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A determinant formula for the relative class number of an imaginary abelian number field

Mikihito Hirabayashi

Abstract. We give a new formula for the relative class number of an imaginary abelian number field $K$ by means of determinant with elements being integers of a cyclotomic field generated by the values of an odd Dirichlet character associated to $K$. We prove it by a specialization of determinant formula of Hasse.

1 Introduction

There are lots of formulas for the relative class number of an imaginary abelian number field $K$ by means of determinant (see [5] for bibliography). In this paper we give such a new formula. We prove it by a specialization of the determinant formula for generalized group matrix which appears in [2, §13]. The key idea is a transformation of generalized Bernoulli numbers and a transformation of their product over the odd characters to one over the even characters. In our formula, elements of the determinant are integers of a cyclotomic field generated by the values of an odd Dirichlet character associated to $K$, whereas elements of the determinants are rational numbers for known formulas. We may regard our formula as an imaginary version of Hasse’s formula [2, §16, (3)], which expresses the class number of a real abelian number field by means of determinant with elements being logarithms of cyclotomic units of its cyclic subfields.

2 Results

Let $K$ be an imaginary abelian number field of degree $n$ and with conductor $f$, and let $K_0$ be the maximal real subfield of $K$. Let $H_0$ be the subgroup of the group $(\mathbb{Z}/f\mathbb{Z})^\times$ of reduced residue classes modulo $f$ corresponding to $K_0$. Let $X_0$ be the set of Dirichlet characters associated to $K_0$.

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We assume that the Dirichlet characters \( \chi \) associated to \( K \), which we call characters of \( K \) for short, are primitive and that, as usual, \( \chi(x) = 0 \) for an integer \( x \) not relatively prime to the conductor \( f(\chi) \) of \( \chi \).

We classify the group \( \mathcal{X}_0 \) by the following equivalence \( \sim \): for characters \( \chi, \psi \in \mathcal{X}_0 \) let \( \chi \sim \psi \) if and only if there exists an integer \( m \) such that \( m \) is relatively prime to \( n_\chi \) and that \( \psi = \chi^m \), where \( n_\chi \) is the order of \( \chi \). We call the classes classified by this equivalence Frobenius classes. Let \( \{ \psi_0 \} \) be a system of representatives of the Frobenius classes. For a representative \( \psi_0 \) let \( t_{\psi_0} \) be an integer such that the quotient group \( (\mathbf{Z}/f\mathbf{Z})^\times/H_{\psi_0} \) is generated by a class represented by \( t_{\psi_0} \mod f \), where \( H_{\psi_0} = \{ x \mod f \in (\mathbf{Z}/f\mathbf{Z})^\times; \psi_0(x) = 1 \} \).

We fix an odd character \( \chi_0 \) of \( K \). As we will see, the elements of the determinant of our formula are integers of the field generated by the values of the character \( \chi_0^* \).

For an even character \( \chi_0 \) of \( K \) and for an element \( a \mod f \) of \( (\mathbf{Z}/f\mathbf{Z})^\times \) let

\[
\begin{align*}
\quad & u_{\chi_0}(a) = -\chi_0^*(a) \sum_{\substack{x=1 \quad (x,f)=1 \quad \chi_0(x)=1}}^{f} \, \chi_0^*(x) R_f(ax),
\end{align*}
\]

where \( R_f(a) \) is the least positive residue modulo \( f \) of \( a \). Then we define a matrix \( U \) by

\[
\begin{align*}
\begin{pmatrix}
\quad & u_{\psi_0}(s_{\psi_0}^{-1}(-k))\mod f \end{pmatrix}_{0 \leq k \leq \varphi(n_{\psi_0})-1},
\end{align*}
\]

where \( (s \mod f)H_0 \) runs in the rows over the quotient group \( (\mathbf{Z}/f\mathbf{Z})^\times/H_0 \), which is isomorphic to the Galois group \( G_0 \) of \( K_0 \); \( \psi_0 \) and \( k \) run in the columns: \( \{ \psi_0 \} \) is a system defined above and \( \varphi \) is the Euler totient function. Here, \( t_{\psi_0}^{-k} \mod f \) is the inverse of \( t_{\psi_0}^k \mod f \), i.e., \( t_{\psi_0}^{-k} \) is an integer satisfying \( t_{\psi_0}^{-k} \psi_0 \equiv 1 \mod f \).

