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MODULATION SPACE ESTIMATES FOR SCHRÖDINGER TYPE EQUATIONS WITH TIME-DEPENDENT POTENTIALS

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Abstract. We give a new representation of solutions to a class of time-dependent Schrödinger type equations via the short-time Fourier transform and the method of characteristics. Moreover, we also establish some novel estimates for oscillatory integrals which are associated with the fractional power of negative Laplacian $(-\Delta)^{\kappa/2}$ with $1 \leqslant \kappa \leqslant 2$. Consequently the classical Hamiltonian corresponding to the previous Schrödinger type equations is studied. As applications, a series of new boundedness results for the corresponding propagator are obtained in the framework of modulation spaces. The main results of the present article include the case of wave equations.

Keywords: Schrödinger type equation; short-time Fourier transform; modulation space; classical Hamiltonian; complex interpolation

MSC 2010: 35Q40, 42B35

1. Introduction and main results

This work is concerned with a series of new estimates for the solution to the time-dependent Schrödinger type equation

$$\begin{cases} \mathrm{i}\partial_t u(t,x) = (-\Delta)^{\kappa/2} u(t,x) + V(t,x)u(t,x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^n \end{cases} \tag{1.1}$$

in the framework of modulation spaces, which were first introduced by Feichtinger [3] and several equivalent characterizations have been studied extensively (see [5]). Here $1 \leqslant \kappa \leqslant 2$, $\mathrm{i} = \sqrt{-1}$, $u(t,x)$ is a complex valued function of $(t,x) \in \mathbb{R} \times \mathbb{R}^n$, u_0 is a complex valued function defined on $\mathbb{R}^n$ and $V(t,x)$ is a real valued function of

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$(t,x) \in \mathbb{R} \times \mathbb{R}^n$; $\partial_t = \partial/\partial t$, $\Delta = \sum\limits_{i=1}^{n} \partial^2/\partial x_i^2$, $(-\Delta)^{\kappa/2} = \mathcal{F}^{-1}|\cdot|^{\kappa}\mathcal{F}$ with $\mathcal{F}$ and $\mathcal{F}^{-1}$ denoting the Fourier and inverse Fourier transforms, respectively, defined by

$$\mathcal{F}g(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x\cdot\xi} g(x)\,\mathrm{d}x \quad \text{and} \quad \mathcal{F}^{-1}g(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi} g(\xi)\,\mathrm{d}\xi$$

for all $g \in \mathcal{S}(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz class of rapidly decreasing functions on $\mathbb{R}^n$.

Under the condition $V(t,x) = 0$, it is well known that the solution $u(t,x)$ of (1.1) can be written as

$$u(t,x) = (\mathrm{e}^{-\mathrm{i}t(-\Delta)^{\kappa/2}} u_0)(x) = \mathcal{F}^{-1}_{\xi\to x}[\mathrm{e}^{-\mathrm{i}t|\xi|^{\kappa}} \mathcal{F}u_0(\xi)](x),$$

and the unimodular Fourier multiplier $\mathrm{e}^{-\mathrm{i}(-\Delta)^{\kappa/2}}$ generally does not preserve any Lebesgue space L^q, except for $q = 2$. For example, $\mathrm{e}^{-\mathrm{i}(-\Delta)}$ is bounded on L^q if and only if $q = 2$ (see [6]). It is then natural to look for some appropriate substitution of Lebesgue spaces for the study of the unimodular Fourier multiplier, one of which are the so-called *modulation spaces*.

To define the modulation spaces, let us first recall the definitions of the short-time Fourier transform (STFT) and its adjoint operator which were introduced by Córdoba-Fefferman [2]. Fix $\varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$, then the short-time Fourier transform of $g \in \mathcal{S}'(\mathbb{R}^n)$ with respect to the window function φ is defined by

$$V_\varphi g(x,\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}y\cdot\xi}\overline{\varphi(y-x)} g(y)\,\mathrm{d}y,$$

where $\mathcal{S}'(\mathbb{R}^n)$ denotes the dual space of $\mathcal{S}(\mathbb{R}^n)$.

Let G be a function on $\mathbb{R}^n \times \mathbb{R}^n$, then the adjoint operator V_φ^* of V_φ is defined by

$$V_\varphi^* G(x) = \iint_{\mathbb{R}^{2n}} \mathrm{e}^{\mathrm{i}x\cdot\xi}\varphi(x-y)G(y,\xi)\,\mathrm{d}y\,\bar{\mathrm{d}}\xi$$

with $\bar{\mathrm{d}}\xi = (2\pi)^{-n}\,\mathrm{d}\xi$. It is known that for all $\psi, \varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ satisfying $\langle\psi,\varphi\rangle \neq 0$, we have the inversion formula

$$\frac{1}{\langle\psi,\varphi\rangle} V_\psi^* V_\varphi g = g, \quad g \in \mathcal{S}'(\mathbb{R}^n)$$

(see [5], Corollary 11.2.7). Throughout this paper, we denote

$$V_{\varphi(t,\cdot)}u(t,x,\xi) = V_{\varphi(t,\cdot)}[u(t,\cdot)](x,\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}y\cdot\xi}\overline{\varphi(t,y-x)} u(t,y)\,\mathrm{d}y,$$

where u and φ are functions on $\mathbb{R} \times \mathbb{R}^n$. Below, we shall present Feichtinger's definition on modulation spaces by STFT.

Let $1 \leqslant p, q \leqslant \infty$ and $\psi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$. We define the modulation space $M_\psi^{p,q}(\mathbb{R}^n)$ as a Banach space which consists of all $g \in \mathcal{S}'(\mathbb{R}^n)$ such that their norm satisfies

$$\|g\|_{M_\psi^{p,q}} = \big\| \|V_\psi g(x,\xi)\|_{L_x^p} \big\|_{L_\xi^q} < \infty.$$

Note that this definition of $M_\psi^{p,q}(\mathbb{R}^n)$ is independent of the choice of ψ, i.e., $M_\psi^{p,q}(\mathbb{R}^n) = M_\varphi^{p,q}(\mathbb{R}^n)$ for all $\psi, \varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ (see [3], Theorem 6.1). Thus we may choose a suitable window function ψ to estimate the modulation space norms in this paper. For the sake of convenience, we denote $M^{p,q} = M_\psi^{p,q}(\mathbb{R}^n)$. Furthermore, we have the complex interpolation theory for modulation spaces as follows: Let $0 < \theta < 1$ and $1 \leqslant p_i, q_i \leqslant \infty$ with $q_2 < \infty$, $i = 1, 2$. Set $1/p = (1-\theta)/p_1 + \theta/p_2$, $1/q = (1-\theta)/q_1 + \theta/q_2$, then $(M^{p_1,q_1}, M^{p_2,q_2})_{[\theta]} = M^{p,q}$ (see [4], Corollary 2.3).

As mentioned before, the unimodular Fourier multiplier is not a bounded operator on any L^q in general except for $q = 2$. However, a recent work by Bényi-Gröchenig-Okoudjou-Rogers [1] has shown that $\mathrm{e}^{-\mathrm{i}t(-\Delta)^{\kappa/2}}$ with $\kappa \in [0, 2]$ preserves the $M^{p,q}$-norm for any $1 \leqslant p, q \leqslant \infty$, which is proved by the stationary phase method. More precisely, we have the estimate

$$\|u(t,\cdot)\|_{M_{\varphi_0}^{p,q}} \leqslant C(1+|t|)^{n/2} \|u_0\|_{M_{\varphi_0}^{p,q}}, \quad u_0 \in \mathcal{S}(\mathbb{R}^n) \tag{1.2}$$

for all $1 \leqslant p, q \leqslant \infty$ and $t \in \mathbb{R}$, where C is a positive constant, $\varphi_0 \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ and $u(t,x)$ is the solution of (1.1) with $V(t,x) = 0$.

Furthermore, by using the unit-cube decomposition to the frequency spaces, Wang and Hudzik [11] established the following two inequalities:

$$\|u(t,\cdot)\|_{M_{\varphi_0}^{p,q}} \leqslant C(1+|t|)^{-n(1/2-1/p)} \|u_0\|_{M_{\varphi_0}^{p',q}}, \quad u_0 \in \mathcal{S}(\mathbb{R}^n) \tag{1.3}$$

and

$$\|u(t,\cdot)\|_{M_{\varphi_0}^{p,q}} \leqslant C(1+|t|)^{n(1/2-1/p)} \|u_0\|_{M_{\varphi_0}^{p,q}}, \quad u_0 \in \mathcal{S}(\mathbb{R}^n) \tag{1.4}$$

for all $t \in \mathbb{R}$, $1 \leqslant q \leqslant \infty$ and $2 \leqslant p \leqslant \infty$, where $1/p' + 1/p = 1$, C is a positive constant, $\varphi_0 \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ and $u(t,x)$ is the solution of (1.1) with $V(t,x) = 0$.

On the other hand, K. Kato, M. Kobayashi and S. Ito applied the properties of the wave packet transform and the method of characteristics to obtain some new estimates for (1.1) as follows, which cover (1.2), (1.3) and (1.4) when $\kappa = 2$ (see [7]).

Theorem A (Kato-Kobayashi-Ito [7], [8]). *Let* $1 \leqslant p, q \leqslant \infty$, $\varphi_0 \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ *and* $\kappa = 2$.

(i) *Suppose* $V(t,x) = 0$. *Then*

$$\|u(t,\cdot)\|_{M^{p,q}_{\varphi(t,\cdot)}} = \|u_0\|_{M^{p,q}_{\varphi_0}}, \quad u_0 \in \mathcal{S}(\mathbb{R}^n)$$

holds for all $t \in \mathbb{R}$.

(ii) *Suppose* $V(t,x) = \pm|x|^2$. *Then*

$$\|u(t,\cdot)\|_{M^{p,p}_{\varphi(t,\cdot)}} = \|u_0\|_{M^{p,p}_{\varphi_0}}, \quad u_0 \in \mathcal{S}(\mathbb{R}^n)$$

holds for all $t \in \mathbb{R}$.

In (i) and (ii), $\varphi(t,x)$ and $u(t,x)$ are the solutions of (1.1) with $\varphi(0,x) = \varphi_0(x)$ and $u(0,x) = u_0(x)$, respectively.

Our purpose in the present paper is to obtain similar estimates to Theorem A for the general time-dependent Schrödinger type equations, whose potentials $V(t,x)$ satisfy the following assumptions.

Assumptions. Throughout this paper, we shall assume that $V(t,x)$ is a real valued C^2-function of $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ and $C^{2[n/2]+4}$ in x, such that for all multi-indices α with $2 \leqslant |\alpha| \leqslant 2[n/2]+4$ or $1 \leqslant |\alpha| \leqslant 2[n/2]+4$ there exists $C_\alpha > 0$ such that

$$|\partial_x^\alpha V(t,x)| \leqslant C_\alpha, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \tag{1.5}$$

where C_α is a constant depending only on α.

The next theorem is one of our main results.

Theorem 1.1. *Let* $1 \leqslant p \leqslant \infty$, $1 \leqslant \kappa \leqslant 2$, $\mathcal{F}(\varphi_0) \in C_c^\infty(\mathbb{R}^n \setminus \{0\}) \setminus \{0\}$ *and* $T > 0$. *Set* $\varphi(t,x) = \mathrm{e}^{-\mathrm{i}t(-\Delta)^{\kappa/2}}\varphi_0(x)$. *If* $V \in C^2(\mathbb{R} \times \mathbb{R}^n)$ *satisfies* (1.5) *for all multi-indices* α *with* $2 \leqslant |\alpha| \leqslant 2[n/2]+4$, *then there exists* $C > 0$ *such that*

$$\|u(t,\cdot)\|_{M^{p,p}_{\varphi(t,\cdot)}} \leqslant C\|u_0\|_{M^{p,p}_{\varphi_0}}, \quad u_0 \in \mathcal{S}(\mathbb{R}^n)$$

for all $t \in [-T,T]$, *where* $u(t,x)$ *is the solution of* (1.1) *in* $C(\mathbb{R}; L^2(\mathbb{R}^n))$ *with* $u(0,x) = u_0(x)$, *and* C *is a positive constant depending only on* κ, n *and* T.

