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OPTIMAL CONVERGENCE AND A POSTERIORI ERROR ANALYSIS OF THE ORIGINAL DG METHOD FOR ADVECTION-REACTION EQUATIONS

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Abstract. We consider the original DG method for solving the advection-reaction equations with arbitrary velocity in d space dimensions. For triangulations satisfying the *flow* condition, we first prove that the optimal convergence rate is of order $k+1$ in the L_2-norm if the method uses polynomials of order k. Then, a very simple derivative recovery formula is given to produce an approximation to the derivative in the flow direction which superconverges with order $k+1$. Further we consider a residual-based a posteriori error estimate and give the global upper bound and local lower bound on the error in the DG-norm, which is stronger than the L_2-norm. The key elements in our a posteriori analysis are the saturation assumption and an interpolation estimate between the DG spaces. We show that the a posteriori error bounds are efficient and reliable. Finally, some numerical experiments are presented to illustrate the theoretical analysis.

Keywords: discontinuous Galerkin method; advection-reaction equation; optimal convergence rate; a posteriori error estimate

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1. Introduction

This paper investigates the optimal convergence and a posteriori error estimates of the original discontinuous Galerkin (DG) method [17] for the advection-reaction

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equation governed by

$$\boldsymbol{\beta} \cdot \nabla u + \alpha u = f, \quad x \in \Omega, \tag{1.1}$$

with the inflow boundary condition given on $\Gamma_- = \{x \in \partial\Omega\colon\ \boldsymbol{\beta} \cdot n(x) < 0\}$, where $\Omega \subset \mathbb{R}^d$ is a bounded polyhedral domain, $n(x)$ is the outward unit normal at the point $x \in \partial\Omega$.

The first mathematical analysis of this DG method was given by Lesaint and Raviart [15]. They showed that the DG scheme can be solved in an explicit fashion and the convergence order is of $O(h^k)$ if the method uses polynomials of order k. Later on, Johnson and Pitkaranta [14] improved this convergence order to $O(h^{k+1/2})$. Peterson in [16], for a constant vector $\boldsymbol{\beta}$ and a particular type of two-dimensional mesh, further proved that the $O(h^{k+1/2})$-order convergence is sharp, namely, the convergence order of the original DG method is suboptimal in the general case. Also, see Richter's recent work [19].

On the other hand, in a diametrically opposed effort, some optimal error estimates are achieved on special meshes. In 1988, Richter [18] showed that, in the two-dimensional case, the L_2-error estimate

$$\|u - u_h\| \leqslant C h^{k+1} \|u\|_{H^{k+2}(\Omega)} \tag{1.2}$$

holds for semi-uniform triangle meshes with the curious assumption that all element edges are bounded away from the characteristic direction $\boldsymbol{\beta}$ of the hyperbolic equation, that is, the triangulation T_h satisfies

$$|\boldsymbol{\beta} \cdot n(x)| \geqslant c_0 > 0, \quad x \in \partial K \quad \forall\, K \in T_h, \tag{1.3}$$

where n is the outward unit normal on the element boundary ∂K. Obviously, condition (1.3) is less significant in the practical case, and the regularity required for the exact solution in (1.2) is not optimal.

The first optimal convergence is obtained by Cockburn et al. in [7] under the assumptions that $\boldsymbol{\beta}$ is a constant vector and the triangulation T_h satisfies the so-called *flow* condition:

(1.4) Each simplex K has a unique outflow face e_K^+ with respect to $\boldsymbol{\beta}$ and there are no hanging nodes on each interior outflow face e_K^+,

where a face e of simplex K is called the outflow (inflow) with respect to $\boldsymbol{\beta}$ if $\boldsymbol{\beta}\cdot n|_e > 0$ (< 0). They showed that

$$\|u - u_h\| \leqslant C h^{k+1} |u|_{H^{k+1}(T_h)}, \tag{1.5}$$

where $|u|^2_{H^{k+1}(T_h)} = \sum_{K \in T_h} |u|^2_{H^{k+1}(K)}$. This estimate is optimal in both the convergence order and the regularity requirement. Moreover, Cockburn et al. in [7] also propose a postprocessing method for the approximation to the directional derivative $\partial_{\boldsymbol{\beta}} u = \boldsymbol{\beta} \cdot \nabla u$ and have proved the following superconvergence estimate

$$\|P_h(\partial_{\boldsymbol{\beta}} u) - \partial_{\boldsymbol{\beta},h} u_h\| \leqslant C h^{k+1} |u|_{H^{k+1}(T_h)}, \tag{1.6}$$

where P_h is the L_2-projection onto the DG space and $\partial_{\boldsymbol{\beta},h} u_h$ is an approximation to $\partial_{\boldsymbol{\beta}} u$ obtained by the postprocessing procedure.

The first part of this paper aims to further improve the optimal convergence results in [7]. Under the same *flow* condition assumption (1.4), we prove that, for an arbitrary vector $\boldsymbol{\beta}(x)$, the optimal convergence result (1.5) still holds, and our argument is more skillful. Meanwhile, we also present a very simple derivative recovery formula by using the DG solution u_h to recover the derivative $\partial_{\boldsymbol{\beta}} u$ and prove that

$$\|\partial_{\boldsymbol{\beta}} u - R_h(\partial_{\boldsymbol{\beta}} u)\| \leqslant C h^{k+1} |u|_{H^{k+1}(T_h)}, \tag{1.7}$$

where $R_h(\partial_{\boldsymbol{\beta}} u)$ is the recovery value of $\partial_{\boldsymbol{\beta}} u$. Obviously, estimate (1.7) is a better result compared with estimate (1.6), noting that using (1.6) to derive a result like (1.7) requires $u \in H^{k+2}(T_h)$.

Now let us turn to a posteriori error estimates of the original DG method for problem (1.1). An early attempt to derive a posteriori error estimates for advection-dominated advection-diffusion problems was made by Eriksson and Johnson in [9], using regularization and duality techniques. Improved energy norm techniques were then proposed by Verfürth in [20], where semi-robust estimates were obtained. However, for the pure advection-reaction problems, the estimates based on the ideas in [9], [20] fail. This is mainly due to the fact that the elliptic problem has smoothing properties, whereas the advection equation does not. This lack of smoothing and symmetry of the advection-reaction problem is what makes standard techniques using coercivity no longer work well. Another possibility is to use the duality technique to derive a posteriori error estimates. Indeed, Houston and Süli in [12], [13] used this technique to establish two types of a posteriori error estimates (labelled as Type I and Type II) by means of some target functionals of the exact solution. The main idea in [12], [13] is to relate the a posteriori error estimate of the functional to the solution of the dual problem, which is set in terms of this functional. However, in both Type I and Type II estimates, the error estimators contain the unknown solution of the dual problem that must be solved analytically or numerically. Such a posteriori error estimates are clearly difficult to use in adaptive computations. Recently, Burman in [4] also gave some residual-based a posteriori error bounds in the graph norm for linear finite element. The key assumption in [4] is the so-called *saturation* assumption (see [3]): There exists a constant $0 < \delta < 1$ such that

$$\|u - u_{h^*}\|^* \leqslant \delta \|u - u_h\|^*, \tag{1.8}$$

where u_h and u_{h^*} are two DG solutions associated with the triangulation T_h and the refined triangulation T_{h^*}, respectively, and $\|\cdot\|^*$ is a mesh-dependent norm. The *saturation* assumption has been used extensively in a posteriori error analysis. See, for example, [1], [2], [3], [4], [8]. It is typically needed in situations where the standard techniques for a posteriori error estimates based on coercivity or smoothing fail.