With the notation above we have the following

**Theorem 1.** For an imaginary abelian number field \( K \) of degree \( n \) and with conductor \( f \), we have

\[
\det U = \pm \left( \frac{2f}{Q} \right)^{n/2} c g^* h^* w
\]

where \( h^* \) is the relative class number of \( K \), \( Q \) is the Hasse unit index of \( K \), \( w \) is the number of roots of unity in \( K \), and \( g^* \) is defined by

\[
g^* = \prod_{\chi_1} \prod_{p|f} (1 - \chi_1(p))
\]

where the products \( \prod_{\chi_1} \) and \( \prod_{p|f} \) are taken over the odd characters \( \chi_1 \) of \( K \) and the prime numbers \( p \) dividing \( f \), respectively, and \( c \) is a natural number expressed by

\[
c = \prod_{p|n_0} p^\frac{1}{2} \sum_{p^r|n_0} \left( \frac{q(m)}{p^r} - \frac{n_0}{p^r} \right),
\]

where the product \( \prod_{p|n_0} \) and the sum \( \sum_{p^r|n_0} \) are taken over prime numbers \( p \) dividing \( n_0 = n/2 \) and the powers of \( p \) dividing \( n_0 \), respectively, and \( q(m) \) is the number of solutions of \( x^m = 1 \) in \( G_0 \).
We remark here that the elements \( u_{\chi_0}(a) \) and the matrix \( U \) depend on the character \( \chi_1^* \), as we see in the examples below, and that, in addition, \( U \) depends on the choice of integers \( t_{\psi_0} \). In fact, we have different \( U \)'s for different \( t_{\psi_0} \)'s in the case of \( K = \mathbb{Q}(\zeta_7) \), the 7th cyclotomic field. Moreover, we note that the matrix \( U \) never coincides with any matrix in known formulas, because \( U \) always contains a constant column corresponding to the principal character \( \psi_0 = 1 \).

As seen by definition, the number \( g^* \) may be zero and then remains a problem of how to construct such a formula in Theorem 1 in case of \( g^* = 0 \).

For the cyclotomic fields of prime power conductor we have the following corollaries.

**Corollary 1.** For the cyclotomic field \( K = \mathbb{Q}(\zeta_{p^\nu}) \) of conductor \( p^\nu \) (\( p \geq 1 \), \( p \) an odd prime, we have

\[
\det U = \det \left( u_{\psi_0} \left( g^t \left( t_{\psi_0}^k \right) \right) \right)_{0 \leq i \leq \frac{p^\nu - 1 \cdot (p-1)}{2}; 0 \leq k \leq \varphi(n_{\psi_0})-1} = \pm (2p^\nu)^{\frac{p^\nu - 1 \cdot (p-1)}{2} - 1}h^*,
\]

where \( g \) is a primitive root modulo \( p^\nu \).

For the field \( K = \mathbb{Q}(\zeta_{p^\nu}) \) we can take \( t_{\psi_0} = g \) for every \( \psi_0 \neq 1 \) and \( t_{\psi_0} = 1 \) for \( \psi_0 = 1 \).

**Corollary 2.** For the cyclotomic field \( K = \mathbb{Q}(\zeta_{2^\nu}) \) of conductor \( 2^\nu \) (\( \nu \geq 2 \)) we have

\[
\det U = \left( u_{\psi_0} \left( 5^t \left( t_{\psi_0}^k \right) \right) \right)_{0 \leq i \leq 2^\nu - 1; 0 \leq k \leq \varphi(n_{\psi_0})-1} = \pm 2^{(\nu+1)2^\nu - 2 - \nu}h^*.
\]

For the field \( K = \mathbb{Q}(\zeta_{2^\nu}) \) we can take \( t_{\psi_0} = 5 \) for every \( \psi_0 \neq 1 \) and \( t_{\psi_0} = 1 \) for \( \psi_0 = 1 \).

Here we give examples. We adopt the basic characters which Hasse used in [2].

For an odd prime \( p \) let \( \chi_p \) be an odd character modulo \( p \) of order \( p - 1 \) and \( \psi_{p^\nu} \) (\( \nu \geq 2 \)) an even character modulo \( p^\nu \) of order \( p^\nu - 1 \); in addition \( \psi_{p^\nu} = \psi_{p^\nu-1} \).

For the prime 2 let \( \chi_4 \) be the odd character modulo 4 and \( \psi_{2^\nu} \) (\( \nu \geq 3 \)) an even character modulo \( 2^\nu \) of order \( 2^{\nu-2} \); in addition \( \psi_{2^\nu}^2 = \psi_{2^\nu-1} \). The subscript of a basic character denotes the conductor.