We remark that the $M^{p,p}$ norm generally cannot be replaced by the $M^{p,q}$ norm in the above theorem. Indeed, when $\kappa = 2$ and $V(t,x) = |x|^2$, it follows that

$$\left\|u\Big(\frac{\pi}{4},\cdot\Big)\right\|_{M^{p,q}_{\varphi(\pi/4,\cdot)}} = 2^{n/q}\big\|\|V_{\varphi_0}u_0(\xi,x)\|_{L^p_x}\big\|_{L^q_\xi}$$

(see [8]), but the inequality $\big\|\|V_{\varphi_0}u_0(\xi,x)\|_{L^p_x}\big\|_{L^q_\xi} \leqslant C\big\|\|V_{\varphi_0}u_0(x,\xi)\|_{L^p_x}\big\|_{L^q_\xi}$ fails in general. However, if the assumption (1.5) is strengthened to include the case of $|\alpha| = 1$, we can replace the $M^{p,p}$ norm by the $M^{p,q}$ norm as follows.

Theorem 1.2. *Let $1 \leqslant p,q \leqslant \infty$, $1 \leqslant \kappa \leqslant 2$, $\mathcal{F}(\varphi_0) \in C_c^\infty(\mathbb{R}^n \setminus \{0\}) \setminus \{0\}$ and $T > 0$. Set $\varphi(t,x) = \mathrm{e}^{-\mathrm{i}t(-\Delta)^{\kappa/2}}\varphi_0(x)$. If $V \in C^2(\mathbb{R}\times\mathbb{R}^n)$ satisfies (1.5) for all multi-indices α with $1 \leqslant |\alpha| \leqslant 2[n/2]+4$, then there exists $C > 0$ such that*

$$\|u(t,\cdot)\|_{M^{p,q}_{\varphi(t,\cdot)}} \leqslant C\|u_0\|_{M^{p,q}_{\varphi_0}}, \quad u_0 \in \mathcal{S}(\mathbb{R}^n)$$

for all $t \in [-T,T]$, where $u(t,x)$ is the solution of (1.1) in $C(\mathbb{R}; L^2(\mathbb{R}^n))$ with $u(0,x) = u_0(x)$, and C is a positive constant depending only on κ, n and T.

Our paper is organized as follows. In Section 2, we set up some key lemmas concerning the classical orbit associated with the classical Hamiltonian corresponding to the time-dependent Schrödinger type equation (1.1). Furthermore, two important inequalities are provided, which play a crucial role in the proof of our main results. In Section 3, we obtain a new representation (3.10) of solutions to (1.1) and use it to prove Theorem 1.1. Finally, the proof of Theorem 1.2 is given in Section 4.

Throughout this paper, $\mathbb{R}$ and $\mathbb{N}$ will stand for the sets of reals and positive integers, respectively. The letter C denotes a positive constant, which may be different at different places. For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$, we write

$$x\cdot\xi = \langle x,\xi\rangle = \sum_{i=1}^n x_i\xi_i, \quad |x| = \langle x,x\rangle^{1/2} \quad \text{and} \quad \langle x\rangle = (|x|^2+1)^{1/2}.$$

2. Key lemmas

In this section, we give two inequalities which play a key role in the proof of main results, and study the classical Hamiltonian corresponding to (1.1).

Lemma 2.1. *Let $1 \leqslant \kappa \leqslant 2$, then there exists $C_\kappa > 0$ such that*

$$\big||a|^{\kappa-2}a - |b|^{\kappa-2}b\big| \leqslant C_\kappa|a-b|^{\kappa-1} \tag{2.1}$$

for all $a \in \mathbb{R}^n$, $b \in \mathbb{R}^n$, where C_κ depends only on κ.

P r o o f. We note that the inequality (2.1) is trivial when $\kappa = 1$ or $\kappa = 2$ or $\min\{|a|, |b|, |a-b|\} = 0$. Letting $1 < \kappa < 2$, $\min\{|a|, |b|, |a-b|\} > 0$, we have

$$\begin{aligned} ||a|^{\kappa-2}a - |b|^{\kappa-2}b|^2 &= |a|^{2\kappa-2} + |b|^{2\kappa-2} - 2|a|^{\kappa-2}|b|^{\kappa-2}\langle a, b\rangle \\ &= |a|^{2\kappa-2} + |b|^{2\kappa-2} - |a|^{\kappa-2}|b|^{\kappa-2}(|a|^2 + |b|^2 - |a-b|^2) \\ &= (|a|^\kappa - |b|^\kappa)(|a|^{\kappa-2} - |b|^{\kappa-2}) + |a|^{\kappa-2}|b|^{\kappa-2}|a-b|^2 \\ &= |b|^{2\kappa-2}\Big\{\Big(\frac{|a|^\kappa}{|b|^\kappa} - 1\Big)\Big(\frac{|a|^{\kappa-2}}{|b|^{\kappa-2}} - 1\Big) + \frac{|a|^{\kappa-2}}{|b|^{\kappa-2}}\frac{|a-b|^2}{|b|^2}\Big\}. \end{aligned}$$

Setting $x = |a|/|b|$, $p = |a-b|/|b|$ and

$$F(x) = (x^\kappa - 1)(x^{\kappa-2} - 1) + x^{\kappa-2}p^2 = x^{2\kappa-2} - x^{\kappa-2}(x^2 - p^2 + 1) + 1,$$

it is sufficient to prove the inequality

$$F(x) \leqslant C_\kappa p^{2\kappa-2}, \tag{2.2}$$

if $|p-1| \leqslant x \leqslant |p+1|$. Differentiating (2.2), we have

$$F'(x) = x^{\kappa-3}\{(\kappa-2)(p^2-1) + (2\kappa-2)x^\kappa - \kappa x^2\} = x^{\kappa-3}G(x),$$

where

$$G(x) = (\kappa-2)(p^2-1) + (2\kappa-2)x^\kappa - \kappa x^2,$$

hence

$$G'(x) = 2\kappa x^{\kappa-1}(\kappa - 1 - x^{2-\kappa}).$$

Let $x_0 = (\kappa-1)^{1/(2-\kappa)}$, we have $G'(x) > 0$ for all $x \in (0, x_0)$, and $G'(x) < 0$ for all $x \in (x_0, \infty)$. Thus,

$$\sup_{x\geqslant 0} G(x) = G(x_0) = (2-\kappa)\{(\kappa-1)^{2/(2-\kappa)} + 1 - p^2\}.$$

Setting

$$p_0 = \{(\kappa-1)^{2/(2-\kappa)} + 1\}^{1/2},$$

the proof of (2.2) can be divided into three cases.

Case 1. If $p \geqslant p_0$, then

$$F'(x) = x^{\kappa-3}G(x) \leqslant x^{\kappa-3}G(x_0) = (2-\kappa)x^{\kappa-3}(p_0^2 - p^2) \leqslant 0$$

for all $x > 0$. It follows that

$$\begin{aligned}\sup_{|p-1|\leqslant x\leqslant |p+1|} F(x) &= F(|p-1|)\\ &= |p-1|^{2\kappa-2} - |p-1|^{\kappa-2}(|p-1|^2 - p^2 + 1) + 1\\ &= \Big(|p-1|^{\kappa-1} + \frac{p-1}{|p-1|}\Big)^2\\ &= \begin{cases} ((p-1)^{\kappa-1}+1)^2 \leqslant 4p^{2\kappa-2}, & p \geqslant 1;\\ (1-(1-p)^{\kappa-1})^2 \leqslant p^{2\kappa-2}, & 0 < p < 1.\end{cases}\end{aligned}$$

Case 2. If $p < p_0$ and $x_0 \leqslant |p-1|$, we have

$$\sup_{|p-1|\leqslant x\leqslant |p+1|} F(x) \leqslant \sup_{x\geqslant x_0} F(x) \leqslant \sup_{x\geqslant x_0} x^{\kappa-2}p^2 \leqslant {x_0}^{\kappa-2}p^2 \leqslant ({x_0}^{\kappa-2}{p_0}^{4-2\kappa})p^{2\kappa-2},$$

where $C_\kappa = {x_0}^{\kappa-2}{p_0}^{4-2\kappa}$ depends only on κ.

Case 3. If $p < p_0$ and $x_0 > |p-1|$, we shall show that

$$\sup_{|p-1|\leqslant x\leqslant x_0} F(x) \leqslant \max\{F(|p-1|), F(x_0)\}.$$

Indeed, if we suppose that there exists a point $x_1 \in (|p-1|, x_0)$ such that

$$F(x_1) > \max\{F(|p-1|), F(x_0)\},$$

it follows from the Mean-Value Theorem for derivatives that there exist $\xi_1 \in (|p-1|, x_1)$ and $\xi_2 \in (x_1, x_0)$ such that

$$\begin{aligned}G(\xi_1) = {\xi_1}^{3-\kappa}F'(\xi_1) &= {\xi_1}^{3-\kappa}\Big\{\frac{F(x_1)-F(|p-1|)}{x_1-|p-1|}\Big\} > 0 > {\xi_2}^{3-\kappa}\Big\{\frac{F(x_0)-F(x_1)}{x_0-x_1}\Big\}\\ &= {\xi_2}^{3-\kappa}F'(\xi_2) = G(\xi_2),\end{aligned}$$

which contradicts $G'(x) > 0$ for all $x \in (0, x_0)$. Thus we obtain

$$\begin{aligned}\sup_{|p-1|\leqslant x\leqslant |p+1|} F(x) \leqslant \sup_{x\geqslant |p-1|} F(x) &= \max\Big\{\sup_{|p-1|\leqslant x\leqslant x_0} F(x), \sup_{x\geqslant x_0} F(x)\Big\}\\ &\leqslant \max\Big\{F(|p-1|), \sup_{x\geqslant x_0} F(x)\Big\} \leqslant \max\{4, C_\kappa\}p^{2\kappa-2},\end{aligned}$$

where the last inequality follows from the first two cases.

It is evident that (2.2) follows from the Cases 1–3. This completes the proof of Lemma 2.1. □

Remark 2.2. If $\kappa \notin [1,2]$, the constant C_κ in (2.1) also depends on the vectors a and b.

Lemma 2.3. *Let $1 < \kappa \leqslant 2$ and define*

$$S(y,\xi) = |y-\xi|^{\kappa} - |\xi|^{\kappa} + \kappa|\xi|^{\kappa-2}\langle \xi, y\rangle \tag{2.3}$$

for all $y \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$. Then we have

$$\left|\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot y} S(y,\xi)\varphi(y)\,\mathrm{d}y\right| \leqslant C\langle x\rangle^{-n-\kappa+\varepsilon} \tag{2.4}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$, where ε is an arbitrarily small positive number, and C is a positive constant depending only on φ, κ, n and ε.

P r o o f. We note that the inequality (2.4) is trivial when $\kappa = 2$. Let $1 < \kappa < 2$. Applying (2.1), we have

$$\begin{aligned} |S(y,\xi)| &= \left|\int_0^1 \partial_t |ty-\xi|^{\kappa}\,\mathrm{d}t + \kappa|\xi|^{\kappa-2}\langle\xi,y\rangle\right| \\ &= \left|\int_0^1 \kappa|ty-\xi|^{\kappa-2}\langle ty-\xi, y\rangle\,\mathrm{d}t + \kappa|\xi|^{\kappa-2}\langle\xi,y\rangle\right| \\ &= \left|\int_0^1 \kappa\langle |ty-\xi|^{\kappa-2}(ty-\xi) - |-\xi|^{\kappa-2}(-\xi), y\rangle\,\mathrm{d}t\right| \\ &\leqslant \int_0^1 C_\kappa |ty|^{\kappa-1}|y|\,\mathrm{d}t = C_\kappa|y|^{\kappa}, \end{aligned} \tag{2.5}$$

$$|\partial_{y_i} S(y,\xi)| = \kappa\big||y-\xi|^{\kappa-2}(y_i-\xi_i) - |-\xi|^{\kappa-2}(-\xi_i)\big| \leqslant C_\kappa|y|^{\kappa-1} \tag{2.6}$$

for $1 \leqslant i \leqslant n$. Furthermore,

$$|\partial_y^{\alpha} S(y,\xi)| \leqslant C(\alpha,\kappa)|y-\xi|^{\kappa-|\alpha|} \tag{2.7}$$

for all multi-indices α with $|\alpha| \geqslant 2$, where $C(\alpha,\kappa)$ is a positive constant depending only on α and κ.