The second part of this paper aims to propose an efficient and reliable, residual-based a posteriori error estimate for the original DG method. The key ingredient of our analysis is to use the *saturation* assumption and establish an interpolation estimate between the two DG spaces associated with the triangulations T_h and $T_{h/2}$. We give the global upper bound and the local lower bound on the error $u - u_h$ in the DG-norm (see (2.9)), which is stronger than the L_2-norm.

The paper is organized as follows. In Section 2, we review the original DG method and some basic results. In Section 3, we derive the optimal convergence in the L_2-norm and establish the superconvergent derivative recovery formula for the approximation of $\partial_{\boldsymbol{\beta}} u$. Section 4 is devoted to the a posteriori error analysis, and in Section 5 some numerical examples are presented to illustrate our theoretical analysis.

Throughout this paper, we use the usual Sobolev space and norm notations, and use letter C to represent a generic positive constant, which is independent of the mesh size h.

2. The problem and its DG approximation

Consider the following advection-reaction equation [11]:

$$\begin{aligned} \mathcal{L}u \equiv \boldsymbol{\beta} \cdot \nabla u + \alpha u = f \quad &\text{in } \Omega, \\ u = g \quad &\text{on } \Gamma_-, \end{aligned} \tag{2.1}$$

where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_d)^{\mathrm{T}}$ is a vector function, α, f, and g are some known functions. As usual, we assume that $\boldsymbol{\beta} \in [W^1_\infty(\Omega)]^d$, $\alpha \in L_\infty(\Omega)$, $f \in L_2(\Omega)$, $g \in L_2(\Gamma_-)$, and

$$\alpha - \frac{1}{2}\operatorname{div} \boldsymbol{\beta} = \sigma \geqslant \sigma_0 > 0, \quad x \in \Omega. \tag{2.2}$$

Let $T_h = \bigcup\{K\}$ be a shape regular triangulation of domain Ω parameterized by mesh size $h = \max h_K$, where K is the simplex and h_K is the diameter of K. We say that the triangulation T_h is shape-regular, if the elements of T_h are affine equivalent and there exists a positive constant γ independent of $K \in T_h$ such that

$$h_K/\varrho_K \leqslant \gamma \quad \forall\, K \in T_h,$$

where ϱ_K denotes the diameter of the biggest ball included in K.

With the triangulation T_h, we associate the finite-dimensional space

$$S_h = \{v \in L_2(\Omega)\colon\ v|_K \in P_k(K) \quad \forall\, K \in T_h\}, \tag{2.3}$$

where $P_k(K)$ is composed of polynomials of degree at most k on each element K. Also, we denote the piecewise smooth function space on T_h by

$$H^s(T_h) = \{v \in L_2(\Omega)\colon\ v|_K \in H^s(K) \quad \forall\, K \in T_h\}, \quad s \geqslant 1.$$

In order to cope with the discontinuity of functions across element interfaces, we introduce the jump of function $\varphi \in H^1(T_h)$ on ∂K by

$$[\varphi] = \varphi^+ - \varphi^- \quad \text{and} \quad [\varphi]|_{\partial\Omega} = \varphi^+,$$

where φ^+ and φ^- are the traces of φ on ∂K from the interior and the exterior of K, respectively. We will also use the notations

$$(u,v)_h = \sum_{K\in T_h} (u,v)_K = \sum_{K\in T_h} \int_K uv \,\mathrm{d}x, \quad \langle u,v\rangle_S = \sum_{K\in T_h} \int_{S\cap\partial K} uv \,\mathrm{d}s,$$

where S is some collection of faces of elements.

In what follows, we denote by $\mathcal{L}^* = -\boldsymbol{\beta}\cdot\nabla + \alpha - \operatorname{div}\boldsymbol{\beta}$ the adjoint operator of $\mathcal{L}$, and by ∂K_+ and ∂K_- the outflow boundary and the inflow boundary of K, respectively, where $\partial K_\pm = \{x \in \partial K\colon\ \pm\boldsymbol{\beta}\cdot n(x) > 0\}$, and n represents the outward unit normal on the boundaries concerned.

Now we define the DG approximation of problem (2.1) by finding $u_h \in S_h$ such that

$$a_h(u_h, v_h) = (f, v_h) - \langle \boldsymbol{\beta}\cdot n g, v_h\rangle_{\Gamma_-} \quad \forall\, v_h \in S_h, \tag{2.4}$$

where $a_h(\cdot,\cdot)$ is the bilinear form defined by

$$a_h(w,v) = (w, \mathcal{L}^* v)_h + \sum_{K\in T_h} \int_{\partial K\setminus\Gamma_-} \boldsymbol{\beta}\cdot n\hat{w}v \,\mathrm{d}s,\ \ w,v \in H^1(T_h), \tag{2.5}$$

and $\hat{w}$ is the numerical trace of w given by the upwind value:

$$\hat{w} = \begin{cases} w^- & \text{on } \partial K_-, \\ w^+ & \text{on } \partial K_+. \end{cases}$$

Equality (2.4) is the well-known DG scheme introduced originally by Reed and Hill in [17]. By using integration by parts and the definition of $\hat{w}$, an equivalent form of $a_h(w, v)$ can be derived as follows:

$$a_h(w, v) = (\mathcal{L}w, v)_h - \sum_{K \in T_h} \int_{\partial K_-} \boldsymbol{\beta} \cdot n[w] v \,\mathrm{d}s, \quad w, v \in H^1(T_h), \tag{2.6}$$

which will be used sometimes for convenience. It is easy to derive the following identity for $w \in H^1(T_h)$,

$$a_h(w, w) = (\sigma w, w) + \frac{1}{4} \sum_{K \in T_h} \int_{\partial K \setminus \partial\Omega} |\boldsymbol{\beta} \cdot n| [w]^2 \,\mathrm{d}s + \frac{1}{2} \int_{\partial\Omega} |\boldsymbol{\beta} \cdot n| w^2 \,\mathrm{d}s. \tag{2.7}$$

In addition, following [5], [10], we may show the following inf-sup condition:

$$|\!|\!| w_h |\!|\!|_h \leqslant C \sup_{v_h \in S_h} \frac{a_h(w_h, v_h)}{|\!|\!| v_h |\!|\!|_h}, \tag{2.8}$$

where $|\!|\!| \cdot |\!|\!|_h$ is the DG-norm:

$$|\!|\!| v |\!|\!|_h^2 = \sigma_0 \|v\|_h^2 + \sum_{K \in T_h} h_K \|\boldsymbol{\beta} \cdot \nabla v\|_{L_2(K)}^2 + \sum_{K \in T_h} \int_{\partial K} |\boldsymbol{\beta} \cdot n| [v]^2 \,\mathrm{d}s, \quad v \in H^1(T_h). \tag{2.9}$$

Introduce the L_2-projection operator $P_h : L_2(\Omega) \to S_h$ (restricted to $K \in T_h$, $P_h u \in P_k(K)$) defined by

$$(u - P_h u, v)_K = 0 \quad \forall\, v \in P_k(K),\ K \in T_h. \tag{2.10}$$

The operator P_h can be a continuous linear operator mapping $H^{k+1}(K)$ into $P_k(K)$ and $P_h v = v$ for all $v \in P_k(K)$. Hence, by the interpolation theory of Sobolev space [6], we have the standard approximation result

$$\|u - P_h u\|_{L_2(K)} + h_K^{1/2} \|u - P_h u\|_{L_2(\partial K)} \leqslant C h_K^{k+1} |u|_{H^{k+1}(K)}, \quad k \geqslant 0,\ K \in T_h. \tag{2.11}$$

Denote by w^c the piecewise constant approximation of the function w, that is,

$$w^c = \frac{1}{|K|} \int_K w \,\mathrm{d}x, \quad |w - w^c|_{\infty,K} \leqslant h_K |\nabla w|_{\infty,K}, \quad K \in T_h. \tag{2.12}$$

The following finite element inverse inequality will be used throughout this paper:

$$\|\nabla v_h\|_{L_2(K)} + h_K^{-1/2} \|v_h\|_{L_2(\partial K)} \leqslant C h_K^{-1} \|v_h\|_{L_2(K)}, \quad v_h \in P_k(K),\ K \in T_h. \tag{2.13}$$

We now give the standard error estimate which will be used in Section 4.