For the following calculation of the values of \( u_{\chi_0}(a) \), we use the identity

\[
\sum_{x=1}^{f} \frac{\chi_1^*(x)R_f(ax)}{(x,f)=1} = \sum_{x=1}^{[f/2]} \frac{\chi_1^*(x)(2R_f(ax) - f)}{(x,f)=1, \chi_0(x)=1}.
\]

**Example 1.** Let \( K = \mathbb{Q}(\zeta_5) \), i.e., \( p = 5 \), \( \rho = 1 \). Take \( g = 2 \) and \( \chi_1^* = \chi_5 \). Then \( \{\psi_0\} = \{1, \chi_5^2\} \) and

\[
u_1(a) = -\chi_5(a) \left( 2R_5(a) - 5 + i \left( 2R_5(2a) - 5 \right) \right), \]
\[
u_2(a) = -\chi_5(a) \left( 2R_5(a) - 5 \right).
\]
Consequently
\[ U = \begin{pmatrix} u_1(1) & u_{\chi_1^3}(1) \\ u_1(2) & u_{\chi_2^5}(2) \end{pmatrix} = \begin{pmatrix} 3 + i & 3 \\ 3 + i & i \end{pmatrix} \]
and hence \( \det U = -2 \cdot 5 \). Otherwise, by Corollary [1] and [2, Tafel II], \( \det U = \pm (2 \cdot 5)^{\frac{p-2}{2}} \cdot 1 = \pm 2 \cdot 5 \).

Taking \( g = 2 \) and \( \chi_1^* = \chi_5^3 \), we have
\[ U = \begin{pmatrix} 3 - i & 3 \\ 3 - i & -i \end{pmatrix} \]
and hence \( \det U = -2 \cdot 5 \).

**Example 2.** Let \( K = \mathbb{Q}(\zeta_{23}) \), i.e., \( p = 2, \rho = 3 \). Take \( \chi_1^* = \chi_4 \). Then \( \{\psi_0\} = \{1, \psi_{23}\} \) and
\[ u_1(a) = -2\chi_4(a)(R_{23}(a) - R_{23}(3a)), \]
\[ u_{\psi_{23}}(a) = -2\chi_4(a)(R_{23}(a) - 4). \]

Consequently
\[ U = \begin{pmatrix} u_1(1) & u_{\psi_{23}}(1) \\ u_1(5) & u_{\psi_{23}}(5) \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 4 & -2 \end{pmatrix} \]
and hence \( \det U = -2^5 \). Otherwise, by Corollary [2] and [2, Tafel II], \( \det U = \pm 2(3+1)^{2-2} \cdot 1 = \pm 2^5 \).

Taking \( \chi_1^* = \chi_4\psi_8 \), we have
\[ U = \begin{pmatrix} 8 & 6 \\ 8 & 2 \end{pmatrix} \]
and hence \( \det U = -2^5 \).

**Example 3.** Let \( K = \mathbb{Q}(\sqrt{-3}, \sqrt{5}) \). Take \( \chi_1^* = \chi_3 \). Then \( \{\psi_0\} = \{1, \chi_5^2\} \) and
\[ u_1(a) = -2\chi_3(a)(R_{15}(a) - R_{15}(2a) + R_{15}(4a) + R_{15}(7a) - 15), \]
\[ u_{\chi_3^2}(a) = -2\chi_3(a)(R_{15}(a) + R_{15}(4a) - 15). \]

Consequently
\[ U = \begin{pmatrix} u_1(1) & u_{\chi_3^2}(1) \\ u_1(2) & u_{\chi_3^2}(2) \end{pmatrix} = \begin{pmatrix} 10 & 20 \\ 10 & -10 \end{pmatrix} \]
and hence \( \det U = -2^2 \cdot 3 \cdot 5^2 \). Otherwise, since \( c = 1, g^* = 2, w = 2 \cdot 3 \) and \( Q = 1 \), which is obtained by [2, Tafel II], we have by Theorem [1]
\[ \det U = \pm \frac{(2f)^{n/2}c^g^*}{Qw}h^* = \pm \frac{(2 \cdot 15)^2 \cdot 1 \cdot 2}{1 \cdot 2 \cdot 3} \cdot 1 = \pm 2^2 \cdot 3 \cdot 5^2. \]

Taking \( \chi_1^* = \chi_3\chi_5^2 \), we have
\[ U = \begin{pmatrix} 30 & 20 \\ 30 & 10 \end{pmatrix} \]
and hence \( \det U = -2^2 \cdot 3 \cdot 5^2 \).
3 The determinant of a generalized group matrix

In the second chapter of the book [2] Hasse gave two transformations of the class number formula for a real abelian number field; the first transformation is an application of summations $\sum s \chi(s)u_\chi(s)$ to the group matrix, $A_f(x)$ an ordinary distribution (cf. [2 p.18] or [3 Lemma 12.15]), and the second transformation is one for summations $\sum s \chi(s)u_\chi(s)$ and for the matrix $U_\phi$ (see Lemma 1).

By the first transformation, replacing the distribution $A_f(x)$ in [2 p.18] with

$$A_f(x) = -\frac{\left( R_f(x) - \frac{1}{2} \right)}{f},$$

we can obtain the formula of Girstmair [1] with Maillet determinant for the relative class number of an imaginary abelian number field with conductor $f$.

