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TRAVEL GROUPOIDS ON INFINITE GRAPHS

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Abstract. The notion of travel groupoids was introduced by L. Nebeský in 2006 in connection with a study on geodetic graphs. A travel groupoid is a pair of a set V and a binary operation $*$ on V satisfying two axioms. We can associate a graph with a travel groupoid. We say that a graph G has a travel groupoid if the graph associated with the travel groupoid is equal to G. Nebeský gave a characterization of finite graphs having a travel groupoid.

In this paper, we study travel groupoids on infinite graphs. We answer a question posed by Nebeský, and we also give a characterization of infinite graphs having a travel groupoid.

Keywords: travel groupoid; geodetic graph; infinite graph

MSC 2010: 20N02, 05C63, 05C12

1. Introduction

A *groupoid* is the pair $(V,*)$ of a nonempty set V and a binary operation $*$ on V. The notion of travel groupoids was introduced by L. Nebeský [5] in 2006 in connection with his study on geodetic graphs [1], [2], [3] and signpost systems [4]. First, let us recall the definition of travel groupoids.

A *travel groupoid* is a groupoid $(V,*)$ satisfying the following axioms (t1) and (t2):

(t1) $(u*v)*u=u$, for all $u,v\in V$,

(t2) if $(u*v)*v=u$, then $u=v$ for all $u,v\in V$.

Note that a travel groupoid is an idempotent groupoid, i.e., $x*x=x$ holds for any $x\in V$ ([5], Proposition 1).

Let $(V,*)$ be a travel groupoid, and let G be a graph. We say that $(V,*)$ *is on* G or that G *has* $(V,*)$ if $V(G)=V$ and $E(G)=\{\{u,v\};\ u,v\in V,\ u\neq v,$ and $u*v=v\}$. It follows immediately from the definition that if $(V,*)$ is a travel

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groupoid on a graph G, then u and $u * v$ are adjacent in G for two distinct elements u and v of V ([5], Proposition 3). Thus the following holds.

Lemma 1. *Let G be a graph and let $(V, *)$ be a travel groupoid on G. For any two distinct elements u and v in V, we have $u * v \in N_G(u)$, where $N_G(u)$ denotes the set of vertices adjacent to u in G.*

Nebeský showed the following theorem which characterizes finite graphs having travel groupoids.

Theorem 2 ([5], Theorem 6)**.** *Let G be a finite graph. Then, G has a travel groupoid if and only if either G is connected or G is disconnected and no component of G is a tree.*

Nebeský posed the following question.

Question 3 ([5], Question 3)**.** Does there exist an infinite graph G with no finite components such that G has no travel groupoid?

In this paper, we study travel groupoids on infinite graphs. In Section 2, we answer the above question by Nebeský. In Section 3, we give a characterization of infinite graphs having travel groupoids, which is an extension of Theorem 2.

2. Answer to a question by Nebeský

An *infinite star* is a graph S_∞ defined by

$$V(S_\infty) = \{v_i;\ i \in \{0\} \cup \mathbb{N}\} \quad \text{and} \quad E(S_\infty) = \{\{v_0, v_i\};\ i \in \mathbb{N}\},$$

where $\mathbb{N} = \{1, 2, \ldots\}$ denotes the set of positive integers.

Theorem 4. *Let G be the disjoint union of an infinite star S_∞ and an infinite connected graph H. Then G has no travel groupoids.*

P r o o f. Suppose that there exists a travel groupoid $(V, *)$ on G, where $V = V(G)$. Take any vertex w in H. Then $w \neq v_0$. Since $N_G(v_0) = \{v_i;\ i \in \mathbb{N}\}$, we have $v_0 * w \in \{v_i;\ i \in \mathbb{N}\}$ by Lemma 1. Let $v_j := v_0 * w$. Since $N_G(v_j) = \{v_0\}$, we have $v_j * w = v_0$ by Lemma 1. Therefore it follows that $(v_0 * w) * w = v_j * w = v_0$ while $w \neq v_0$. Thus $(V, *)$ does not satisfy Axiom (t2), which is a contradiction. Hence the theorem holds. $\square$

Let G be the disjoint union of an infinite star S_∞ and an infinite connected graph H. Then G has no finite connected component. By Theorem 4, there is no travel groupoid on G. Hence the answer to Question 3 is YES.