Lemma 2.1. *Let T_h be a shape-regular triangulation, u and u_h the solutions of problem* (2.1) *and* (2.4)*, respectively, $u \in H^1(\Omega) \cap H^{k+1}(T_h)$. Then, we have*

$$\|\!|u - u_h|\!\|_h \leqslant C h^{k+1/2} |u|_{H^{k+1}(T_h)}, \quad k \geqslant 0.$$

P r o o f. Let $v_h \in S_h$. Then, from the error equation

$$a_h(u - u_h, v_h) = 0 \quad \forall\, v_h \in S_h \tag{2.14}$$

we can derive that

$$\begin{aligned} a_h(u_h - P_h u, v_h) &= a_h(u - P_h u, v_h) \\ &= (u - P_h u, \mathcal{L}^* v_h)_h + \sum_{K \in T_h} \langle \boldsymbol{\beta} \cdot n(u - \widehat{P_h u}), v_h \rangle_{\partial K \setminus \Gamma_-} \\ &= E_1 + E_2. \end{aligned} \tag{2.15}$$

Since $\boldsymbol{\beta}^c \nabla v_h \in S_h$, we have

$$\begin{aligned} E_1 &= -(u - P_h u, (\boldsymbol{\beta} - \boldsymbol{\beta}^c) \cdot \nabla v_h)_h + ((\alpha - \operatorname{div} \boldsymbol{\beta})(u - P_h u), v_h)_h \\ &\leqslant \|u - P_h u\|_{L_2(T_h)} (|\boldsymbol{\beta}|_{1,\infty} \|h_K \nabla v_h\|_{L_2(T_h)} + |\alpha - \operatorname{div} \boldsymbol{\beta}|_\infty \|v_h\|_{L_2(T_h)}) \\ &\leqslant C \|u - P_h u\| \|v_h\|, \end{aligned}$$

where we have used the inverse inequality. Next, note that when $x \in \partial K_- \setminus \Gamma_-$, there must exist an adjacent element K' such that $x \in \partial K_- \cap \partial K'_+$ and $\boldsymbol{\beta} \cdot n|_{x \in \partial K_-} = -\boldsymbol{\beta} \cdot n'|_{x \in \partial K'_+}$. Thus, we have

$$\begin{aligned} E_2 &= \sum_{K \in T_h} (\langle \boldsymbol{\beta} \cdot n(u - P_h u), v_h \rangle_{\partial K_+} + \langle \boldsymbol{\beta} \cdot n(u - P_h u)_h^-, v_h \rangle_{\partial K_- \setminus \Gamma_-}) \\ &= \sum_{K \in T_h} \langle \boldsymbol{\beta} \cdot n(u - P_h u), [v_h] \rangle_{\partial K_+ \setminus \Gamma_+} + \langle \boldsymbol{\beta} \cdot n(u - P_h u), v_h \rangle_{\Gamma_+} \\ &\leqslant \sum_{K \in T_h} |\boldsymbol{\beta}|_\infty^{1/2} \|u - P_h u\|_{L_2(\partial K_+)} (\| |\boldsymbol{\beta} \cdot n|^{1/2} [v_h] \|_{L_2(\partial K_+)} + \| |\boldsymbol{\beta} \cdot n|^{1/2} v_h \|_{L_2(\partial K \cap \Gamma_+)}) \\ &\leqslant 2 |\boldsymbol{\beta}|_\infty^{1/2} \bigg(\sum_{K \in T_h} \|u - P_h u\|_{L_2(\partial K)}^2 \bigg)^{1/2} \|\!|v_h|\!\|_h, \end{aligned}$$

where $\Gamma_+ = \partial\Omega \setminus \Gamma_-$. Therefore, it follows from substituting E_1 and E_2 into (2.15), using approximation property (2.11), inf-sup condition (2.8), and the triangle inequality that the desired estimate is available. $\square$

3.1. Optimal convergence. We first give a simple and useful lemma.

Lemma 3.1. *Let e_K^0 be the collection of faces of K that are neither inflow nor outflow faces. Then we have*

$$|(\boldsymbol{\beta}\cdot n)(x)| \leqslant h_K|\nabla\boldsymbol{\beta}|_{\infty,K} \quad \forall\, x\in e,\ e\in e_K^0. \tag{3.1}$$

P r o o f. Let $e \in e_K^0$ be a face. Because e is neither an inflow nor an outflow face, then there must exist points $x_1, x_2 \in e$ such that $(\boldsymbol{\beta}\cdot n)(x_1) \geqslant 0$ and $(\boldsymbol{\beta}\cdot n)(x_2) \leqslant 0$. Therefore, by the continuity of $(\boldsymbol{\beta}\cdot n)(x)$ on e, there exists a point $x_0 = (1-\theta)x_1 + \theta x_2 \in e$ with some $\theta \in [0,1]$ such that $(\boldsymbol{\beta}\cdot n)(x_0) = 0$. Thus, we have

$$|(\boldsymbol{\beta}\cdot n)(x)| = |\boldsymbol{\beta}(x)\cdot n(x) - \boldsymbol{\beta}(x_0)\cdot n(x_0)| \leqslant |\boldsymbol{\beta}(x)-\boldsymbol{\beta}(x_0)| \leqslant h_K|\nabla\boldsymbol{\beta}|_{\infty,K}, \quad x\in e,$$

noting that the outward unit norm $n(x) = n(x_0)$ on face e. □

In order to obtain the optimal convergence, we still need to introduce a special projection mapping $H^1(T_h)$ into S_h. Define the projection function $\mathcal{P}u \in S_h$, restricted to $K \in T_h$, $\mathcal{P}u \in P_k(K)$ such that

$$\int_K (u-\mathcal{P}u)v\,\mathrm{d}x = 0 \quad \forall\, v\in P_{k-1}(K), \tag{3.2}$$

$$\int_{e_K^+} (u-\mathcal{P}u)v\,\mathrm{d}s = 0 \quad \forall\, v\in P_k(e_K^+), \tag{3.3}$$

where e_K^+ is an outflow face of K and the first condition is vacuous if $k=0$. Note that although K may have several outflow faces for general meshes, we only select one of them to define the projection in (3.3). This projection has been used in some articles (see e.g. [7]), but the authors of the present paper did not find a strict proof of its existence and approximation property in existing literature. We here give the proof in detail.