We shall divide the proof of Lemma 2.3 into four steps.

Step 1. (2.4) holds for $|x| \leqslant 1$.

P r o o f of Step 1. Using (2.5), we have

$$\begin{aligned} \left|\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot y} S(y,\xi)\varphi(y)\,\mathrm{d}y\right| &\leqslant \int_{\mathbb{R}^n} |S(y,\xi)\varphi(y)|\,\mathrm{d}y \\ &\leqslant \int_{\mathbb{R}^n} C_\kappa|y|^{\kappa}|\varphi(y)|\,\mathrm{d}y \leqslant C(\varphi,\kappa) \leqslant C\langle x\rangle^{-n-\kappa+\varepsilon} \end{aligned} \tag{2.8}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $|x| \leqslant 1$, where C is a positive constant depending only on φ, κ, n and ε. □

Step 2. Let $|\alpha| = 2$. Then

$$\left|\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot y}\partial_y^\alpha S(y,\xi)\varphi(y)\,\mathrm{d}y\right| \leqslant C|x|^{-n-\kappa+\varepsilon+2} \tag{2.9}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $|x| \geqslant 1$, where ε is an arbitrarily small positive number, and C is a positive constant depending only on φ, κ, n and ε.

P r o o f of Step 2. Define the derivative operator

$$L(x,D) = \frac{x\cdot\nabla_y}{\mathrm{i}|x|^2}.$$

Then we have

$$L(x,D)\mathrm{e}^{\mathrm{i}x\cdot y} = \mathrm{e}^{\mathrm{i}x\cdot y}.$$

The conjugate operator is

$$L^*(x,D) = -\frac{x\cdot\nabla_y}{\mathrm{i}|x|^2}.$$

We note that

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot y}\partial_y^\alpha S(y,\xi)\varphi(y)\,\mathrm{d}y &= \mathrm{e}^{\mathrm{i}x\cdot\xi}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot y}\partial_y^\alpha S(y+\xi,\xi)\varphi(y+\xi)\,\mathrm{d}y\\
&= \mathrm{e}^{\mathrm{i}x\cdot\xi}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot y}\varrho\Big(\frac{y}{\delta}\Big)\partial_y^\alpha S(y+\xi,\xi)\varphi(y+\xi)\,\mathrm{d}y\\
&\quad + \mathrm{e}^{\mathrm{i}x\cdot\xi}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot y}\Big(1-\varrho\Big(\frac{y}{\delta}\Big)\Big)\partial_y^\alpha S(y+\xi,\xi)\varphi(y+\xi)\,\mathrm{d}y\\
&\triangleq I_1 + I_2,
\end{aligned}$$

where $\delta > 0$ is to be chosen later, and ϱ is a $C_c^\infty(\mathbb{R}^n)$ function satisfying $0 \leqslant \varrho \leqslant 1$ and

$$\varrho(y) = \begin{cases} 0, & |y| > 2;\\ 1, & |y| \leqslant 1.\end{cases}$$

It is clear that

$$|I_1| \leqslant C(\kappa,\varphi)\int_{|y|\leqslant 2\delta} |y|^{\kappa-2}\,\mathrm{d}y \leqslant C(n,\kappa,\varphi)\delta^{n+\kappa-2}.$$

To estimate I_2, we can take a sufficiently large integer $N > 0$, apply (2.7) and integrate by parts to obtain that

$$
\begin{aligned}
|I_2| &\leqslant \int_{\mathbb{R}^n} \Big| \mathrm{e}^{\mathrm{i}x\cdot y}(L^*)^N \Big\{ \Big(1-\varrho\Big(\frac{y}{\delta}\Big)\Big) \partial_y^\alpha S(y+\xi,\xi)\varphi(y+\xi) \Big\} \Big| \,\mathrm{d}y \\
&\leqslant C(\kappa,N)|x|^{-N} \int_{|y|\geqslant\delta} \sum_{j=0}^{N} |y|^{\kappa-2-N+j} \sum_{|\beta|=j} |\partial_y^\beta \varphi(y+\xi)| \,\mathrm{d}y \\
&\quad + C(\kappa,N)|x|^{-N} \sum_{l=1}^{N} \delta^{-l} \int_{\delta\leqslant|y|\leqslant2\delta} \sum_{j=0}^{N-l} |y|^{\kappa-2-N+l+j} \sum_{|\beta|=j} |\partial_y^\beta \varphi(y+\xi)| \,\mathrm{d}y \\
&\leqslant C(\kappa,N)|x|^{-N} \int_{|y|\geqslant\delta} \sum_{j=0}^{N} |y|^{\kappa-2-N+j} \sum_{|\beta|=j} |\partial_y^\beta \varphi(y+\xi)| \,\mathrm{d}y \\
&\leqslant C(\kappa,N)|x|^{-N} \sum_{|\beta|\leqslant N} \int_{|y|\geqslant\delta} (|y|^{\kappa-2}+|y|^{\kappa-2-N}) |\partial_y^\beta \varphi(y+\xi)| \,\mathrm{d}y \\
&\leqslant C(\kappa,N)|x|^{-N}(\delta^{\kappa-2}+\delta^{\kappa-2-N}) \sum_{|\beta|\leqslant N} \int_{\mathbb{R}^n} |\partial_y^\beta \varphi(y+\xi)| \,\mathrm{d}y \\
&\leqslant C(\kappa,\varphi,N)|x|^{-N}(1+\delta^{-N})\delta^{\kappa-2}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
|I_1|+|I_2| &\leqslant C(n,\kappa,\varphi,N)(\delta^{n+\kappa-2}+|x|^{-N}\delta^{\kappa-2}+|x|^{-N}\delta^{\kappa-2-N}) \\
&\leqslant C(n,\kappa,\varphi,N)\{(1+|x|^{-N})\delta^{n+\kappa-2}+2|x|^{-N}\delta^{\kappa-2-N}\} \\
&\leqslant 2C(n,\kappa,\varphi,N)(\delta^{n+\kappa-2}+|x|^{-N}\delta^{\kappa-2-N})
\end{aligned}
$$

for $|x| \geqslant 1$. Taking $\delta^{n+\kappa-2} = |x|^{-N}\delta^{\kappa-2-N}$, we obtain $\delta = |x|^{-N/(N+n)}$, and it follows that

$$
|I_1|+|I_2| \leqslant C(n,\kappa,\varphi,N)|x|^{-(N/(N+n))(n+\kappa-2)} \leqslant C(n,\kappa,\varepsilon,\varphi)|x|^{-n-\kappa+\varepsilon+2}
$$

for $|x| \geqslant 1$, if we take a sufficiently large positive integer N which depends only on κ, n and ε. This proves (2.9). □

Step 3. Let $|\alpha| = 1$. Then

$$
\Big| \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot y} \partial_y^\alpha S(y,\xi)\varphi(y)\,\mathrm{d}y \Big| \leqslant C|x|^{-n-\kappa+\varepsilon+1} \tag{2.10}
$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $|x| \geqslant 1$, where ε is an arbitrarily small positive number, and C is a positive constant depending only on φ, κ, n and ε.

P r o o f of Step 3. Denote the derivative operator $L = L(x, D)$ and $L^* = L^*(x, D)$, which are defined in the proof of Step 2.

By induction and integration by parts, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot y}\partial_y^\alpha S(y,\xi)\varphi(y)\,\mathrm{d}y &= \int_{\mathbb{R}^n} L(\mathrm{e}^{\mathrm{i}x\cdot y})\partial_y^\alpha S(y,\xi)\varphi(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot y} L^*(\partial_y^\alpha S(y,\xi)\varphi(y))\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot y} L^*(\partial_y^\alpha S(y,\xi))\varphi(y)\,\mathrm{d}y + \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot y}\partial_y^\alpha S(y,\xi) L^*(\varphi(y))\,\mathrm{d}y \\
&= \sum_{j=0}^{N-1}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot y} L^*(\partial_y^\alpha S(y,\xi))(L^*)^j\varphi(y)\,\mathrm{d}y + \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot y}\partial_y^\alpha S(y,\xi)(L^*)^N\varphi(y)\,\mathrm{d}y.
\end{aligned}$$

Taking a sufficiently large positive integer $N = N(n,\kappa,\varepsilon)$ such that $N > n+\kappa-1-\varepsilon$, then it follows from (2.5) that

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot y}\partial_y^\alpha S(y,\xi)(L^*)^N\varphi(y)\,\mathrm{d}y\right| &\leqslant C_\kappa \sum_{|\beta|=N} |x|^{-N}\int_{\mathbb{R}^n} |y|^{\kappa-1}|\partial_y^\beta\varphi(y)|\,\mathrm{d}y \\
&\leqslant C(N,\kappa,\varphi)|x|^{-N} = C(n,\kappa,\varphi,\varepsilon)|x|^{-N} \\
&\leqslant C(n,\kappa,\varepsilon,\varphi)|x|^{-n-\kappa+1+\varepsilon},
\end{aligned}$$

and it follows from (2.9) that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot y} L^*(\partial_y^\alpha S(y,\xi))(L^*)^j\varphi(y)\,\mathrm{d}y\right| \\
&\quad\leqslant \sum_{|\beta_1|=2}\sum_{|\beta_2|=j} |x|^{-j-1}\left|\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot y}\partial_y^{\beta_1} S(y,\xi)\partial_y^{\beta_2}\varphi(y)\,\mathrm{d}y\right| \\
&\quad\leqslant \sum_{|\beta_2|=j} C(n,\kappa,\varepsilon,\partial_y^{\beta_2}\varphi)|x|^{-n-\kappa+\varepsilon+1-j} \\
&\quad\leqslant C(n,\kappa,\varepsilon,N,\varphi)|x|^{-n-\kappa+\varepsilon+1} = C(n,\kappa,\varepsilon,\varphi)|x|^{-n-\kappa+1+\varepsilon}
\end{aligned}$$

for $|x| \geqslant 1$ and $0 \leqslant j \leqslant N-1$. Summing up, we complete the proof of Step 3. □

Step 4. Let $|x| \geqslant 1$. Then we have

$$\left|\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot y} S(y,\xi)\varphi(y)\,\mathrm{d}y\right| \leqslant C|x|^{-n-\kappa+\varepsilon}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$, where ε is an arbitrarily small positive number, and C is a positive constant depending only on φ, κ, n and ε.

P r o o f of Step 4. Denote the operator L and L^* as in the proof of Step 3. Then it follows by induction and integration by parts that

$$\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot y} S(y,\xi)\varphi(y)\,\mathrm{d}y = \sum_{j=0}^{N-1}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot y} L^*(S(y,\xi))(L^*)^j\varphi(y)\,\mathrm{d}y + \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot y} S(y,\xi)(L^*)^N\varphi(y)\,\mathrm{d}y.$$

Proceeding as in the proof of Step 3, we can apply (2.5) and (2.10) to obtain the desired inequality in Step 4. □

It is evident that (2.4) follows from the Steps 1–4. This completes the proof of Lemma 2.3. □

Remark 2.4. If $\kappa = 1$, the desired inequality (2.4) still holds. Indeed, we have obtained in the proof of Step 4 that

$$\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot y} S(y,\xi)\varphi(y)\,\mathrm{d}y = \sum_{j=0}^{N-1}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot y} L^*(S(y,\xi))(L^*)^j\varphi(y)\,\mathrm{d}y + \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot y} S(y,\xi)(L^*)^N\varphi(y)\,\mathrm{d}y.$$

Let $|\alpha| = 1$. Then we may apply the same method to

$$\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot y}\partial_y^\alpha S(y,\xi)\varphi(y)\,\mathrm{d}y$$

as in the proof of Step 2, and obtain (2.10) in the case of $\kappa = 1$. Thus the inequality (2.4) with $\kappa = 1$ follows from an argument similar to that of Step 4.

