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FINITE GROUPS WHOSE SET OF NUMBERS OF SUBGROUPS OF POSSIBLE ORDER HAS EXACTLY 2 ELEMENTS

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Abstract. Counting subgroups of finite groups is one of the most important topics in finite group theory. We classify the finite non-nilpotent groups G whose set of numbers of subgroups of possible orders $n(G)$ has exactly two elements. We show that if G is a non-nilpotent group whose set of numbers of subgroups of possible orders has exactly 2 elements, then G has a normal Sylow subgroup of prime order and G is solvable. Moreover, as an application we give a detailed description of non-nilpotent groups with $n(G) = \{1, q+1\}$ for some prime q. In particular, G is supersolvable under this condition.

Keywords: finite group; number of subgroups of possible orders

MSC 2010: 20E45

1. Introduction

All groups considered in this paper are finite and G always denotes a group. We denote by $\pi(G)$ the set of prime divisors of G and by $\nu_p(G)$ the number of Sylow p-subgroups of G for a prime $p \in \pi(G)$. Further unexplained notation and terminology are standard, readers may refer to [4].

One of the most important topics in group theory is to count the subgroups of finite groups. Recall that the problem was completely solved in abelian case by establishing an explicit expression of the number of subgroups of abelian groups in [1]. For non-abelian p-groups, M. Tărnăuceanu in [8] gave an explicit formula for the number of subgroups having a cyclic maximal subgroup. In 1995, J. Zhang [9] studied groups by Sylow numbers. Also W. B. Guo [2] in 1996 considered the

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influence of Sylow numbers on groups, which is the indices of the normalizers of Sylow subgroups in the whole group. Let $n(G)$ be the set of numbers of subgroups of possible orders of G. In 2010, H. Qu, Y. Sun and Q. Zhang [5] classified p-groups in which every element in $n(G)$ is no more than $2p^2 + p + 1$. In this paper, we focus on groups where $n(G)$ has exactly 2 elements. Our idea springs from F. Tang's result in [7]. He characterized groups whose set of numbers of conjugacy class size of all subgroups has exactly two elements, he obtained that a non-nilpotent group whose set of conjugacy class sizes of all subgroups is $\{1, m\}$, then:

(1) $m = p \in \pi(G)$. If G has no central decomposition factors, then m is the largest prime divisor;

(2) $G \cong (C_p \rtimes H) \times Z$, then Z is the central decomposition factors, H is a normal p'-subgroup of G, H is either cyclic or the direct product of Q_8 with an odd cyclic group. In particular, G is supersolvable.

In the present paper, we use an elementary and skillful method of applying Sylow's Theorem to give a description of non-nilpotent groups G with $n(G) = \{1, m\}$ for some integer $m > 1$. Our result is:

Theorem 1.1. *Let G be a non-nilpotent group. If $n(G) = \{1, m\}$, then the following statements hold:*

(1) $m = p$ with some prime p. Moreover, G has a normal Sylow p-subgroup P of order p.

(2) G is solvable. Moreover, $G = (P \rtimes H) \times A$, where $A \leqslant Z(G)$ is a cyclic Hall subgroup and H is a nilpotent Hall $\pi(p-1)$-subgroup of G.

(3) $H/C_H(P) \leqslant \mathrm{Aut}(P)$. Moreover, if there is an $R \in \mathrm{Syl}_r(H)$ such that $C_R(P)$ is a maximal subgroup of R, then R is cyclic.

Further, as an application, we give a description of groups with $n(G) = \{1, q+1\}$, where q is a prime:

Corollary 1.2. *Let G be a non-nilpotent group. Then $n(G) = \{1, q+1\}$ with q a prime if and only if $G = (P \rtimes Q) \times A$, $q = 2$, $|P| = 3$ and Q is a cyclic 2-subgroup of G with $(6, |A|) = 1$, and A is cyclic. In particular, G is supersolvable.*

2. Preliminaries

Before taking up the problem, we present here some results which will be used in the sequel.

Lemma 2.1. *Let G be a group and $q \in \pi(G)$. If G has a normal subgroup K such that $q \nmid |K|$, then $\nu_q(G/K) \mid \nu_q(G)$.*

Proof. This follows immediately from [6], Lemma 2.5(3). $\square$

Lemma 2.2 ([4], Theorem 3.8.3). *Assume that G is a group of order p^n. If $s_k(G) = 1$ for each $2 \leqslant k \leqslant n-1$, then G is cyclic, where $s_k(G)$ is the number of subgroups of order p^k in G.*

Lemma 2.3 ([3], Theorem 9.3.1). *Let G be a solvable group of order mn with $(m,n) = 1$. Then the number h_m of subgroups of order m may be expressed as a product of factors, each of which* (a) *is congruent to 1 modulo some prime factor of m, and* (b) *is a power of a prime and divides one of the orders of the chief factors of G.*

3. Proof of the main theorem

First we show that G is solvable. Write $\pi(m) = \{p_1, \ldots, p_n\}$, where $p_1, \ldots, p_n$ are distinct prime divisors of m. Since m is also a Sylow number of G, we see clearly that $m \mid |G|$ by Sylow's Theorem, yielding that $p_i \in \pi(G)$. Let P_i be a Sylow p_i-subgroup of G with $i \in \{1, \ldots, n\}$. Again by Sylow's Theorem, we obtain that $p_i \nmid \nu_{p_i}(G) \in n(G) = \{1, m\}$ since $\nu_{p_i}(G) \equiv 1 \pmod{p_i}$. Then $\nu_{p_i}(G) = 1$, implying $P_i \trianglelefteq G$. Consequently, G has a normal nilpotent Hall $\pi(m)$-subgroup, say K. Further, the theorem of Schur-Zassenhaus yields that G has a $\pi(m)$-complement, say H_1.