Theorem 3.1. *The projection function $\mathcal{P}u$ is well posed and satisfies the approximation property*

$$\|u-\mathcal{P}u\|_{L_2(K)} + h_K^{1/2}\|u-\mathcal{P}u|_{L_2(\partial K)} \leqslant Ch_K^{k+1}|u|_{H^{k+1}(K)}, \quad k\geqslant 0,\ K\in T_h, \tag{3.4}$$

where C is a constant independent of the element K.

P r o o f. Let us begin by proving the unique existence of the function $\mathcal{P}u \in P_k(K)$ satisfying (3.2)–(3.3). Since

$$\begin{aligned}\dim(P_{k-1}(K)) + \dim(P_k(e_K^+)) &= \binom{k-1+d}{d} + \binom{k+d-1}{d-1} \\ &= \binom{k+d}{d} = \dim(P_k(K)),\end{aligned}$$

we see that the linear system (3.2)–(3.3) is square, so we only need to show that $\mathcal{P}u = 0$ if $u = 0$. Without loss of generality, we assume that the face e_K^+ in (3.3) lies on the hyperplane $x_1 = 0$ and $x_1 < 0$ when $x \in K$ (otherwise we may use the affine transformation $F\colon K \to \widehat{K}$ such that $e_{\widehat{K}}^+$ lies on $\hat{x}_1 = 0$, and $\hat{x}_1 < 0$ when $\hat{x} \in \widehat{K}$). Let $u = 0$. Then we have from (3.3) that $\mathcal{P}u|_{e_K^+} = 0$ and hence, $\mathcal{P}u = x_1 p$ for some polynomial $p \in P_{k-1}(K)$. Taking $v = p$ in (3.2), we get

$$(x_1 p, p)_K = (x_1, p^2)_K = 0,$$

since $x_1 < 0$ on K, we conclude that $p = 0$. This implies that $\mathcal{P}u = 0$ on K.

Now we are in the position to prove the approximation property (3.4). Let P_h be the L_2-projection defined by (2.10). From (3.3) we see that

$$\|\mathcal{P}u - P_h u\|_{L_2(e_K^+)} = \|\mathcal{P}(u - P_h u)\|_{L_2(e_K^+)} \leqslant \|u - P_h u\|_{L_2(e_K^+)}. \tag{3.5}$$

Introduce the polynomial space

$$P_k^0(K) = \{v \in P_k(K)\colon\ (v, p)_K = 0 \quad \forall\, p \in P_{k-1}(K)\}.$$

It is easy to see that $\|\cdot\|_{L_2(e_K^+)}$ defines a norm on the space $P_k^0(K)$ (see the argument of the unique existence) and this norm is equivalent to the norm $\|\cdot\|_{L_2(K)}$, since $P_k^0(K)$ is a finite dimensional space. Then, by using the norm equivalence on the reference element and a simple scaling argument, we have

$$\|v\|_{L_2(K)} \leqslant C h_K^{1/2} \|v\|_{L_2(e_K^+)} \quad \forall\, v \in P_k^0(K),$$

which, together with (3.5) and $\mathcal{P}u - P_h u \in P_k^0(K)$, implies

$$\|\mathcal{P}u - P_h u\|_{L_2(K)} \leqslant C h_K^{1/2} \|\mathcal{P}u - P_h u\|_{L_2(e_K^+)} \leqslant C h_K^{1/2} \|u - P_h u\|_{L_2(e_K^+)}.$$

This completes the proof by using the triangle inequality and the approximation property (2.11). $\square$

Now we can give the first main result of this paper.

Theorem 3.2. *Assume that T_h is a shape-regular triangulation satisfying flow condition* (1.4)*, and let u and u_h be the solutions of problems* (2.1) *and* (2.4)*, respectively, $u \in H^1(\Omega) \cap H^{k+1}(T_h)$. Then we have the following optimal convergence estimate:*

$$\|u - u_h\| \leqslant Ch^{k+1}|u|_{H^{k+1}(T_h)}, \quad k \geqslant 0. \tag{3.6}$$

Proof. First, by a similar derivation to that of (2.15), we obtain for $v_h \in S_h$ that

$$a_h(u_h - \mathcal{P}u, v_h) = (u - \mathcal{P}u, \mathcal{L}^* v_h)_h + \sum_{K\in T_h} \langle \boldsymbol{\beta}\cdot n(u - \widehat{\mathcal{P}u}), v_h\rangle_{\partial K\setminus\Gamma_-} = F_1 + F_2, \tag{3.7}$$

and (noting that $\boldsymbol{\beta}^c \cdot \nabla v_h \in P_{k-1}(K)$)

$$\begin{aligned} F_1 &= -(u - \mathcal{P}u, (\boldsymbol{\beta} - \boldsymbol{\beta}^c)\cdot\nabla v_h)_h + ((\alpha - \operatorname{div}\boldsymbol{\beta})(u - \mathcal{P}u), v_h)_h \\ &\leqslant C\|u - \mathcal{P}u\|\,\|v_h\| \leqslant Ch^{k+1}|u|_{H^{k+1}(T_h)}\|v_h\|. \end{aligned}$$

It remains to estimate F_2. We begin by writing F_2 as

$$\begin{aligned} F_2 = &\sum_{K\in T_h} \langle \boldsymbol{\beta}\cdot n(u - \widehat{\mathcal{P}u}), v_h\rangle_{(\partial K\setminus e_K^0)\setminus\Gamma_-} \\ &+ \sum_{K\in T_h} \langle \boldsymbol{\beta}\cdot n(u - \widehat{\mathcal{P}u}), v_h\rangle_{e_K^0\setminus\Gamma_-} = S_1 + S_2. \end{aligned} \tag{3.8}$$

Since the numerical trace $\widehat{\mathcal{P}u}$ is continuous across the interfaces of elements in T_h and an interior face of K is the outflow face if and only if it is an inflow face of some adjacent element, we obtain

$$S_1 = \sum_{K\in T_h} \langle \boldsymbol{\beta}\cdot n(u - \mathcal{P}u), [v_h]\rangle_{e_K^+}.$$

From the *flow* condition (1.4) we know that $\boldsymbol{\beta}^c \cdot n[v_h]|_{e_K^+} \in P_k(e_K^+)$. Then, we have from (3.3) that

$$S_1 = \sum_{K\in T_h} \langle (\boldsymbol{\beta} - \boldsymbol{\beta}^c)\cdot n(u - \mathcal{P}u), [v_h]\rangle_{e_K^+}.$$

Hence, by using the approximation properties and the inverse inequality, we obtain

$$S_1 \leqslant \sum_{K\in T_h} Ch_K|\nabla\boldsymbol{\beta}|_{\infty,K}\|u - \mathcal{P}u\|_{L_2(e_K^+)}\|v_h\|_{L_2(e_K^+)} \leqslant Ch^{k+1}|u|_{H^{k+1}(T_h)}\|v_h\|.$$

Next, from Lemma 3.1 we know that

$$
\begin{aligned}
S_2 &\leqslant \sum_{K\in T_h} h_K |\nabla\boldsymbol{\beta}|_{\infty,K} \|u-\widehat{\mathcal{P}u}\|_{L_2(e_K^0)} \|v_h\|_{L_2(e_K^0)} \\
&\leqslant \sum_{K\in T_h} h_K |\nabla\boldsymbol{\beta}|_{\infty,K} \|u-\mathcal{P}u\|_{L_2(\partial K)} \|v_h\|_{L_2(\partial K)} \leqslant Ch^{k+1}|u|_{H^{k+1}(T_h)}\|v_h\|.
\end{aligned}
$$

Now, substituting S_1 and S_2 into (3.8) to obtain the estimate for F_2, and then substituting F_1 and F_2 into (3.7), we arrive at

$$
a_h(u_h-\mathcal{P}u, v_h) \leqslant Ch^{k+1}|u|_{H^{k+1}(T_h)}\|v_h\| \quad \forall\, v_h \in S_h. \tag{3.9}
$$

It follows from taking $v_h = u_h - \mathcal{P}u$ and noting that $a_h(v_h, v_h) \geqslant \sigma_0\|v_h\|^2$ that the proof is completed by using the triangle inequality and approximation property (3.4). □

3.2. The approximation of $\partial_{\boldsymbol{\beta}} u$ by the derivative recovery technique. In this subsection, we consider the approximation of the directional derivative $\partial_{\boldsymbol{\beta}} u = \boldsymbol{\beta}\cdot\nabla u$ by using the post-processing technique.