To derive Theorem 1.1 and Theorem 1.2 given in Section 1, we also need to consider the classical orbit associated with the classical Hamiltonian corresponding to (1.1), which is described as follows:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s} f(s) = \kappa|g(s)|^{\kappa-2}g(s), \quad f(t) = x; \\ \dfrac{\mathrm{d}}{\mathrm{d}s} g(s) = -\nabla_x V(s, f(s)), \quad g(t) = \xi \end{cases} \tag{2.11}$$

where $g\colon\ \mathbb{R}\to\mathbb{R}^n$ and $f\colon\ \mathbb{R}\to\mathbb{R}^n$.

The following two lemmas will show some properties of the solutions to (2.11).

Lemma 2.5. *Let* $V \in C^2(\mathbb{R} \times \mathbb{R}^n)$ *satisfy* (1.5) *for all multi-indices* α *with* $|\alpha| = 2$. *Then there exists a unique solution of* (2.11) *with* $\kappa \in [1,2]$. *The solution* $(f,g)(s,t,x,\xi)$ *is* C^1 *in* $(s,t,x,\xi) \in \mathbb{R}^2 \times \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. *Furthermore, denote the* $2n \times 2n$ *Jacobian matrix as*

$$M(s,t,x,\xi) = \Big(\frac{\partial(f,g)}{\partial(x,\xi)}\Big)(s,t,x,\xi);$$

then $\det M(s,t,x,\xi) = 1$ *for all* $(s,t,x,\xi) \in \mathbb{R}^2 \times \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$.

P r o o f. Note that the equations (2.11) are equivalent to the integral equations

$$\text{(2.12)} \qquad \begin{cases} f(s) = x + \kappa \displaystyle\int_t^s |g(\tau)|^{\kappa-2} g(\tau)\,\mathrm{d}\tau; \\ g(s) = \xi - \displaystyle\int_t^s \nabla_x V(\tau, f(\tau))\,\mathrm{d}\tau. \end{cases}$$

Thus the existence, uniqueness and differentiability of the solution $(f,g)(s,t,x,\xi)$ are easily seen by Picard's iteration scheme from (2.12) (see [10]).

Denote $f(s) = (f_1(s), f_2(s), \ldots, f_n(s))_{1\times n}$ and $g(s) = (g_1(s), g_2(s), \ldots, g_n(s))_{1\times n}$, where $f_i\colon \mathbb{R} \to \mathbb{R}$ and $g_i\colon \mathbb{R} \to \mathbb{R}$, for all $1 \leqslant i \leqslant n$.

For all $(s,t,x,\xi) \in \mathbb{R}^2 \times \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, we differentiate (2.12) by the derivative operator $\partial_s \partial_{(x,\xi)}$ and obtain

$$\text{(2.13)} \qquad \begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s}(\nabla_x f_i(s), \nabla_\xi f_i(s))_{1\times 2n} \\ &\quad = \kappa(\kappa-2) g_i(s) |g(s)|^{\kappa-4} \sum_{j=1}^n g_j(s) (\nabla_x g_j(s), \nabla_\xi g_j(s))_{1\times 2n} \\ &\qquad + \kappa |g(s)|^{\kappa-2} (\nabla_x g_i(s), \nabla_\xi g_i(s))_{1\times 2n}, \end{aligned}$$

$$\text{(2.14)} \qquad \begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s}(\nabla_x g_i(s), \nabla_\xi g_i(s))_{1\times 2n} \\ &\quad = -\sum_{j=1}^n (\partial_{x_j}\partial_{x_i} V)(s, f(s)) (\nabla_x f_j(s), \nabla_\xi f_j(s))_{1\times 2n} \end{aligned}$$

for all $1 \leqslant i \leqslant n$. Let $w^{(i)}(s)$ be the $1 \times 2n$ matrix defined by

$$\begin{cases} w^{(i)}(s) = (\nabla_x f_i(s), \nabla_\xi f_i(s))_{1\times 2n}, & 1 \leqslant i \leqslant n; \\ w^{(n+i)}(s) = (\nabla_x g_i(s), \nabla_\xi g_i(s))_{1\times 2n}, & 1 \leqslant i \leqslant n. \end{cases}$$

Set $a_{ij} = \kappa(\kappa-2)g_i(s)|g(s)|^{\kappa-4}g_j(s)$, $b_{ij} = -(\partial_{x_j}\partial_{x_i}V)(s,f(s))$ and $c = \kappa|g(s)|^{\kappa-2}$ for all $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$. By (2.13), we have

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}w^{(i)}(s) = \displaystyle\sum_{j=1}^{n} a_{ij}w^{(n+j)}(s) + cw^{(n+i)}(s), \quad 1 \leqslant i \leqslant n; \\ \dfrac{\mathrm{d}}{\mathrm{d}s}w^{(n+i)}(s) = \displaystyle\sum_{j=1}^{n} b_{ij}w^{(j)}(s), \quad 1 \leqslant i \leqslant n. \end{cases}$$

Hence, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}(\det M(s,t,x,\xi)) = \frac{\mathrm{d}}{\mathrm{d}s}\begin{vmatrix} w^{(1)} \\ w^{(2)} \\ \vdots \\ w^{(2n)} \end{vmatrix}_{2n\times 2n} = \sum_{i=1}^{n}\begin{vmatrix} w^{(1)} \\ \vdots \\ w^{(i-1)} \\ \frac{\mathrm{d}}{\mathrm{d}s}w^{(i)} \\ w^{(i+1)} \\ \vdots \\ w^{(2n)} \end{vmatrix}_{2n\times 2n} + \sum_{i=1}^{n}\begin{vmatrix} w^{(1)} \\ \vdots \\ w^{(n+i-1)} \\ \frac{\mathrm{d}}{\mathrm{d}s}w^{(n+i)} \\ w^{(n+i+1)} \\ \vdots \\ w^{(2n)} \end{vmatrix}_{2n\times 2n}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\begin{vmatrix} w^{(1)} \\ \vdots \\ w^{(i-1)} \\ w^{(n+j)} \\ w^{(i+1)} \\ \vdots \\ w^{(2n)} \end{vmatrix}_{2n\times 2n} + \sum_{i=1}^{n} c\begin{vmatrix} w^{(1)} \\ \vdots \\ w^{(i-1)} \\ w^{(n+i)} \\ w^{(i+1)} \\ \vdots \\ w^{(2n)} \end{vmatrix}_{2n\times 2n} + \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\begin{vmatrix} w^{(1)} \\ \vdots \\ w^{(n+i-1)} \\ w^{(j)} \\ w^{(n+i+1)} \\ \vdots \\ w^{(2n)} \end{vmatrix}_{2n\times 2n} = 0.$$

Therefore

$$\det M(s,t,x,\xi) = \det M(t,t,x,\xi) = \det I_{2n} = 1,$$

where I_{2n} is the $2n \times 2n$ identity matrix. □

Lemma 2.6. *Let $(f,g)(s,t,x,\xi)$ be solutions to (2.11) with $\kappa \in [1,2]$. If $V \in C^2(\mathbb{R}\times\mathbb{R}^n)$ satisfies (1.5) for all multi-indices α with $1 \leqslant |\alpha| \leqslant 2$, then there exist $C_1, C_2 > 0$ such that*

$$\frac{1}{\langle y - f(s,t,x,\xi)\rangle} \leqslant \frac{C_1(1+|t-s|^{\kappa})}{\langle y - x + \kappa(t-s)|\xi|^{\kappa-2}\xi\rangle} \tag{2.15}$$

and

$$\frac{1}{\langle \eta - g(s,t,x,\xi)\rangle} \leqslant \frac{C_2(1+|t-s|)}{\langle \eta - \xi\rangle} \tag{2.16}$$

for all $(y,\eta)\in\mathbb{R}^n\times\mathbb{R}^n$, *where* $C_1=C_1(\kappa,n)$ *depends only on* κ *and* n, $C_2=C_2(n)$ *depends only on* n.

Proof. Since $(f,g)(s,t,x,\xi)$ are solutions to (2.11), using (2.12) we have

$$f(s,t,x,\xi)=x+\kappa\int_t^s|g(\tau,t,x,\xi)|^{\kappa-2}g(\tau,t,x,\xi)\,\mathrm{d}\tau, \tag{2.17}$$

and

$$g(s,t,x,\xi)=\xi-\int_t^s\nabla_xV(\tau,f(\tau,t,x,\xi))\,\mathrm{d}\tau. \tag{2.18}$$

Note that it follows from (1.5) that

$$|\nabla_xV(\tau,f(\tau,t,x,\xi))|\leqslant C(n) \tag{2.19}$$

for all $(\tau,t,x,\xi)\in\mathbb{R}^2\times\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})$, where $C(n)$ is a constant depending only on n. Thus, by (2.17), (2.18) and (2.19), we may apply (2.1) to obtain

$$\begin{aligned}
\big|y-x+\kappa(t-s)|\xi|^{\kappa-2}\xi\big| &\leqslant |y-f(s,t,x,\xi)|+\big|f(s,t,x,\xi)-x+\kappa(t-s)|\xi|^{\kappa-2}\xi\big|\\
&\leqslant |y-f(s,t,x,\xi)|+\left|\int_t^s\kappa\big||g(\tau,t,x,\xi)|^{\kappa-2}g(\tau,t,x,\xi)-|\xi|^{\kappa-2}\xi\big|\,\mathrm{d}\tau\right|\\
&\leqslant |y-f(s,t,x,\xi)|+C_\kappa\left|\int_t^s|g(\tau,t,x,\xi)-\xi|^{\kappa-1}\,\mathrm{d}\tau\right|\\
&= |y-f(s,t,x,\xi)|+C_\kappa\left|\int_t^s\left|\int_t^\tau\nabla_xV(\sigma,f(\sigma,t,x,\xi))\,\mathrm{d}\sigma\right|^{\kappa-1}\mathrm{d}\tau\right|\\
&\leqslant |y-f(s,t,x,\xi)|+C(n,\kappa)\left|\int_t^s|\tau-t|^{\kappa-1}\,\mathrm{d}\tau\right|\\
&= |y-f(s,t,x,\xi)|+C(n,\kappa)|t-s|^\kappa.
\end{aligned}$$

Then

$$\begin{aligned}
\langle y-x+\kappa(t-s)|\xi|^{\kappa-2}\xi\rangle &\leqslant \langle|y-f(s,t,x,\xi)|+C(n,\kappa)|t-s|^\kappa\rangle\\
&\leqslant C(n,\kappa)\langle|y-f(s,t,x,\xi)|+|t-s|^\kappa\rangle\\
&\leqslant C(n,\kappa)\langle y-f(s,t,x,\xi)\rangle\langle|t-s|^\kappa\rangle,
\end{aligned}$$

which yields (2.15).