Let q be an arbitrary prime in $\pi(H_1)$ and let Q_1 be a Sylow q-subgroup of H_1. We see easily that Q_1 is also a Sylow q-subgroup of G. Moreover, $\nu_q(G/K) \mid \nu_q(G)$ by Lemma 2.1, leading to $\nu_q(H_1) \mid \nu_q(G) \in \{1, m\}$ since $G/K \cong H_1$. On the other hand, $\nu_q(H_1) \mid |H_1|$ according to Sylow's Theorem, which implies that $\nu_q(H_1) = 1$ as $(m, |H_1|) = 1$. As a result, $Q_1 \trianglelefteq H_1$, implying that H_1 is also nilpotent. As a consequence, G is solvable by Wielandt's Theorem, as required.

We now show that m is a prime power. Assume that this is false. Then $n \geqslant 2$, where n is the one appearing in the first paragraph. Moreover, as is proved in the first paragraph, $K_{p_i} \trianglelefteq G$ for each $p_i \in \pi(m)$; we see clearly that $K_{p_i}H_1$ is a Hall subgroup of G and thus $|G\colon N_G(K_{p_i}H_1)| \in n(G)$. Notice that $K_{p_i} \leqslant N_G(K_{p_i}H_1)$. This shows that $p_i \nmid |G\colon N_G(K_{p_i}H_1)| \in \{1, m\}$, yielding $|G\colon N_G(K_{p_i}H_1)| = 1$. Consequently, $K_{p_i}H_1 \trianglelefteq G$, and thus $G = K_{p_i}H_1 \times K_{(\pi(m)-\{p_i\})}$, where $K_{(\pi(m)-\{p_i\})}$ is a Hall $(\pi(m) - \{p_i\})$-subgroup of G. Furthermore, $G = K \times H_1$ is nilpotent, contrary to our assumption. Hence $n = 1$ and m is a prime power, as required. Write $m = p^a$ with a prime p and a positive integer a.

Let P be the Sylow p-subgroup of G. We claim that P is a cyclic group of order p. Otherwise, there exists an integer $k \in \{2, \dots, a\}$ such that $s_k(P) \neq 1$ by Lemma 2.2, which implies that $s_k(P) \equiv 1 \pmod p$ according to [4], Theorem 1.7.2. On the other hand, $P \trianglelefteq G$ implies that $s_k(P) \in n(G)$, leading to $s_i(P) = m = p^a$. This contradiction shows that P is cyclic. Now we consider the action of H_1 on P by conjugation. By [4], Theorem 3.13.4(b), we obtain that $P = [P, H_1] \times C_P(H_1)$. Note that P is cyclic. Then either $[P, H_1] = 1$ or $C_P(H_1) = 1$. If the former holds, then G is nilpotent, a contradiction to our assumption. Hence $C_P(H_1) = 1$, yielding $N_G(H_1) = H_1$ and $m = |G\colon N_G(H_1)| = |P|$. It follows by Lemma 2.3(b) that $|P|$ divides some order of a chief factor of G, implying that P is isomorphic to a chief factor of G. Since P is cyclic, we conclude that $a = 1$ and thus $m = |P| = p$, hence (1) holds.

Let $q \nmid p-1$ be a prime in $\pi(H_1)$ and Q a Sylow q-subgroup of H_1. We prove that Q is cyclic. Clearly, Q is also a Sylow q-subgroup of G. By Sylow's Theorem, we see that $\nu_q(G) \equiv 1 \pmod q$, showing that $Q \trianglelefteq G$ since $q \nmid (p-1)$. On the other hand, for every $k \in \{2, \dots, t\}$ we obtain that $s_i(Q) \equiv 1 \pmod q$ by [4], Theorem 1.7.2. Hence it follows that Q is cyclic by Lemma 2.2. As a result, we may write $G = (P \rtimes H) \times A$, where $A \leqslant Z(G)$ is a cyclic Hall subgroup of G and H is a nilpotent Hall $\pi(p-1)$-subgroup of G; hence (2) holds.

Because H is not normal in G, there exists a Sylow r-subgroup R of H satisfying $R \ntrianglelefteq G$, which gives $R_0 := C_R(P) < R$. As a result, $R/R_0 \leqslant \mathrm{Aut}(P)$ is cyclic, which indicates that $H/C_H(P) \leqslant \mathrm{Aut}(P)$ is cyclic.

Assume that $R_0 := C_R(P)$ is a maximal subgroup of R. We assert that R is cyclic. Easily, it is sufficient to show that R_0 is the unique maximal subgroup of R. If not, then R has at least $r+1$ maximal subgroups by [4], Theorem 1.7.2. On the other hand, $R_0 \trianglelefteq G$ indicates that $\bigcap_{R_i \in \mathrm{Syl}_r(H)} R_i = R_0$ and thus $R_i \cap R_j = R_0$ for distinct $i, j \in \{1, \dots, p\}$. Hence G has at least $rp+1$ subgroups of order $|R_0|$ by [4], Theorem 1.7.2, a contradiction. Therefore, R is cyclic and (3) holds. Theorem 1.1 is established. $\square$

4. Proof of corollary

The sufficiency is obvious, we only prove the necessity. By Theorem 1.1, we see that $q+1 = p$ is a prime, implying that $p = 3$ and $q = 2$. Moreover, (3) of the theorem implies that $C_Q(P) \leqslant \mathrm{Aut}(P) \cong C_2$, which gives that $C_Q(P)$ is the maximal subgroup of Q since G is non-nilpotent. Again by applying (3) of the theorem, Q is cyclic, this completes the proof. $\square$

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