Let u_h be the DG solution. The error order of $\|\partial_{\boldsymbol{\beta}}(u-u_h)\|_{L_2(T_h)}$ will be, in general, one order lower than that of $\|u-u_h\|$. In order to improve the approximation accuracy, some post-processing methods have been used, e.g., see [7]. However, these methods usually cost much additional computation. Here we provide a very simple derivative recovery formula which gives a superconvergent approximation to $\partial_{\boldsymbol{\beta}} u$.

Theorem 3.3. *Let T_h be an arbitrary shape-regular triangulation, u and u_h the solutions of problem* (2.1) *and* (2.4), *respectively. Define the recovery formula of the derivative $\partial_{\boldsymbol{\beta}} u$ by*

$$
R_h(\partial_{\boldsymbol{\beta}} u) = f - \alpha u_h \quad \text{in } \Omega. \tag{3.10}
$$

Then we have

$$
\partial_{\boldsymbol{\beta}} u - R_h(\partial_{\boldsymbol{\beta}} u) = -\alpha(u-u_h). \tag{3.11}
$$

P r o o f. Equality (3.11) comes from (2.1) and (3.10) directly. □

A direct result of Theorem 3.3 is

$$
\|\partial_{\boldsymbol{\beta}} u - R_h(\partial_{\boldsymbol{\beta}} u)\| \leqslant \|\alpha(u-u_h)\| \leqslant Ch^{k+s}|u|_{H^{k+1}(T_h)}, \tag{3.12}
$$

where $s = 1/2$ for general meshes, and $s = 1$ for the meshes satisfying *flow* condition (1.4), see Lemma 2.1 and Theorem 3.2.

Remark 3.1. Compared with the post-processing formula in [7], Theorem 2.3, our formula (3.10) is simpler and dependent only on u_h and problem data f and α.

4. A posteriori error analysis

It is very important for the finite element method to have a computable a posteriori error bound so that we can assess the accuracy of the finite element solution and enhance the computation efficiency by adaptive algorithms in practical applications. In this section, we will establish an efficient and reliable a posteriori error estimator for the DG method (2.4) associated with general meshes and special meshes.

In what follows, we assume that $|\boldsymbol{\beta}(x)| \geqslant \boldsymbol{\beta}_{\min} > 0$. Since

$$\begin{aligned}\boldsymbol{\beta}^c|_K &= \frac{1}{K}\int_K (\beta_1,\dots,\beta_d)^{\mathrm{T}}\,\mathrm{d}x = (\beta_1(\xi_1),\dots,\beta_d(\xi_d))^{\mathrm{T}}\\ &= \boldsymbol{\beta}(\xi_1) + (\beta_1(\xi_1),\dots,\beta_d(\xi_d))^{\mathrm{T}} - \boldsymbol{\beta}(\xi_1),\end{aligned}$$

we have when h is small

$$\begin{aligned}|\boldsymbol{\beta}^c(x)| &\geqslant |\boldsymbol{\beta}(\xi_1)| - \Big(\sum_{i=1}^d |\beta_i(\xi_i) - \beta_i(\xi_1)|^2\Big)^{1/2}\\ &\geqslant \boldsymbol{\beta}_{\min} - h_K|\nabla\boldsymbol{\beta}|_{\infty,K} > 0, \quad x \in K.\end{aligned} \tag{4.1}$$

Denote by T_{h^*} a uniform refinement of T_h with mesh size $h^* = \alpha_0 h$, $0 < \alpha_0 < 1$, and a corresponding DG space S_{h^*} such that $S_h \subset S_{h^*}$. For a $K \in T_h$, let us denote by $\mathring{F}_K = \{e \in \partial K^* \setminus \partial K\colon\ K^* \subset K\}$ the set of interior faces of simplices $K^* \subset K$, i.e., the subgrid faces that are not included in a face of K. We assume that the subdivision is such that

$$|\boldsymbol{\beta}^c \cdot n_e| > 0 \quad \forall\, e \in \mathring{F}_K,\ K \in T_h. \tag{4.2}$$

It is easy to see that in two-dimensional space there is always at least one subdivision of each K such that condition (4.2) holds. For instance, compare the two subdivisions in Figure 4.1. If $|\boldsymbol{\beta}^c \cdot n_e| = 0$ for some face in one of the subdivisions, it will be larger than zero on all faces in the other, since no two interior faces are parallel between the two types of refinements.

Introduce the auxiliary problem: Find $u_{h^*} \in S_{h^*}$ such that

$$a_{h^*}(u_{h^*}, v_{h^*}) = (f, v_{h^*}) - \langle \boldsymbol{\beta}\cdot ng,\ v_{h^*}\rangle_{\Gamma_-} \quad \forall\, v_{h^*} \in S_{h^*}. \tag{4.3}$$

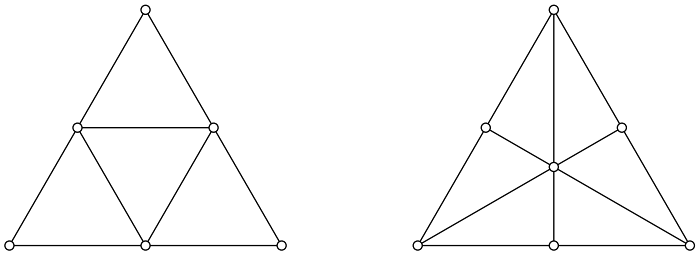

Figure 4.1. Two different types of subgrids such that $|\boldsymbol{\beta}^c \cdot n|$ cannot vanish on the interior faces in both Type I (left) and Type II (right) simultaneously.

Let u_h be the DG solution of problem (2.4), and note that $a_h(u_h, v_h) = a_{h^*}(u_h, v_h)$ (see (2.6)). Then, we have the orthogonal equation

$$a_{h^*}(u_{h^*} - u_h, v_h) = 0 \quad \forall\, v_h \in S_h \subset S_{h^*}. \tag{4.4}$$

Let $v \in S_h \oplus H^1(\Omega)$. Since

$$\boldsymbol{\beta} \cdot \nabla v \in L_2(K), \quad [v]|_e = 0, \quad e \in \mathring{F}_K, \quad \frac{1}{2} h_K \leqslant h_{K^*} < h_K, \quad K^* \subset K, \quad \forall\, K \in T_h,$$

we have

$$\frac{1}{2} \|v\|_h^2 \leqslant \|v\|_{h^*}^2 \leqslant \|v\|_h^2 \quad \forall\, v \in S_h \oplus H^1(\Omega). \tag{4.5}$$

The key of our a posteriori analysis is to introduce the so-called *saturation* assumption, which has been used widely in existing literature (see [1], [2], [3], [4], [8]): There exists a constant $\delta < 1$, independent of h, such that

$$\|u - u_{h^*}\|_{h^*} \leqslant \delta \|u - u_h\|_{h^*}. \tag{4.6}$$

Clearly, by Lemma 2.1, assumption (4.6) is expected to hold for smooth u and h sufficiently small. Under this assumption, we have

$$\|u - u_h\|_{h^*} \leqslant \|u - u_{h^*}\|_{h^*} + \|u_{h^*} - u_h\|_{h^*} \leqslant \delta \|u - u_h\|_{h^*} + \|u_{h^*} - u_h\|_{h^*},$$

which, together with (4.5), implies

$$\|u - u_h\|_h \leqslant \sqrt{2}(1 - \delta)^{-1} \|u_{h^*} - u_h\|_{h^*}. \tag{4.7}$$

Thus, the a posteriori error estimate of $u - u_h$ is converted to an estimate of the error $u_{h^*} - u_h$. The following approximation result is basic in our analysis.