Similarly, by (2.18) and (2.19) we have

$$\begin{aligned}
|\eta-\xi| &\leqslant |\eta-g(s,t,x,\xi)|+|g(s,t,x,\xi)-\xi|\\
&\leqslant |\eta-g(s,t,x,\xi)|+\left|\int_t^s\nabla_xV(\sigma,f(\sigma,t,x,\xi))\,\mathrm{d}\sigma\right|\\
&\leqslant |\eta-g(s,t,x,\xi)|+C(n)|t-s|.
\end{aligned}$$

Then

$$\begin{aligned}\langle \eta - \xi \rangle &\leqslant \langle |\eta - g(s,t,x,\xi)| + C(n)|t-s| \rangle \\ &\leqslant C(n)\langle |\eta - g(s,t,x,\xi)| + |t-s| \rangle \\ &\leqslant C(n)\langle \eta - g(s,t,x,\xi)\rangle \langle t-s \rangle,\end{aligned}$$

which implies (2.16). □

3. Proof of Theorem 1.1

We only consider the case $t \in [0,T]$, since the case $t \in [-T,0]$ can be treated similarly. Let $\varphi(t,x) = \mathrm{e}^{-\mathrm{i}t(-\Delta)^{\kappa/2}}\varphi_0(x)$. It follows by STFT and integration by parts that

$$\mathrm{i}\partial_t V_{\varphi(t,\cdot)}u(t,x,\xi) = V_{\varphi(t,\cdot)}(\mathrm{i}\partial_t u)(t,x,\xi) - V_{\mathrm{i}\partial_t\varphi(t,\cdot)}u(t,x,\xi) \tag{3.1}$$

and

$$\begin{aligned} &V_{\varphi(t,\cdot)}((-\Delta)^{\kappa/2}u)(t,x,\xi) \\ &\quad = |\xi|^\kappa V_{\varphi(t,\cdot)}u(t,x,\xi) - \mathrm{i}\kappa|\xi|^{\kappa-2}\xi\cdot\nabla_x V_{\varphi(t,\cdot)}u(t,x,\xi) + \widetilde{R}u(t,x,\xi).\end{aligned} \tag{3.2}$$

Here we denote

$$\widetilde{R}u(t,x,\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}y\cdot\xi}\psi(t,y-x,\xi)u(t,y)\,\mathrm{d}y \tag{3.3}$$

with

$$\psi(t,x,\xi) = \mathcal{F}^{-1}_{z\to x}\{S(z,\xi)\mathcal{F}_{w\to z}[\overline{\varphi(t,w)}]\} \tag{3.4}$$

where $S(z,\xi)$ is defined by (2.3). Using integration by parts, we may apply Taylor's Theorem to the potential $V(t,\cdot)$ and obtain

$$\begin{aligned} &V_{\varphi(t,\cdot)}(Vu)(t,x,\xi) \\ &= \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}y\cdot\xi}\overline{\varphi(t,y-x)}u(t,y) \\ &\quad\times\Big(V(t,x) + \nabla_x V(t,x)\cdot(y-x) + \sum_{k,j=1}^n (y_k-x_k)(y_j-x_j)\widetilde{V}_{kj}(t,x,y)\Big)\,\mathrm{d}y \\ &= \{V(t,x) + \mathrm{i}\nabla_x V(t,x)\cdot\nabla_\xi - \nabla_x V(t,x)\cdot x\}V_{\varphi(t,\cdot)}u(t,x,\xi) + Ru(t,x,\xi),\end{aligned} \tag{3.5}$$

where

$$Ru(t,x,\xi) = \sum_{k,j=1}^n \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}y\cdot\xi}\overline{\varphi(t,y-x)}u(t,y)(y_k-x_k)(y_j-x_j)\widetilde{V}_{kj}(t,x,y)\,\mathrm{d}y \tag{3.6}$$

and

$$\widetilde{V}_{kj}(t,x,y)=\int_0^1 \partial_{x_k}\partial_{x_j}V(t,x+\theta(y-x))(1-\theta)\,\mathrm{d}\theta. \tag{3.7}$$

Since $\mathrm{i}\partial_t\varphi(t,x)=(-\Delta)^{\kappa/2}\varphi(t,x)$, we have

$$V_{\mathrm{i}\partial_t\varphi(t,\cdot)}u(t,x,\xi)=V_{(-\Delta)^{\kappa/2}\varphi(t,\cdot)}u(t,x,\xi). \tag{3.8}$$

Combining (3.1), (3.2), (3.5) and (3.8), we may transform the initial value problem (1.1) via STFT with the window function $\varphi(t,x)$ into a first order partial differential equation

$$\begin{cases}(\mathrm{i}\partial_t+\mathrm{i}\kappa|\xi|^{\kappa-2}\xi\cdot\nabla_x-\mathrm{i}\nabla_xV(t,x)\cdot\nabla_\xi-|\xi|^\kappa-V(t,x)\\ \quad+\nabla_xV(t,x)\cdot x)V_{\varphi(t,\cdot)}u(t,x,\xi)-Ru(t,x,\xi)-Hu(t,x,\xi)=0,\\ V_{\varphi(0,\cdot)}u(0,x,\xi)=V_{\varphi_0}u_0(x,\xi).\end{cases}$$

where

$$Hu(t,x,\xi)=\widetilde{R}u(t,x,\xi)-V_{(-\Delta)^{\kappa/2}\varphi(t,\cdot)}u(t,x,\xi). \tag{3.9}$$

By the method of characteristics, we have

$$\begin{aligned}V_{\varphi(t,\cdot)}u(t,x,\xi)&\\ &=\mathrm{e}^{-\mathrm{i}\int_0^t h(s,t,x,\xi)\,\mathrm{d}s}(V_{\varphi_0}u_0(f(0,t,x,\xi),g(0,t,x,\xi))\\ &\quad-\mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}\int_0^\sigma h(s,t,x,\xi)\,\mathrm{d}s}\{Ru+Hu\}(\sigma,f(\sigma,t,x,\xi),g(\sigma,t,x,\xi))\,\mathrm{d}\sigma),\end{aligned} \tag{3.10}$$

where $f(s,t,x,\xi)$ and $g(s,t,x,\xi)$ are solutions to (2.11), and

$$h(s,t,x,\xi)=|g(s,t,x,\xi)|^\kappa+V(s,f(s,t,x,\xi))-\nabla_xV(s,f(s,t,x,\xi))\cdot f(s,t,x,\xi).$$

Then it follows from (3.10) that

$$|V_{\varphi(t,\cdot)}u(t,x,\xi)|\leqslant|I_1|+\int_0^t|I_2|\,\mathrm{d}\sigma+\int_0^t|I_3|\,\mathrm{d}\sigma, \tag{3.11}$$

where

$$\begin{aligned}I_1&=V_{\varphi_0}u_0(f(0,t,x,\xi),g(0,t,x,\xi)),\\ I_2&=Ru(\sigma,f(\sigma,t,x,\xi),g(\sigma,t,x,\xi)),\\ I_3&=Hu(\sigma,f(\sigma,t,x,\xi),g(\sigma,t,x,\xi)).\end{aligned} \tag{3.12}$$

Applying the Minkowski inequality, we obtain

$$\tag{3.13} \begin{aligned} \|u(t,\cdot)\|_{M^{p,p}_{\varphi(t,\cdot)}} &= \|V_{\varphi(t,\cdot)}u(t,x,\xi)\|_{L^p_{x,\xi}} \\ &\leqslant \|I_1\|_{L^p_{x,\xi}} + \int_0^t \|I_2\|_{L^p_{x,\xi}}\,\mathrm{d}\sigma + \int_0^t \|I_3\|_{L^p_{x,\xi}}\,\mathrm{d}\sigma. \end{aligned}$$

Now we make the change of variables $\Xi = g(0,t,x,\xi)$ and $X = f(0,t,x,\xi)$. Then it follows from Lemma 2.5 and the inverse mapping theorem that

$$\Big|\frac{\partial(x,\xi)}{\partial(X,\Xi)}\Big| = 1$$

for all $(t,X,\Xi) \in \mathbb{R}\times\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})$. So we can easily obtain $\|I_1\|_{L^\infty_{x,\xi}} = \|u_0\|_{M^{\infty,\infty}_{\varphi_0}}$, and

$$\tag{3.14} \|I_1\|_{L^p_{x,\xi}} = \Big(\iint_{\mathbb{R}^{2n}} |V_{\varphi_0}u_0(X,\Xi)|^p\Big|\frac{\partial(x,\xi)}{\partial(X,\Xi)}\Big|\,\mathrm{d}X\,\mathrm{d}\Xi\Big)^{1/p} = \|u_0\|_{M^{p,p}_{\varphi_0}}$$

for all $p \in [1,\infty)$.

As for I_2, by (3.6) and the inverse formula of STFT for u we obtain

$$\begin{aligned} Ru(t,x,\xi) = \frac{1}{\|\varphi(t,\cdot)\|^2_{L^2}} \sum_{k,j=1}^n \iiint_{\mathbb{R}^{3n}} &\mathrm{e}^{\mathrm{i}y\cdot(\eta-\xi)}\varphi_{kj}(t,y-x)\widetilde{V}_{kj}(t,x,y)\varphi(t,y-z) \\ &\times V_{\varphi(t,\cdot)}u(t,z,\eta)\,\mathrm{d}y\,\mathrm{d}z\,\bar{\mathrm{d}}\eta, \end{aligned}$$

where $\varphi_{kj}(t,y) = y_k y_j \overline{\varphi(t,y)}$. Take a large positive integer $N = [n/2]+1$ such that $2N > n$. Note that

$$(1-\Delta_y)^N \mathrm{e}^{\mathrm{i}y\cdot(\eta-g(\sigma,t,x,\xi))} = \langle \eta - g(\sigma,t,x,\xi)\rangle^{2N}\mathrm{e}^{\mathrm{i}y\cdot(\eta-g(\sigma,t,x,\xi))}.$$

By integration by parts, we obtain

$$\begin{aligned} &\|I_2\|_{L^p_{x,\xi}} = \|Ru(\sigma,f(\sigma,t,x,\xi),g(\sigma,t,x,\xi))\|_{L^p_{x,\xi}} \\ &\leqslant \frac{1}{\|\varphi(\sigma,\cdot)\|^2_{L^2}} \sum_{k,j=1}^n \Big\| \iiint_{\mathbb{R}^{3n}} |(1-\Delta_y)^N\{\varphi_{kj}(\sigma,y-f(\sigma,t,x,\xi))\varphi(\sigma,y-z) \\ &\qquad\qquad \times \widetilde{V}_{kj}(\sigma,f(\sigma,t,x,\xi),y)\}|\frac{|V_{\varphi(\sigma,\cdot)}u(\sigma,z,\eta)|}{\langle\eta-g(\sigma,t,x,\xi)\rangle^{2N}}\,\mathrm{d}y\,\mathrm{d}z\,\bar{\mathrm{d}}\eta\Big\|_{L^p_{x,\xi}} \\ &\leqslant \frac{1}{\|\varphi(\sigma,\cdot)\|^2_{L^2}} \sum_{k,j=1}^n \sum_{|\beta_1|+|\beta_2|+|\beta_3|\leqslant 2N} \Big\| \iiint_{\mathbb{R}^{3n}} |\partial_y^{\beta_1}\varphi_{kj}(\sigma,y-f(\sigma,t,x,\xi)) \\ &\quad \times \partial_y^{\beta_2}\varphi(\sigma,y-z)\partial_y^{\beta_3}\widetilde{V}_{kj}(\sigma,f(\sigma,t,x,\xi),y)|\frac{|V_{\varphi(\sigma,\cdot)}u(\sigma,z,\eta)|}{\langle\eta-g(\sigma,t,x,\xi)\rangle^{2N}}\,\mathrm{d}y\,\mathrm{d}z\,\bar{\mathrm{d}}\eta\Big\|_{L^p_{x,\xi}}. \end{aligned}$$

