+1)\geqslant 0$ and $\overline{w}_{\alpha,\beta}$ is nondecreasing. □

Remark 3.9. The proof of Lemma 3.8 does not work for $p>1$, $q\in(0,1/2)$, $a>0$, i.e., for $\alpha>0$, $\beta\in(-2,-1)$, because in that case we have $\overline{u}'_{\alpha,\beta}(1) = \alpha\overline{v}_{\alpha,\beta}(1) < 0$, implying that $\overline{u}_{\alpha,\beta} < 0$ in the right neighbourhood of 1. In addition, numerical calculations suggest that if $p>1$ is big enough and $q\in(0,1/2)$ is small enough, then $\overline{L}_1 + \overline{L}_2$ has a stationary point where a minimum is attained.

Joining the results of Lemmata 3.4, 3.5 and 3.8, we immediately obtain the following assertion:

Theorem 3.10. *Assume* $a, l > 0$ *and either* $p = 1$, $q \in (0,1)$ *or* $p > 1$, $q \in [1/2, (p+1)/2)$. *There exists such a number* $\overline{L}(\Theta) > 0$ *(see Lemma* 3.2 *and Definition* 3.3*) that* (1.1) *has four sign-changing nonantisymmetric solutions for* $l > \overline{L}(\Theta)$ *and none for* $l \leqslant \overline{L}(\Theta)$. *(See Figure* 2.*)*

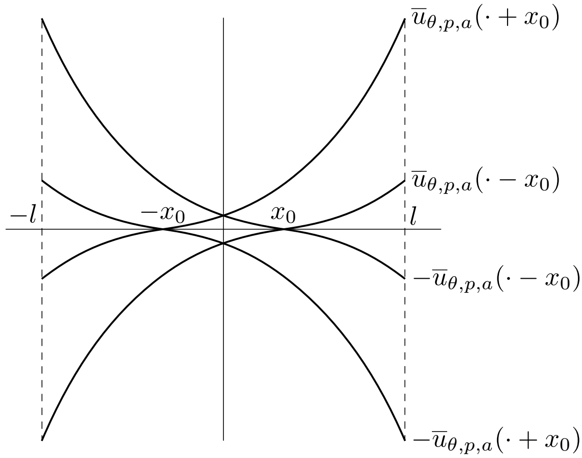

Figure 2. The four sign-changing nonantisymmetric solutions of (1.1) in the case dealt with in Theorem 3.10. (Here $x_0 = (\overline{L}_2(\theta) - \overline{L}_1(\theta))/2$, see Lemma 3.4.)

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References

[1] *F. Andreu, J. M. Mazón, J. Toledo, J. D. Rossi*: Porous medium equation with absorption and a nonlinear boundary condition. Nonlinear Anal., Theory Methods Appl., Ser. A *49* (2002), 541–563.

[2] *J. M. Arrieta, A. Rodríguez-Bernal*: Localization on the boundary of blow-up for reaction-diffusion equations with nonlinear boundary conditions. Commun. Partial Differ. Equations *29* (2004), 1127–1148.

[3] *B. Brighi, M. Ramaswamy*: On some general semilinear elliptic problems with nonlinear boundary conditions. Adv. Differ. Equ. *4* (1999), 369–390.

[4] *S. Cano-Casanova*: On the positive solutions of the logistic weighted elliptic BVP with sublinear mixed boundary conditions. Spectral Theory and Nonlinear Analysis with Applications to Spatial Ecology. Proceedings of the international seminar on spectral theory and nonlinear analysis, Universidad Complutense de Madrid, 2004 (S. Cano-Casanova et al., eds.). World Scientific, Hackensack, 2005, pp. 1–15.

[5] *M. Chipot, M. Fila, P. Quittner*: Stationary solutions, blow up and convergence to stationary solutions for semilinear parabolic equations with nonlinear boundary conditions. Acta Math. Univ. Comen., New Ser. *60* (1991), 35–103.

[6] *M. Chipot, P. Quittner*: Equilibria, connecting orbits and a priori bounds for semilinear parabolic equations with nonlinear boundary conditions. J. Dyn. Differ. Equations *16* (2004), 91–138.

[7] *M. Chipot, M. Ramaswamy*: Semilinear elliptic problems with nonlinear boundary conditions. Differ. Equ. Dyn. Syst. *6* (1998), 51–75.
[8] *M. Fila, B. Kawohl*: Large time behavior of solutions to a quasilinear parabolic equation with a nonlinear boundary condition. Adv. Math. Sci. Appl. *11* (2001), 113–126.
[9] *M. Fila, J. J. L. Velázquez, M. Winkler*: Grow-up on the boundary for a semilinear parabolic problem. Nonlinear elliptic and parabolic problems: a special tribute to the work of Herbert Amann. Proc. Conf., Zürich, 2004 (M. Chipot, J. Escher, eds.). Progress in Nonlinear Differential Equations and Their Applications 64, Birkhäuser, Basel, 2005, pp. 137–150.
[10] *J. García-Melián, C. Morales-Rodrigo, J. D. Rossi, A. Suárez*: Nonnegative solutions to an elliptic problem with nonlinear absorption and a nonlinear incoming flux on the boundary. Ann. Mat. Pura Appl. (4) *187* (2008), 459–486.
[11] *A. W. Leung, Q. Zhang*: Reaction diffusion equations with non-linear boundary conditions, blowup and steady states. Math. Methods Appl. Sci. *21* (1998), 1593–1617.
[12] *J. López Gómez, V. Márquez, N. Wolanski*: Dynamic behavior of positive solutions to reaction-diffusion problems with nonlinear absorption through the boundary. Rev. Unión Mat. Argent. *38* (1993), 196–209.
[13] *C. Morales-Rodrigo, A. Suárez*: Some elliptic problems with nonlinear boundary conditions. Spectral Theory and Nonlinear Analysis with Applications to Spatial Ecology. Proceedings of the international seminar on spectral theory and nonlinear analysis, Universidad Complutense de Madrid, 2004 (S. Cano-Casanova et al., eds.). World Scientific, Hackensack, 2005, pp. 175–199.
[14] *S. Peres*: Solvability of a nonlinear boundary value problem. Acta Math. Univ. Comen., New Ser. *82* (2013), 69–103.
[15] *P. Quittner*: On global existence and stationary solutions for two classes of semilinear parabolic problems. Commentat. Math. Univ. Carol. *34* (1993), 105–124.
[16] *A. Rodríguez-Bernal, A. Tajdine*: Nonlinear balance for reaction-diffusion equations under nonlinear boundary conditions: dissipativity and blow-up. J. Differ. Equations *169* (2001), 332–372.
[17] *J. D. Rossi*: Elliptic problems with nonlinear boundary conditions and the Sobolev trace theorem. Stationary Partial Differential Equations. Vol. II. (M. Chipot, P. Quittner, eds.). Handbook of Differential Equations, Elsevier/North Holland, Amsterdam, 2005, pp. 311–406.

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