Lemma 4.1. *Let $i_h\colon S_{h^*}\to S_h$ be an interpolation or projection operator such that $i_h v = v$ for all $v\in P_k(K)$. Then we have*

$$\sum_{K\in T_h} h_K^{-1}\|v_{h^*}-i_h v_{h^*}\|_{0,K}^2 \leqslant C|\!|\!|v_{h^*}|\!|\!|_{h^*}^2 \quad \forall\, v_{h^*}\in S_{h^*}.$$

P r o o f. For any simplex $K\in T_h$, introduce the piecewise polynomial space $P_*(K)=\{(v_{h^*}-i_h v_{h^*})|_K\colon\ v_{h^*}\in S_{h^*}\}$ and the notation

$$\|v_{h^*}-i_h v_{h^*}\|_{*,K}^2 \equiv \sum_{K^*\subset K}\|\boldsymbol{\beta}^c\cdot\nabla v_{h^*}\|_{0,K^*}^2+\sum_{e\in\mathring{F}_K}\int_e|\boldsymbol{\beta}^c\cdot n|[v_{h^*}]^2\,\mathrm{d}s.$$

We first prove that $\|\cdot\|_{*,K}$ is a norm on $P_*(K)$. Let $\|v_{h^*}-i_h v_{h^*}\|_{*,K}=0$. Then it follows from (4.2) that

$$\frac{\partial v_{h^*}}{\partial\boldsymbol{\beta}^c}\Big|_{K^*}=0,\ [v_{h^*}]\big|_e=0 \quad \forall\, K^*\subset K,\ e\in\mathring{F}_K,$$

which implies that $v_{h^*}=c_l$ on $l\cap K$, where l is any line that is parallel to $\boldsymbol{\beta}^c$ and c_l is a constant. Since $\boldsymbol{\beta}^c$ is not parallel to any face $e\in\mathring{F}_K$ (see (4.2)), we see that the constant c_l crosses each face $e\in\mathring{F}_K$. Then $v_{h^*}\in P_*(K)$ implies that v_{h^*} is a polynomial on K such that $v_{h^*}-i_h v_{h^*}=0$, noting that $i_h v=v$ if $v\in P_k(K)$. This shows that $\|\cdot\|_{*,K}$ is a norm on $P_*(K)$. Now, using the norm equivalence on the reference element and a scaling argument, we obtain

$$h_K^{-1}\|v_{h^*}-i_h v_{h^*}\|_{0,K}^2\leqslant C\Big(\sum_{K^*\subset K}h_K\|\boldsymbol{\beta}^c\cdot\nabla v_{h^*}\|_{0,K^*}^2+\sum_{e\in\mathring{F}_K}\int_e|\boldsymbol{\beta}^c\cdot n|[v_{h^*}]^2\,\mathrm{d}s\Big). \tag{4.8}$$

Noting that $\boldsymbol{\beta}^c=\boldsymbol{\beta}+O(h_K)|\boldsymbol{\beta}|_{1,\infty,K}$, it follows from (4.8) and the inverse inequality that

$$\begin{aligned}
&h_K^{-1}\|v_{h^*}-i_h v_{h^*}\|_{0,K}^2\\
&\qquad\leqslant C\sum_{K^*\subset K}\Big(\|v_{h^*}\|_{0,K^*}^2+h_{K^*}\|\boldsymbol{\beta}\cdot\nabla v_{h^*}\|_{0,K^*}^2+\int_{\partial K^*}|\boldsymbol{\beta}\cdot n|[v_{h^*}]^2\,\mathrm{d}s\Big),
\end{aligned}$$

which implies the conclusion of Lemma 4.1. □

Let $\pi_K^{(k)}w\in P_k(K)$ be an approximation of the function w on K, for example, the L_2-projection or interpolation approximation. Similarly, $\pi_e^{(k)}w\in P_k(e)$ is an approximation of w on the face $e \subset \partial K$. Introduce several quantities,

$$\eta_1(u_h) = \Big(\sum_{K\in T_h} h_K \|(f-\mathcal{L}u_h) - \pi_K^{(k)}(f-\mathcal{L}u_h)\|_{0,K}^2\Big)^{1/2},$$

$$\eta_2(u_h) = \Big(\sum_{K\in T_h} \int_{\partial K\setminus\partial\Omega} |\boldsymbol{\beta}\cdot n|^2 |[u_h]|^2 \,\mathrm{d}s\Big)^{1/2},$$

$$\eta_3(u_h) = \Big(\sum_{K\in T_h} \int_{\partial K\cap\Gamma_-} |\boldsymbol{\beta}\cdot n|^2 (g-u_h)^2 \,\mathrm{d}s\Big)^{1/2}.$$

Obviously, these quantities are computable in terms of the DG solution u_h. Now we can give the second main result of this paper.

Theorem 4.1. *Let u and u_h be the solutions of problems* (2.1) *and* (2.4), *respectively, $u \in H^1(\Omega)$ and $|\boldsymbol{\beta}| \geqslant \boldsymbol{\beta}_{\min} > 0$. Then, we have the following a posteriori error estimate:*

$$\|\!|u-u_h|\!\|_h \leqslant C(\eta_1(u_h)+\eta_2(u_h)+\eta_3(u_h)), \quad k \geqslant 0. \tag{4.9}$$

P r o o f. According to (4.7), we only need to estimate $\|\!|u_{h^*}-u_h|\!\|_{h^*}$. Using the expression (2.6) of $a_h(u,v)$, equations (4.3) and (4.4), we gain for $v_{h^*} \in S_{h^*}$ that

$$\begin{aligned} a_{h^*}(u_{h^*}-u_h, v_{h^*}) &= a_{h^*}(u_{h^*}-u_h, v_{h^*}-P_h v_{h^*}) \\ &= a_{h^*}(u_{h^*}, v_{h^*}-P_h v_{h^*}) - a_{h^*}(u_h, v_{h^*}-P_h v_{h^*}) \\ &= (f, v_{h^*}-P_h v_{h^*}) - \int_{\Gamma_-} \boldsymbol{\beta}\cdot n g(v_{h^*}-P_h v_{h^*})\,\mathrm{d}s - a_h(u_h, v_{h^*}-P_h v_{h^*}) \\ &= (f-\mathcal{L}u_h, v_{h^*}-P_h v_{h^*})_h + \sum_{K\in T_h} \int_{\partial K_-\setminus\Gamma_-} \boldsymbol{\beta}\cdot n[u_h](v_{h^*}-P_h v_{h^*})\,\mathrm{d}s \\ &\quad - \int_{\Gamma_-} \boldsymbol{\beta}\cdot n(g-u_h)(v_{h^*}-P_h v_{h^*})\,\mathrm{d}s \\ &= T_1+T_2+T_3. \end{aligned} \tag{4.10}$$