Since $|\partial_y^{\beta_3}\widetilde{V}_{kj}(\sigma, f(\sigma,t,x,\xi), y)| \leqslant C_{\beta_3}$ for $C_{\beta_3} > 0$, we make the change of variables $\Xi = g(\sigma,t,x,\xi)$ and $X = f(\sigma,t,x,\xi)$ when $p \in [1,\infty)$, and then apply Lemma 2.5 and Young's inequality twice to obtain that

$$
\begin{aligned}
(3.15)\quad \|I_2\|_{L^p_{x,\xi}} &\leqslant \frac{1}{\|\varphi(\sigma,\cdot)\|_{L^2}^2} \sum_{k,j=1}^{n} \sum_{|\beta_1|+|\beta_2|+|\beta_3|\leqslant 2N} \\
&\quad C_{\beta_3}\bigg\{ \iint_{\mathbb{R}^{2n}} \bigg(\iint_{\mathbb{R}^{2n}} \frac{|\partial_y^{\beta_1}\varphi_{kj}(\sigma, y-X)|}{\langle \eta - \Xi\rangle^{2N}} \\
&\quad \times \int_{\mathbb{R}^n} |\partial_y^{\beta_2}\varphi(\sigma, y-z) V_{\varphi(\sigma,\cdot)} u(\sigma,z,\eta)| \,\mathrm{d}z \,\bar{\mathrm{d}}\eta \,\mathrm{d}y \bigg)^p \Big| \frac{\partial(x,\xi)}{\partial(X,\Xi)} \Big| \,\mathrm{d}X \,\mathrm{d}\Xi \bigg\}^{1/p} \\
&\leqslant \frac{1}{\|\varphi_0\|_{L^2}^2} \sum_{k,j=1}^{n} \sum_{|\beta_1|+|\beta_2|+|\beta_3|\leqslant 2N} C_{\beta_3} \|\partial_y^{\beta_1}\varphi_{kj}(\sigma,y)\|_{L^1_y} \|\langle\eta\rangle^{-2N}\|_{L^1_\eta} \\
&\quad \times \|\partial_y^{\beta_2}\varphi(\sigma,\cdot)\|_{L^1} \|V_{\varphi(\sigma,\cdot)} u(\sigma,z,\eta)\|_{L^p_{z,\eta}}.
\end{aligned}
$$

Since $\mathcal{F}\varphi_0 \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$, it follows by integration by parts that

$$
\begin{aligned}
(3.16)\quad |\partial_y^{\beta_2}\varphi(\sigma,y)| &= \Big| \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}y\cdot\xi} \mathrm{e}^{-\mathrm{i}\sigma|\xi|^\kappa} (\mathrm{i}\xi)^{\beta_2} \mathcal{F}\varphi_0(\xi) \,\bar{\mathrm{d}}\xi \Big| \\
&= \Big| \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}y\cdot\xi} (L^*)^m \{\mathrm{e}^{-\mathrm{i}\sigma|\xi|^\kappa} (\mathrm{i}\xi)^{\beta_2} \mathcal{F}\varphi_0(\xi)\} \,\bar{\mathrm{d}}\xi \Big| \\
&\leqslant C(\beta_2, m, T, \kappa) \langle y\rangle^{-m}
\end{aligned}
$$

for all $m \in \mathbb{N}$ and $\sigma \in [0,T]$, where the derivative operator L^* is defined by

$$
L^* = -\frac{y\cdot\nabla_\xi}{\mathrm{i}|y|^2},
$$

and $C(\beta_2, m, T, \kappa)$ depends only on β_2, m, T and κ.

Similarly, we have

$$
(3.17)\quad |\partial_y^{\beta_1}\varphi_{kj}(\sigma,y)| \leqslant C(\beta_1, m, T, \kappa)\langle y\rangle^{-m}
$$

for all $m \in \mathbb{N}$ and $\sigma \in [0,T]$, where $C(\beta_1, m, T, \kappa)$ depends only on β_1, m, T and κ. From (3.15), when $p \in [1,\infty)$, we have

$$
(3.18)\quad \|I_2\|_{L^p_{x,\xi}} \leqslant C(n,\kappa,T) \|u(\sigma,\cdot)\|_{M^{p,p}_{\varphi(\sigma,\cdot)}}
$$

for all $\sigma \in [0,t]$ and $t \in [0,T]$. The case $p = \infty$ can be easily obtained from (3.15) similarly. Now we consider the estimates for I_3. By (3.9) and the inverse formula of

STFT for u, we obtain

$$
\begin{aligned}
&Hu(t,x,\xi)\\
&= \frac{1}{\|\varphi(t,\cdot)\|_{L^2}^2} \iiint_{\mathbb{R}^{3n}} e^{iy\cdot(\eta-\xi)} \widetilde{\psi}(t,y-x,\xi)\varphi(t,y-z)V_{\varphi(t,\cdot)}u(t,z,\eta)\,dy\,dz\,\bar{d}\eta,
\end{aligned}
$$

where

$$
\widetilde{\psi}(t,x,\xi) = \psi(t,x,\xi) - \overline{(-\Delta)^{\kappa/2}\varphi(t,x)}. \tag{3.19}
$$

Using an argument similar to that for I_2 and the change of variables with respect to (x,ξ), we have

$$
\begin{aligned}
\|I_3\|_{L^p_{x,\xi}} &\leqslant \frac{1}{\|\varphi(\sigma,\cdot)\|_{L^2}^2}\\
&\quad \times \Big\| \iiint_{\mathbb{R}^{3n}} |(1-\Delta_y)^N\{\widetilde{\psi}(\sigma,y-f(\sigma,t,x,\xi),g(\sigma,t,x,\xi))\varphi(\sigma,y-z)\}|\\
&\qquad\qquad \times \frac{|V_{\varphi(\sigma,\cdot)}u(\sigma,z,\eta)|}{\langle \eta - g(\sigma,t,x,\xi)\rangle^{2N}}\,dy\,dz\,\bar{d}\eta \Big\|_{L^p_{x,\xi}}\\
&\leqslant \frac{1}{\|\varphi(\sigma,\cdot)\|_{L^2}^2}\\
&\quad \times \sum_{|\beta_1|+|\beta_2|\leqslant 2N} \Big\| \iiint_{\mathbb{R}^{3n}} |\partial_y^{\beta_1}\widetilde{\psi}(\sigma,y-f(\sigma,t,x,\xi),g(\sigma,t,x,\xi))\partial_y^{\beta_2}\varphi(\sigma,y-z)|\\
&\qquad\qquad \times \frac{|V_{\varphi(\sigma,\cdot)}u(\sigma,z,\eta)|}{\langle \eta - g(\sigma,t,x,\xi)\rangle^{2N}}\,dy\,dz\,\bar{d}\eta \Big\|_{L^p_{x,\xi}}\\
&\leqslant \frac{1}{\|\varphi_0\|_{L^2}^2} \sum_{|\beta_1|+|\beta_2|\leqslant 2N} \big\| \|\partial_y^{\beta_1}\widetilde{\psi}(\sigma,y,\xi)\|_{L^\infty_\xi} \big\|_{L^1_y} \|\langle\eta\rangle^{-2N}\|_{L^1_\eta}\\
&\quad \times \|\partial_y^{\beta_2}\varphi(\sigma,\cdot)\|_{L^1} \|V_{\varphi(\sigma,\cdot)}u(\sigma,z,\eta)\|_{L^p_{z,\eta}},
\end{aligned} \tag{3.20}
$$

where the last inequality follows from Young's inequality. Since $\mathcal{F}\varphi_0 \in C_c^\infty(\mathbb{R}^n\setminus\{0\})$, it follows by integration by parts that

$$
\begin{aligned}
|\partial_y^{\beta_1}(-\Delta_y)^{\kappa/2}\varphi(\sigma,y)| &= \left| \int_{\mathbb{R}^n} e^{iy\cdot\xi} e^{-i\sigma|\xi|^\kappa} (i\xi)^{\beta_1}|\xi|^\kappa \mathcal{F}\varphi_0(\xi)\,\bar{d}\xi \right|\\
&= \left| \int_{\mathbb{R}^n} e^{iy\cdot\xi} (L^*)^m\{e^{-i\sigma|\xi|^\kappa}(i\xi)^{\beta_1}|\xi|^\kappa\mathcal{F}\varphi_0(\xi)\}\,\bar{d}\xi \right|\\
&\leqslant C(\beta_1,m,T,\kappa)\langle y\rangle^{-m}
\end{aligned} \tag{3.21}
$$

for all $m \in \mathbb{N}$ and $\sigma \in [0,T]$, where $C(\beta_1, m, T, \kappa)$ depends only on β_1, m, T and κ. Moreover, there exists a function $\varphi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ satisfying $\varphi\mathcal{F}[\overline{\varphi_0}] \equiv \mathcal{F}[\overline{\varphi_0}]$. Thus by (3.4) we obtain

$$\begin{aligned}\partial_y^{\beta_1}\psi(\sigma,y,\xi) &= \mathcal{F}^{-1}_{z\to y}\{S(z,\xi)\mathrm{e}^{\mathrm{i}\sigma|z|^\kappa}(\mathrm{i}z)^{\beta_1}\varphi(z)\mathcal{F}[\overline{\varphi_0}](z)\} \\ &= \overline{\partial_y^{\beta_1}\varphi(\sigma,\cdot)} * \mathcal{F}^{-1}_{z\to y}\{S(z,\xi)\varphi(z)\},\end{aligned} \tag{3.22}$$

where $*$ denotes the convolution operator. Then it follows from Lemma 2.3 and Remark 2.4 that

$$\|\partial_y^{\beta_1}\psi(\sigma,y,\xi)\|_{L^\infty_\xi} \leqslant C|\partial_y^{\beta_1}\varphi(\sigma,\cdot)| * \langle y\rangle^{-n-\kappa+\varepsilon}, \tag{3.23}$$

where ε is an arbitrarily small positive number, and C is a positive constant depending only on φ, κ, n and ε. Thus by Young's inequality and (3.16) with a large number $m \in \mathbb{N}$, we have

$$\|\|\partial_y^{\beta_1}\psi(\sigma,y,\xi)\|_{L^\infty_\xi}\|_{L^1_y} \leqslant C\|\partial_y^{\beta_1}\varphi(\sigma,y)\|_{L^1_y}\|\langle\cdot\rangle^{-n-\kappa+\varepsilon}\|_{L^1} \leqslant C(n,\kappa,T) \tag{3.24}$$

for $\sigma \in [0,T]$, where $C(n,\kappa,T)$ depends only on κ, n and T.

Then it follows from (3.19), (3.21) and (3.24) that

$$\Big\|\|\partial_y^{\beta_1}\widetilde{\psi}(\sigma,y,\xi)\|_{L^\infty_\xi}\Big\|_{L^1_y} \leqslant C(n,\kappa,T). \tag{3.25}$$

Hence, we obtain from (3.20) that

$$\|I_3\|_{L^p_{x,\xi}} \leqslant C(n,\kappa,T)\|u(\sigma,\cdot)\|_{M^{p,p}_{\varphi(\sigma,\cdot)}} \tag{3.26}$$

for all $\sigma \in [0,t]$ and $t \in [0,T]$. From (3.14), (3.18), (3.26) and (3.13), we have

$$\|u(t,\cdot)\|_{M^{p,p}_{\varphi(t,\cdot)}} \leqslant \|u_0\|_{M^{p,p}_{\varphi_0}} + C(n,\kappa,T)\int_0^t \|u(\sigma,\cdot)\|_{M^{p,p}_{\varphi(\sigma,\cdot)}}\,\mathrm{d}\sigma$$

for $t \in [0,T]$. Thus Gronwall's inequality yields

$$\|u(t,\cdot)\|_{M^{p,p}_{\varphi(t,\cdot)}} \leqslant C\|u_0\|_{M^{p,p}_{\varphi_0}} \tag{3.27}$$

for $t \in [0,T]$, where C is a positive constant depending only on κ, n and T. This completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

We only consider the case $t \in [0,T]$, since the case $t \in [-T,0]$ can be treated similarly. In this section, we shall first consider the case $(p,q)=(\infty,1)$, next the case $(p,q)=(1,\infty)$, and finally the general case (p,q).