3. Characterization

In this section, we give an extension of Theorem 2.

Recall that a *geodetic graph* is a graph in which there exists a unique shortest path between any two vertices. Let G be a geodetic graph. Let $V := V(G)$. For two vertices u and v of G, let $A_G(u, v)$ denote the vertex adjacent to u which is on the unique shortest path from u to v in G. Define a binary operation $*$ on V as follows: For all $u, v \in V$, let $u * v := A_G(u, v)$ if $u \neq v$ and $u * v := u$ if $u = v$. This groupoid $(V, *)$ is called the *proper groupoid* of the geodetic graph G. Remark that the proper groupoid of any geodetic graph is a travel groupoid.

Lemma 5. *For every (finite or infinite) tree T, there exists a travel groupoid on T.*

Proof. Since any tree is a geodetic graph, we can define the proper groupoid $(V, *)$ on T. Hence T has a travel groupoid. □

Lemma 6. *For every (finite or infinite) connected graph G, there exists a travel groupoid on G.*

Proof. Let $V := V(G)$. Fix a spanning tree T of the graph G. Let $(V, *_T)$ be the proper groupoid on T. Now we define a groupoid $(V, *)$ as follows. For each edge $\{u, v\} \in E(G)$, let $u * v := v$ and $v * u := u$. For u and v such that $\{u, v\} \notin E(G)$, let $u * v := u *_T v$. Then we can show that $(V, *)$ is a travel groupoid on G as follows. Consider arbitrary two elements u and v in V.

First we check (t1). Put $w := (u * v) * u$. We will show that $w = u$. If $u = v$, then $u * v = u$ and therefore $w = u * u = u$. If u and v are adjacent, then $u * v = v$ and therefore $w = v * u = u$. Assume that u and v are not adjacent in G. Then $u * v = u *_T v$ is the vertex adjacent to u which is on the path from u to v in T. Since $u * v$ and u are adjacent, we have $w = (u * v) * u = u$. Thus (t1) holds.

Second we check (t2). We assume that $u \neq v$. We show that $(u * v) * v \neq u$. If u and v are adjacent in G, then $(u * v) * v = v * v = v \neq u$. Suppose that u and v are not adjacent in G. Then $u * v = u *_T v$. If $u * v$ and v are adjacent in G, then $(u * v) * v = v \neq u$. If $u * v$ and v are not adjacent in G, then $(u * v) * v = (u *_T v) *_T v$ is the vertex which is on the path from u to v in T and the distance from u in T is two, i.e., $(u * v) * v$ is not equal to u. Thus (t2) holds. Hence the lemma holds. □

Theorem 7. *Let G be a (finite or infinite) graph. Then, G has a travel groupoid if and only if either G is connected or G is disconnected and no component of G is a tree with finite diameter.*

Proof. Assume that G is connected or G is disconnected and no component of G is a tree with finite diameter. If G is connected, then, by Lemma 6, there exists a travel groupoid on G. Let G be disconnected. Then every connected component of G contains a cycle or an infinite path. It is easy to see that there exists a mapping f from $V(G)$ into itself such that the following statements hold for every vertex u in G: u and $f(u)$ are adjacent vertices in G and $u \neq f(f(u))$. By Lemma 6, every connected component H of G has a travel groupoid, say, $(V(H), *_H)$. For any two vertices x and y in G, we define $x * y := x *_H y$ if there exists a connected component H of G such that $x, y \in V(H)$, and $x * y := f(x)$ if x and y belong to distinct connected components of G. It is easy to see that $(V(G), *)$ satisfies the axioms (t1) and (t2). Hence G has a travel groupoid.

Conversely, assume that G is disconnected and at least one component T of G is a tree with finite diameter. Suppose, to the contrary, that G has a travel groupoid, say, a travel groupoid $(V, *)$, where $V = V(G)$. Consider $u \in V(T)$ and $v \in V(G) \setminus V(T)$. Since $V(T)$ is finite and T contains neither a cycle nor an infinite path, we see that there exists a positive integer k such that $u *^{k+1} v = u *^{k-1} v$. Therefore, we have $((u *^{k-1} v) * v) * v = u *^{k-1} v$, and so, by (t2), $u *^{k-1} v = v$. Thus u and v belong to the same connected component of G, which is a contradiction. Hence the theorem holds. $\square$

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