Below we estimate the terms T_1, T_2, and T_3. First, by using Lemma 4.1 (taking $i_h = P_h$), we have

$$\begin{aligned} T_1 &= (f-\mathcal{L}u_h - \pi_K^{(k)}(f-\mathcal{L}u_h), v_{h^*}-P_h v_{h^*})_h \\ &\leqslant \eta_1(u_h)\Big(\sum_{K\in T_h} h_K^{-1}\|v_{h^*}-P_h v_{h^*}\|_{0,K}^2\Big)^{1/2} \leqslant C\eta_1(u_h)\|\!|v_{h^*}|\!\|_{h^*}. \end{aligned}$$

Next, using the inverse inequality and Lemma 4.1, we obtain

$$T_2 \leqslant \eta_2(u_h)\Big(\sum_{K\in T_h} \|v_{h^*} - P_h v_{h^*}\|^2_{L_2(\partial K)}\Big)^{1/2} \leqslant C\eta_2(u_h)|\!|\!| v_{h^*}|\!|\!|_{h^*},$$

$$T_3 \leqslant \eta_3(u_h)\Big(\sum_{K\in T_h} \|v_{h^*} - P_h v_{h^*}\|^2_{L_2(\partial K\cap\Gamma_-)}\Big)^{1/2} \leqslant C\eta_3(u_h)|\!|\!| v_{h^*}|\!|\!|_{h^*}.$$

Substituting the estimates T_1, T_2, and T_3 into (4.10) and using the inf-sup condition (2.8) on S_{h^*}, we arrive at the conclusion of Theorem 4.1. $\square$

Now, we consider the special meshes. We assume that the triangulation T_h satisfies *flow* condition (1.4) with respect to $-\boldsymbol{\beta}$, that is:

(4.11) Each simplex K has a unique inflow face e_K^- with respect to $\boldsymbol{\beta}$ and there are no hanging nodes on each interior inflow face e_K^-.

Corresponding to *flow* condition (4.11), we introduce the projection $\mathcal{P}^-$ which satisfies conditions (3.2) and (3.3) with e_K^- replacing e_K^+. Introduce estimate quantities,

$$\widetilde{\eta}_1(u_h) = \Big(\sum_{K\in T_h} h_K\|(f-\mathcal{L}u_h) - \pi_K^{(k-1)}(f-\mathcal{L}u_h)\|^2_{0,K}\Big)^{1/2},$$

$$\widetilde{\eta}_2(u_h) = \Big(\sum_{K\in T_h}\int_{\partial K\setminus\partial\Omega} h_K^2|\nabla\boldsymbol{\beta}|^2_{\infty,K}|[u_h]|^2\,\mathrm{d}s\Big)^{1/2},$$

$$\widetilde{\eta}_3(u_h) = \Big(\sum_{K\in T_h}\int_{\partial K\cap\Gamma_-} (h_K^2|\nabla\boldsymbol{\beta}|^2_{\infty,K}(g-u_h)^2 + |\boldsymbol{\beta}^c\cdot n|^2|(g-\pi_e^{(k)}g)|^2)\,\mathrm{d}s\Big)^{1/2},$$

where $\pi_K^{(k-1)} = 0$ if $k=0$.

Theorem 4.2. *Assume that T_h is a shape-regular triangulation satisfying flow condition* (4.11), *and let u and u_h be the solutions of problems* (2.1) *and* (2.4), *respectively, $u\in H^1(\Omega)$ and $|\boldsymbol{\beta}|\geqslant\boldsymbol{\beta}_{\min}>0$. Then we have*

$$|\!|\!| u-u_h|\!|\!|_h \leqslant C(\widetilde{\eta}_1(u_h)+\widetilde{\eta}_2(u_h)+\widetilde{\eta}_3(u_h)),\ k\geqslant 0. \tag{4.12}$$

P r o o f. We only need to estimate $|\!|\!| u_{h^*}-u_h|\!|\!|_{h^*}$. Using the projection $\mathcal{P}^-$ instead of P_h in (4.10), we obtain for $v_{h^*}\in S_{h^*}$ that

$$\begin{aligned} &a_{h^*}(u_{h^*}-u_h, v_{h^*}) \\ &\quad= (f-\mathcal{L}u_h, v_{h^*}-\mathcal{P}^-v_{h^*})_h + \sum_{K\in T_h}\int_{\partial K_-\setminus\Gamma_-}\boldsymbol{\beta}\cdot n[u_h](v_{h^*}-\mathcal{P}^-v_{h^*})\,\mathrm{d}s \\ &\qquad - \int_{\Gamma_-}\boldsymbol{\beta}\cdot n(g-u_h)(v_{h^*}-\mathcal{P}^-v_{h^*})\,\mathrm{d}s = S_1+S_2+S_3. \end{aligned} \tag{4.13}$$

Now we estimate terms S_1, S_2, and S_3. First, by using Lemma 4.1 and (3.2), we have

$$S_1 = (f - \mathcal{L}u_h - \pi_K^{(k-1)}(f - \mathcal{L}u_h), v_{h^*} - \mathcal{P}^- v_{h^*})_h$$
$$\leqslant \widetilde{\eta}_1(u_h)\Big(\sum_{K\in T_h} h_K^{-1}\|v_{h^*} - \mathcal{P}^- v_{h^*}\|_{0,K}^2\Big)^{1/2} \leqslant C\widetilde{\eta}_1(u_h)|\!|\!|v_{h^*}|\!|\!|_{h^*}.$$

Next, using *flow* condition (4.11) and the orthogonal property of $\mathcal{P}^-$, we may write

$$\int_{\partial K_-\setminus\Gamma_-} \boldsymbol{\beta}\cdot n[u_h](v_{h^*} - \mathcal{P}^- v_{h^*})\,\mathrm{d}s$$
$$= \int_{e_K^-\setminus\Gamma_-} \boldsymbol{\beta}\cdot n[u_h](v_{h^*} - \mathcal{P}^- v_{h^*})\,\mathrm{d}s + \int_{\partial K_-\cap e_K^0} \boldsymbol{\beta}\cdot n[u_h](v_{h^*} - \mathcal{P}^- v_{h^*})\,\mathrm{d}s$$
$$\leqslant \int_{e_K^-\setminus\Gamma_-} (\boldsymbol{\beta} - \boldsymbol{\beta}^c)\cdot n[u_h](v_{h^*} - \mathcal{P}^- v_{h^*})\,\mathrm{d}s + \int_{e_K^0} |\boldsymbol{\beta}\cdot n[u_h](v_{h^*} - \mathcal{P}^- v_{h^*})|\,\mathrm{d}s.$$

Then, it follows from Lemma 3.1, the inverse inequality, and Lemma 4.1 that

$$S_2 \leqslant C\widetilde{\eta}_2(u_h)\Big(\sum_{K\in T_h}\|v_{h^*} - \mathcal{P}^- v_{h^*}\|_{L_2(\partial K)}^2\Big)^{1/2} \leqslant C\widetilde{\eta}_2(u_h)|\!|\!|v_{h^*}|\!|\!|_{h^*}.$$