Take a positive integer $N=[n/2]+1$ so that $2N>n$. Note that

$$(1-\Delta_y)^N \mathrm{e}^{\mathrm{i}y\cdot(\eta-g(0,t,x,\xi))} = \langle \eta - g(0,t,x,\xi)\rangle^{2N}\mathrm{e}^{\mathrm{i}y\cdot(\eta-g(0,t,x,\xi))}.$$

By the inverse formula of STFT for u_0, integration by parts and (2.16) in Lemma 2.6, we obtain

$$\begin{aligned}(4.1)\quad |I_1| &= \frac{1}{\|\varphi_0\|_{L^2}^2}\bigg|\iiint_{\mathbb{R}^{3n}} \mathrm{e}^{\mathrm{i}y\cdot(\eta-g(0,t,x,\xi))}\overline{\varphi_0(y-f(0,t,x,\xi))}\varphi_0(y-z)\\ &\qquad\qquad\qquad \times V_{\varphi_0}u_0(z,\eta)\,\mathrm{d}y\,\mathrm{d}z\,\bar{\mathrm{d}}\eta\bigg|\\ &\leqslant \frac{1}{\|\varphi_0\|_{L^2}^2}\iiint_{\mathbb{R}^{3n}} |(1-\Delta_y)^N\{\overline{\varphi_0(y-f(0,t,x,\xi))}\varphi_0(y-z)\}|\\ &\qquad\qquad\qquad \times \frac{|V_{\varphi_0}u_0(z,\eta)|}{\langle \eta-g(0,t,x,\xi)\rangle^{2N}}\,\mathrm{d}y\,\mathrm{d}z\,\bar{\mathrm{d}}\eta\\ &\leqslant C(1+|t|)^{2N}\sum_{|\beta_1|+|\beta_2|\leqslant 2N}\iiint_{\mathbb{R}^{3n}} |\partial_y^{\beta_1}\varphi_0(y-f(0,t,x,\xi))||\partial_y^{\beta_2}\varphi_0(y-z)|\\ &\qquad\qquad\qquad \times \frac{|V_{\varphi_0}u_0(z,\eta)|}{\langle \eta-\xi\rangle^{2N}}\,\mathrm{d}y\,\mathrm{d}z\,\bar{\mathrm{d}}\eta.\end{aligned}$$

Now we apply Fubini's theorem to obtain

$$\begin{aligned}&\big\|\|I_1\|_{L_x^\infty}\big\|_{L_\xi^1}\\ &\leqslant C(1+|t|)^{2N}\sum_{|\beta_1|+|\beta_2|\leqslant 2N}\|\partial_y^{\beta_2}\varphi_0\|_{L^1}\\ &\quad\times\bigg\|\bigg\|\iint_{\mathbb{R}^{2n}}|\partial_y^{\beta_1}\varphi_0(y-f(0,t,x,\xi))|\frac{\|V_{\varphi_0}u_0(z,\eta)\|_{L_z^\infty}}{\langle\eta-\xi\rangle^{2N}}\,\mathrm{d}y\,\bar{\mathrm{d}}\eta\bigg\|_{L_x^\infty}\bigg\|_{L_\xi^1}\\ &\leqslant C(1+|t|)^{2N}\sum_{|\beta_1|+|\beta_2|\leqslant 2N}\|\partial_y^{\beta_2}\varphi_0\|_{L^1}\|\partial_y^{\beta_1}\varphi_0\|_{L^1}\|\langle\cdot\rangle^{-2N}\|_{L^1}\big\|\|V_{\varphi_0}u_0(z,\eta)\|_{L_z^\infty}\big\|_{L_\eta^1}\\ &\leqslant C(n,T)\|u_0\|_{M_{\varphi_0}^{\infty,1}}\end{aligned}$$

for $t \in [0,T]$, where $C(n,T)$ depends only on n and T. As for I_2, we have

(4.2)

$$|I_2| \leqslant \frac{1}{\|\varphi(\sigma,\cdot)\|_{L^2}^2} \sum_{k,j=1}^{n} \iiint_{\mathbb{R}^{3n}} |(1-\Delta_y)^N \{\varphi_{kj}(\sigma, y - f(\sigma,t,x,\xi))\varphi(\sigma, y-z) \times \widetilde{V}_{kj}(\sigma, f(\sigma,t,x,\xi), y)\}| \frac{|V_{\varphi(\sigma,\cdot)}u(\sigma,z,\eta)|}{\langle \eta - g(\sigma,t,x,\xi)\rangle^{2N}} \,\mathrm{d}y\,\mathrm{d}z\,\bar{\mathrm{d}}\eta$$

$$\leqslant \frac{C(1+|t-\sigma|)^{2N}}{\|\varphi(\sigma,\cdot)\|_{L^2}^2} \sum_{k,j=1}^{n} \sum_{|\beta_1|+|\beta_2|+|\beta_3|\leqslant 2N} \iiint_{\mathbb{R}^{3n}} |\partial_y^{\beta_1}\varphi_{kj}(\sigma, y - f(\sigma,t,x,\xi))| \times |\partial_y^{\beta_2}\varphi(\sigma,y-z)\partial_y^{\beta_3}\widetilde{V}_{kj}(\sigma, f(\sigma,t,x,\xi),y)| \frac{|V_{\varphi(\sigma,\cdot)}u(\sigma,z,\eta)|}{\langle \eta-\xi\rangle^{2N}} \,\mathrm{d}y\,\mathrm{d}z\,\bar{\mathrm{d}}\eta,$$

where φ_{kj} and $\widetilde{V}_{kj}$ are defined in the proof of Theorem 1.1. Since

$$|\partial_y^{\beta_3}\widetilde{V}_{kj}(\sigma, f(\sigma,t,x,\xi),y)| \leqslant C_{\beta_3} \tag{4.3}$$

for $C_{\beta_3} > 0$, we obtain

$$\big\|\|I_2\|_{L_x^\infty}\big\|_{L_\xi^1} \leqslant \frac{C(1+|t-\sigma|)^{2N}}{\|\varphi(\sigma,\cdot)\|_{L^2}^2} \sum_{k,j=1}^{n} \sum_{|\beta_1|+|\beta_2|+|\beta_3|\leqslant 2N} C_{\beta_3} \Big\|\Big\| \iiint_{\mathbb{R}^{3n}} |\partial_y^{\beta_1}\varphi_{kj}(\sigma, y - f(\sigma,t,x,\xi))| \times |\partial_z^{\beta_2}\varphi(\sigma,y-z)| \frac{|V_{\varphi(\sigma,\cdot)}u(\sigma,z,\eta)|}{\langle \eta-\xi\rangle^{2N}} \,\mathrm{d}y\,\mathrm{d}z\,\bar{\mathrm{d}}\eta \Big\|_{L_x^\infty}\Big\|_{L_\xi^1}$$

$$\leqslant \frac{C(1+T)^{2N}}{\|\varphi_0\|_{L^2}^2} \sum_{k,j=1}^{n} \sum_{|\beta_1|+|\beta_2|+|\beta_3|\leqslant 2N} C_{\beta_3} \|\partial_z^{\beta_2}\varphi(\sigma,z)\|_{L_z^1} \|\partial_y^{\beta_1}\varphi_{kj}(\sigma,y)\|_{L_y^1} \times \|\langle\cdot\rangle^{-2N}\|_{L^1} \big\|\|V_{\varphi(\sigma,\cdot)}u(\sigma,z,\eta)\|_{L_z^\infty}\big\|_{L_\eta^1}$$

$$\leqslant C(n,\kappa,T)\|u(\sigma,\cdot)\|_{M^{\infty,1}_{\varphi(\sigma,\cdot)}}$$

for all $t \in [0,T]$ and $\sigma \in [0,t]$, where $C(n,\kappa,T)$ depends only on n, κ and T. Similarly, we have

(4.4)

$$|I_3| \leqslant \frac{C(1+|t-\sigma|)^{2N}}{\|\varphi(\sigma,\cdot)\|_{L^2}^2} \sum_{|\beta_1|+|\beta_2|\leqslant 2N} \iiint_{\mathbb{R}^{3n}} |\partial_y^{\beta_1}\widetilde{\psi}(\sigma, y - f(\sigma,t,x,\xi), g(\sigma,t,x,\xi))| \times |\partial_y^{\beta_2}\varphi(\sigma,y-z)| \frac{|V_{\varphi(\sigma,\cdot)}u(\sigma,z,\eta)|}{\langle \eta-\xi\rangle^{2N}} \,\mathrm{d}y\,\mathrm{d}z\,\bar{\mathrm{d}}\eta,$$

where $\widetilde{\psi}$ is defined by (3.19). Then it follows by Fubini's theorem, Minkowski inequality and (3.25) that

$$\begin{aligned}\big\|\|I_3\|_{L^\infty_x}\big\|_{L^1_\xi} &\leqslant \frac{C(1+|t-\sigma|)^{2N}}{\|\varphi(\sigma,\cdot)\|^2_{L^2}} \sum_{|\beta_1|+|\beta_2|\leqslant 2N} \\ &\qquad \Bigg\|\Bigg\|\iiint_{\mathbb{R}^{3n}} |\partial_y^{\beta_1}\widetilde{\psi}(\sigma, y-f(\sigma,t,x,\xi), g(\sigma,t,x,\xi))| \\ &\qquad\qquad \times |\partial_z^{\beta_2}\varphi(\sigma,y-z)| \frac{|V_{\varphi(\sigma,\cdot)}u(\sigma,z,\eta)|}{\langle \eta-\xi\rangle^{2N}}\,\mathrm{d}y\,\mathrm{d}z\,\bar{\mathrm{d}}\eta\Bigg\|_{L^\infty_x}\Bigg\|_{L^1_\xi} \\ &\leqslant \frac{C(1+T)^{2N}}{\|\varphi_0\|^2_{L^2}} \sum_{|\beta_1|+|\beta_2|\leqslant 2N} \|\partial_z^{\beta_2}\varphi(\sigma,z)\|_{L^1_z}\big\|\|\partial_y^{\beta_1}\widetilde{\psi}(\sigma,y,\xi)\|_{L^\infty_\xi}\big\|_{L^1_y} \\ &\qquad\qquad \times \|\langle\cdot\rangle^{-2N}\|_{L^1}\big\|\|V_{\varphi(\sigma,\cdot)}u(\sigma,z,\eta)\|_{L^\infty_z}\big\|_{L^1_\eta} \\ &\leqslant C(n,\kappa,T)\|u(\sigma,\cdot)\|_{M^{\infty,1}_{\varphi(\sigma,\cdot)}}\end{aligned}$$

for all $t\in[0,T]$ and $\sigma\in[0,t]$, where $C(n,\kappa,T)$ depends only on κ, n and T. Hence, using (3.11) and the Minkowski inequality, we obtain

$$\begin{aligned}\|u(t,\cdot)\|_{M^{\infty,1}_{\varphi(t,\cdot)}} &\leqslant \big\|\|I_1\|_{L^\infty_x}\big\|_{L^1_\xi} + \int_0^t \big\|\|I_2\|_{L^\infty_x}\big\|_{L^1_\xi}\,\mathrm{d}\sigma + \int_0^t \big\|\|I_3\|_{L^\infty_x}\big\|_{L^1_\xi}\,\mathrm{d}\sigma \\ &\leqslant C(n,\kappa,T)\bigg\{\|u_0\|_{M^{\infty,1}_{\varphi_0}} + \int_0^t \|u(\sigma,\cdot)\|_{M^{\infty,1}_{\varphi(\sigma,\cdot)}}\,\mathrm{d}\sigma\bigg\}\end{aligned} \tag{4.5}$$

for all $t\in[0,T]$. Thus we may apply Gronwall's inequality to (4.5) and obtain

$$\|u(t,\cdot)\|_{M^{\infty,1}_{\varphi(t,\cdot)}} \leqslant C\|u_0\|_{M^{\infty,1}_{\varphi_0}} \tag{4.6}$$

for all $t\in[0,T]$ and a positive constant C depending only on κ, n and T, which concludes the case $(p,q)=(\infty,1)$.