Similarly, we have

$$S_3 = -\int_{\Gamma_-} \boldsymbol{\beta}\cdot n(g - u_h)(v_{h^*} - \mathcal{P}^- v_{h^*})\,\mathrm{d}s$$
$$= -\int_{\Gamma_-} (\boldsymbol{\beta} - \boldsymbol{\beta}^c)\cdot n(g - u_h)(v_{h^*} - \mathcal{P}^- v_{h^*})\,\mathrm{d}s$$
$$+ \int_{\Gamma_-} \boldsymbol{\beta}^c\cdot n(g - u_h)(v_{h^*} - \mathcal{P}^- v_{h^*})\,\mathrm{d}s$$
$$\leqslant C\widetilde{\eta}_3(u_h)|\!|\!|v_{h^*}|\!|\!|_{h^*} + \int_{\Gamma_-} \boldsymbol{\beta}^c\cdot n(g - \pi_e^{(k)}g)(v_{h^*} - \mathcal{P}^- v_{h^*})\,\mathrm{d}s \leqslant C\widetilde{\eta}_3(u_h)|\!|\!|v_{h^*}|\!|\!|_{h^*}.$$

Theorem 4.2 follows from substituting estimates S_1, S_2, and S_3 into (4.13) and using the inf-sup condition (2.8) on S_{h^*}. □

If $\boldsymbol{\beta}$ is a constant vector, under the conditions of Theorem 4.2, we can obtain a sharper error upper bound as follows:

$$|\!|\!|u - u_h|\!|\!|_h \leqslant C\Big(\sum_{K\in T_h} h_K\|(f - \alpha u_h) - \pi_K^{(k-1)}(f - \alpha u_h)\|_{0,K}^2$$
$$+ \int_{\Gamma_-} |\boldsymbol{\beta}\cdot n|^2|(g - \pi_e^{(k)}g)|^2\,\mathrm{d}s\Big)^{1/2}. \tag{4.14}$$

We now give the lower bound estimates of the error $u - u_h$.

Theorem 4.3. *Let u and u_h be the solutions of problems* (2.1) *and* (2.4), *respectively, $u \in H^1(\Omega)$. Then, the following local lower bounds hold:*

$$
\begin{aligned}
(4.15)\qquad & h_K\|(f-\mathcal{L}u_h)-\pi_K^{(k)}(f-\mathcal{L}u_h)\|_{0,K}^2 \\
&\quad \leqslant 8h_K\|\alpha(u-u_h)\|_{0,K}^2 + 8h_K\|\boldsymbol{\beta}\cdot\nabla(u-u_h)\|_{0,K}^2,
\end{aligned}
$$

$$
(4.16)\qquad \int_{\partial K}|\boldsymbol{\beta}\cdot n|^2|[u_h]|^2\,\mathrm{d}s = \int_{\partial K}|\boldsymbol{\beta}\cdot n|^2|[u-u_h]|^2\,\mathrm{d}s.
$$

Furthermore, we have the global lower bound estimate:

$$
(4.17)\qquad \eta_1(u_h)+\eta_2(u_h)+\eta_3(u_h) \leqslant C_\alpha |\!|\!| u-u_h |\!|\!|_h,
$$

where $C_\alpha = \max\{8|\alpha|_\infty h/\sigma_0, 8\}$.

Proof. Taking $\pi_K^{(k)} = P_h$, we have

$$
\begin{aligned}
h_K\|(f-\mathcal{L}u_h)-\pi_K^{(k)}(f-\mathcal{L}u_h)\|_{0,K}^2 & \\
&\leqslant 4h_K\|f-\mathcal{L}u_h\|_{0,K}^2 = 4h_K\|\mathcal{L}(u-u_h)\|_{0,K}^2 \\
&\leqslant 4h_K(\|\alpha(u-u_h)\|_{0,K} + \|\boldsymbol{\beta}\cdot\nabla(u-u_h)\|_{0,K})^2,
\end{aligned}
$$

which implies (4.15). Equality (4.16) is obvious. From (4.15) and (4.16) we arrive at (4.17). □

Applying the results of Theorem 4.1 and Theorem 4.3, we have the upper bound and the lower bound estimates, that is, for $\eta(u_h) = \eta_1(u_h)+\eta_2(u_h)+\eta_3(u_h)$, there holds

$$
C_\alpha^{-1}\eta(u_h) \leqslant |\!|\!| u-u_h |\!|\!|_h \leqslant C\eta(u_h),
$$

where the lower bound may be local. This shows that our a posteriori error estimates are reliable and efficient, and the error bounds are almost sharp.

5. Numerical experiments

In this section we present numerical examples to illustrate our theoretical analysis. In our experiments, we take $\Omega = (1,2)\times(1,2)$, $\boldsymbol{\beta} = (x,y)$, $\alpha = 2$, the exact solution $u(x,y) = \sin x \sin y$, and use the linear discontinuous finite element. In order to make the triangulation satisfy *flow* condition (1.4) with respect to $\boldsymbol{\beta}$, we first draw the streamlines of $\boldsymbol{\beta}$ (see Figure 5.1 left), and then construct the meshes so that each element has one edge lying on one of the streamlines (see Figure 5.1 right). Thus, *flow* condition (1.4) holds with respect to $\boldsymbol{\beta}$.

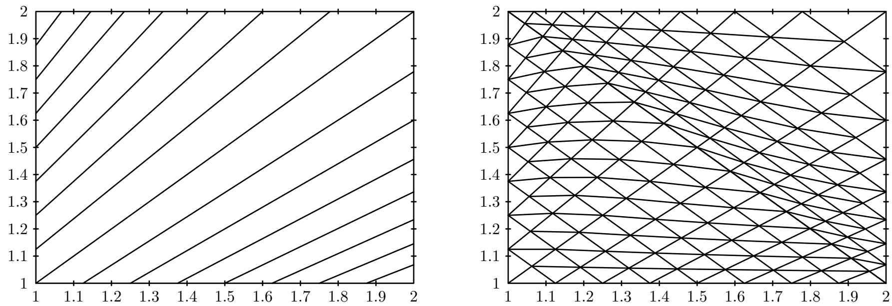

Figure 5.1. Meshes satisfying the flow condition with respect to $\boldsymbol{\beta} = (x, y)$: The streamlines of $\boldsymbol{\beta}$ (left) and the actual mesh (right).

Denote by e_h the error between the exact solution and the DG solution with mesh size h in the L_2-norm, and the numerical convergence order is computed by $r = \ln(e_h/e_{h/2})/\ln 2$. Let $\eta(u_h) = \sum_{i=1}^{3} \eta_i(u_h)$ be the a posteriori error estimator (see Theorem 4.1). Introduce the efficiency index $\sigma = \eta(u_h)/|\!|\!| u - u_h |\!|\!|_h$. In Table 5.1, for successively halving h, we display the errors and the orders of convergence, as well as the index σ for the approximate solutions. As expected, we see that the convergence order is optimal and the error estimator is robust and effective.

	$\|u - u_h\|$			
h	error	order	$\|\!\|\!\| u - u_h \|\!\|\!\|_h$	σ
1/4	5.2327e-3	–	2.5632e-2	1.3305
1/8	1.3172e-3	1.9901	8.4502e-3	1.3149
1/16	3.3002e-4	1.9968	2.7439e-3	1.2490
1/32	8.2567e-5	1.9989	8.9478e-4	1.2560
1/64	2.0648e-5	1.9996	2.9517e-4	1.2191
1/128	5.1624e-6	1.9999	9.8670e-5	1.2251

Table 5.1. Histories of convergence and efficiency indexes

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