Next, we consider the case $(p,q)=(1,\infty)$. Take $\varepsilon=\kappa-1/2>0$, $N=[n/2]+1$ and $\tau=n+1/2$. Thus for all multi-indices β_1 satisfying $|\beta_1|\leqslant 2N$, using (2.15) and (3.16), we have

$$\begin{aligned}&\|\partial_y^{\beta_1}\varphi(\sigma,y-f(\sigma,t,x,\xi))\|_{L^1_x} \\ &\quad\leqslant C(1+|t-\sigma|^\kappa)^\tau \int_{\mathbb{R}^n} \frac{\langle y-f(\sigma,t,x,\xi)\rangle^\tau}{\langle y-x+\kappa(t-\sigma)|\xi|^{\kappa-2}\xi\rangle^\tau}|\partial_y^{\beta_1}\varphi(\sigma,y-f(\sigma,t,x,\xi))|\,\mathrm{d}x \\ &\quad\leqslant C(1+T^\kappa)^\tau\Big(\sup_{\sigma\in[0,T]}\|\langle y\rangle^\tau\partial_y^{\beta_1}\varphi(\sigma,y)\|_{L^\infty_y}\Big)\int_{\mathbb{R}^n}\frac{1}{\langle y-x+\kappa(t-\sigma)|\xi|^{\kappa-2}\xi\rangle^\tau}\,\mathrm{d}x \\ &\quad\leqslant C(n,\kappa,T)\end{aligned} \tag{4.7}$$

for all $t \in [0,T]$ and $\sigma \in [0,t]$, where $C(n,\kappa,T)$ depends only on κ, n and T. Similarly, by (2.15) and (3.17), we have

$$\|\partial_y^{\beta_1}\varphi_{k,j}(\sigma, y - f(\sigma,t,x,\xi))\|_{L^1_x} \leqslant C(n,\kappa,T) \tag{4.8}$$

for $t \in [0,T]$ and $\sigma \in [0,t]$. As for $\partial_y^{\beta_1}\widetilde{\psi}$, by (3.19) we have

$$\begin{aligned}&\big\|\|\langle y\rangle^\tau \partial_y^{\beta_1}\widetilde{\psi}(\sigma,y,\xi)\|_{L^\infty_\xi}\big\|_{L^\infty_y}\\ &\qquad\leqslant \big\|\|\langle y\rangle^\tau \partial_y^{\beta_1}\psi(\sigma,y,\xi)\|_{L^\infty_\xi}\big\|_{L^\infty_y} + \|\langle y\rangle^\tau \partial_y^{\beta_1}(-\Delta_y)^{\kappa/2}\varphi(\sigma,y)\|_{L^\infty_y}.\end{aligned}$$

It follows by (3.21) that

$$\sup_{\sigma\in[0,T]} \|\langle y\rangle^\tau \partial_y^{\beta_1}(-\Delta_y)^{\kappa/2}\varphi(\sigma,y)\|_{L^\infty_y} \leqslant C(n,\kappa,T).$$

On the other hand, by virtue of (3.16) and (3.23) with $\varepsilon = \kappa - 1/2$ we may apply Young's inequality to obtain

$$\begin{aligned}\big\|\|\langle y\rangle^\tau \partial_y^{\beta_1}\psi(\sigma,y,\xi)\|_{L^\infty_\xi}\big\|_{L^\infty_y} &\leqslant C(n,\kappa)\|\langle y\rangle^\tau(|\partial_y^{\beta_1}\varphi(\sigma,\cdot)| * \langle y\rangle^{-n-1/2})\|_{L^\infty_y}\\ &\leqslant C(n,\kappa)\||\langle\cdot\rangle^\tau \partial_y^{\beta_1}\varphi(\sigma,\cdot)| * \langle y\rangle^{\tau-n-1/2}\|_{L^\infty_y}\\ &\leqslant C(n,\kappa)\|\langle y\rangle^\tau \partial_y^{\beta_1}\varphi(\sigma,y)\|_{L^1_y}\|\langle\cdot\rangle^{\tau-n-1/2}\|_{L^\infty}\\ &= C(n,\kappa)\|\langle y\rangle^\tau \partial_y^{\beta_1}\varphi(\sigma,y)\|_{L^1_y} \leqslant C(n,\kappa,T)\end{aligned}$$

for all $\sigma \in [0,T]$. Then it follows that

$$\sup_{\sigma\in[0,T]} \big\|\|\langle y\rangle^\tau \partial_y^{\beta_1}\widetilde{\psi}(\sigma,y,\xi)\|_{L^\infty_\xi}\big\|_{L^\infty_y} \leqslant C(n,\kappa,T).$$

Furthermore, we obtain

$$\begin{aligned}&\|\partial_y^{\beta_1}\widetilde{\psi}(\sigma, y - f(\sigma,t,x,\xi), g(\sigma,t,x,\xi))\|_{L^1_x}\\ &\quad\leqslant C(1+|t-\sigma|^\kappa)^\tau \int_{\mathbb{R}^n} \frac{\langle y - f(\sigma,t,x,\xi)\rangle^\tau}{\langle y - x + \kappa(t-\sigma)|\xi|^{\kappa-2}\xi\rangle^\tau}\\ &\qquad\times |\partial_y^{\beta_1}\widetilde{\psi}(\sigma, y - f(\sigma,t,x,\xi), g(\sigma,t,x,\xi))|\,\mathrm{d}x\\ &\quad\leqslant C(1+T^\kappa)^\tau \Big(\sup_{\sigma\in[0,T]} \big\|\|\langle y\rangle^\tau \partial_y^{\beta_1}\widetilde{\psi}(\sigma,y,\xi)\|_{L^\infty_\xi}\big\|_{L^\infty_y}\Big)\\ &\qquad\times \int_{\mathbb{R}^n} \frac{1}{\langle y - x + \kappa(t-\sigma)|\xi|^{\kappa-2}\xi\rangle^\tau}\,\mathrm{d}x\\ &\quad\leqslant C(n,\kappa,T)\end{aligned} \tag{4.9}$$

for all $t \in [0,T]$ and $\sigma \in [0,t]$, where $C(n,\kappa,T)$ depends only on κ, n and T.

Hence, by (4.1), (4.7) and Fubini's theorem, we have

$$\begin{aligned}\big\|\|I_1\|_{L^1_x}\big\|_{L^\infty_\xi} &\leqslant C(1+|t|)^{2N} \sum_{|\beta_1|+|\beta_2|\leqslant 2N} \\ &\quad \left\| \iiint_{\mathbb{R}^{3n}} \|\partial_y^{\beta_1}\varphi_0(y-f(0,t,x,\xi))\|_{L^1_x} |\partial_y^{\beta_2}\varphi_0(y-z)| \frac{|V_{\varphi_0}u_0(z,\eta)|}{\langle \eta-\xi\rangle^{2N}} \,\mathrm{d}y\,\mathrm{d}z\,\bar{\mathrm{d}}\eta \right\|_{L^\infty_\xi} \\ &\leqslant C(n,\kappa,T)\|u_0\|_{M^{1,\infty}_{\varphi_0}}\end{aligned}$$

for all $t \in [0,T]$. Similarly to the above, by (4.3), (4.2), (4.8) and Fubini's theorem, we obtain

$$\begin{aligned}&\big\|\|I_2\|_{L^1_x}\big\|_{L^\infty_\xi} \\ &\leqslant \frac{C(1+|t-\sigma|)^{2N}}{\|\varphi(\sigma,\cdot)\|^2_{L^2}} \sum_{k,j=1}^{n} \sum_{|\beta_1|+|\beta_2|+|\beta_3|\leqslant 2N} \\ &\quad \left\|\left\| \iiint_{\mathbb{R}^{3n}} |\partial_y^{\beta_1}\varphi_{kj}(\sigma,y-f(\sigma,t,x,\xi))| |\partial_y^{\beta_2}\varphi(\sigma,y-z)\partial_y^{\beta_3}\widetilde{V}_{kj}(\sigma,f(\sigma,t,x,\xi),y)| \right.\right. \\ &\qquad\qquad \left.\left. \times \frac{|V_{\varphi(\sigma,\cdot)}u(\sigma,z,\eta)|}{\langle\eta-\xi\rangle^{2N}}\,\mathrm{d}y\,\mathrm{d}z\,\bar{\mathrm{d}}\eta \right\|_{L^1_x}\right\|_{L^\infty_\xi} \\ &\leqslant C(n,\kappa,T)\|u(\sigma,\cdot)\|_{M^{1,\infty}_{\varphi(\sigma,\cdot)}}\end{aligned}$$

for all $t \in [0,T]$ and $\sigma \in [0,t]$. Furthermore, it follows from (4.4), (4.9) and Fubini's theorem that

$$\begin{aligned}\big\|\|I_3\|_{L^1_x}\big\|_{L^\infty_\xi} &\leqslant \frac{C(1+|t-\sigma|)^{2N}}{\|\varphi(\sigma,\cdot)\|^2_{L^2}} \sum_{|\beta_1|+|\beta_2|\leqslant 2N} \\ &\quad \left\| \iiint_{\mathbb{R}^{3n}} \|\partial_y^{\beta_1}\widetilde{\psi}(\sigma,y-f(\sigma,t,x,\xi),g(\sigma,t,x,\xi))\|_{L^1_x} \right. \\ &\quad \left. \times |\partial_y^{\beta_2}\varphi(\sigma,y-z)| \frac{|V_{\varphi(\sigma,\cdot)}u(\sigma,z,\eta)|}{\langle\eta-\xi\rangle^{2N}}\,\mathrm{d}y\,\mathrm{d}z\,\bar{\mathrm{d}}\eta \right\|_{L^\infty_\xi} \\ &\leqslant C(n,\kappa,T)\|u(\sigma,\cdot)\|_{M^{1,\infty}_{\varphi(\sigma,\cdot)}}\end{aligned}$$

for all $t \in [0,T]$ and $\sigma \in [0,t]$, where $C(n,\kappa,T)$ depends only on κ, n and T.

Hence, by (3.11) and the Minkowski inequality, it follows from the above three inequalities that

$$\begin{aligned}\|u(t,\cdot)\|_{M^{1,\infty}_{\varphi(t,\cdot)}} &\leqslant \big\|\|I_1\|_{L^1_x}\big\|_{L^\infty_\xi} + \int_0^t \big\|\|I_2\|_{L^1_x}\big\|_{L^\infty_\xi}\,\mathrm{d}\sigma + \int_0^t \big\|\|I_3\|_{L^1_x}\big\|_{L^\infty_\xi}\,\mathrm{d}\sigma \\ &\leqslant C(n,\kappa,T)\left\{\|u_0\|_{M^{1,\infty}_{\varphi_0}} + \int_0^t \|u(\sigma,\cdot)\|_{M^{1,\infty}_{\varphi(\sigma,\cdot)}}\,\mathrm{d}\sigma\right\}\end{aligned} \tag{4.10}$$

for $t \in [0, T]$. Thus we may apply Gronwall's inequality to (4.10) and obtain

$$\|u(t,\cdot)\|_{M^{1,\infty}_{\varphi(t,\cdot)}} \leqslant C\|u_0\|_{M^{1,\infty}_{\varphi_0}} \tag{4.11}$$

for all $t \in [0, T]$ and a positive constant C depending only on κ, n and T, which concludes the case $(p, q) = (1, \infty)$.

Finally, combining (3.27), (4.6) and (4.11), the general case (p, q) follows from the complex interpolation theorem for the modulation space, i.e.,

$$\|u(t,\cdot)\|_{M^{p,q}_{\varphi(t,\cdot)}} \leqslant C\|u_0\|_{M^{p,q}_{\varphi_0}}$$

for all $t \in [0, T]$, where C is a positive constant depending only on κ, n and T. This completes the proof of Theorem 1.2. □

Remark. After this paper was submitted, the author was informed kindly by the reviewer of one recent work [9] about the estimates on modulation spaces for Schrödinger equations with smooth potentials. Both the results in the present paper and those in [9] are inspired by [7], [8], and the proofs of main results in this paper and [9] both rely essentially on integration by parts and the inversion formula for the short-time Fourier transform. However, the author would like to point out that the fractional power of negative Laplacian $(-\Delta)^{\kappa/2}$ with $\kappa \neq 2$, which is considered in this article, is much more complicated than the negative Laplacian that is dealt with in [9]. This leads to an additional remainder Hu in (3.10) of our paper, compared with (14) in [9]. To handle this remainder, we need to establish a new inequality (2.1) and apply the estimate for oscillatory integrals in Lemma 2.3, which are both trivial in the case of $\kappa = 2$ in [9]. Moreover, due to [10], the singularity of the classical Hamiltonian (2.11) is considered in Lemma 2.5 and Lemma 2.6, which generalize the corresponding results in the case of the smooth Hamiltonian with $\kappa = 2$ in [9]. Finally, we state that the main results in this paper include the case of wave equations, which correspond to (1.1) with $\kappa = 1$.

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