For the proof of our formula we need the following lemmas. Let $\mathcal{G}$ be an abelian group of order $n$ and $\mathcal{X}$ the group of characters of $\mathcal{G}$. For $\chi \in \mathcal{X}$ let

$$\mathfrak{H}_\chi = \{ x \in \mathcal{G}; \chi(x) = 1 \}.$$

For $s \in \mathcal{G}$ and $\chi \in \mathcal{X}$ let $u_\chi(s)$ be a complex-valued function satisfying the following conditions:

(i) $u_\chi(s) = u_\chi^\nu(s)$ for $s \in \mathcal{G}$ and $\nu \in \mathcal{Z}$ relatively prime to the order $n_\chi$ of $\chi$.

(ii) $u_\chi^s(s) = u_\chi(s')$ for $s, s' \in \mathcal{G}$ with $\chi(s) = \chi(s')$.

We classify the group $\mathcal{X}$ by the Frobenius equivalence defined as in §2. Let $\{ \psi \}$ be a system of representatives of the Frobenius classes of $\mathcal{X}$. For a character $\psi$ let $t_\psi$ be a representative of a generator $t_\psi \mathfrak{H}_\psi$ of the cyclic group $\mathcal{G}/\mathfrak{H}_\psi$. Then we define a matrix $U_\phi$ by

$$U_\phi = (u_\psi(st_\psi^{-k}))_{s \in \mathcal{G}; \psi, 0 \leq k \leq \varphi(n_\psi) - 1},$$

where $s$ runs in the rows, and $\psi$ and $k$ run in the columns.

**Lemma 1.** [2 §14] For the matrix $U_\phi$ we have

$$\det U_\phi = \pm c_\phi \prod_{\chi \in \mathcal{X}} \sum_{s \mod \mathfrak{H}_\chi} \chi(s)u_\chi(s),$$

where $c_\phi$ is a positive number defined by

$$c_\phi = \pm \frac{1}{\det(\chi(s))_{s \in \mathcal{G}; \chi \in \mathcal{X}}} \prod_{\psi} \left( \frac{n_{\psi}}{n_\phi} \right)^{\varphi(n_\psi)} \det(\psi(t_\psi)^{ik})_{1 \leq i \leq n_\psi; 0 \leq k \leq \varphi(n_\psi) - 1}$$

and $s \mod \mathfrak{H}_\chi$ in the sum $\sum_{s \mod \mathfrak{H}_\chi}$ runs over the quotient group $\mathcal{G}/\mathfrak{H}_\chi$.

**Lemma 2.** [2 §14 and §15] For an abelian group $\mathcal{G}$ of order $n$ the number $c_\phi$ is a natural number and holds

$$c_\phi = \prod_{p \mid n} p^{\frac{1}{2} \sum_{p^k \mid n} \left( q(\frac{n}{p^k}) - \frac{n}{p^k} \right)},$$

where the product and summation are taken over the prime numbers $p$ dividing $n$ and over the powers of $p$ dividing $n$, and $q(m)$ is the number of solutions of $x^m = 1$ in $\mathcal{G}$. Therefore $c_\phi = 1$ if and only if $\mathcal{G}$ is cyclic.
4 Proof of Theorem 1

Proof of Theorem 1. We start with the arithmetic class number formula for \( h^* \),

\[
h^* = Qw \prod_{\chi_1} \left( -\frac{1}{2} B_{1,\chi_1} \right).
\]

For any odd character \( \chi_1 \) of \( K \) we have

\[
B_{1,\chi_1} = \frac{1}{f(\chi_1)} \sum_{a=1}^{f(\chi_1)} \chi_1(a) a = \frac{1}{f} \sum_{a=1}^{f} \chi_1(a) a
\]

and like as [4, Lemma 8.7] we have

\[
\sum_{a=1}^{f} \chi_1(a) a = \prod_{p|f} (1 - \chi_1(p)) \cdot \sum_{a=1}^{f} \chi_1(a) a.
\]

In fact, if \( p | f \), we have

\[
\chi_1(p) \sum_{a=1}^{f/p} \chi_1(a) a = \sum_{b=1}^{f/p} \chi_0(pb)(pb) \quad \text{and hence}
\]

\[
\prod_{p|f} (1 - \chi_1(p)) \cdot \sum_{a=1}^{f} \chi_1(a) a = \sum_{a=1}^{f} \chi(a) a + \sum_{d|f} \left( \sum_{d'|d} \mu(d') \right) \chi(d)d
\]

\[
= \sum_{a=1}^{f} \chi(a) a - \sum_{d|f} \chi(d)d = \sum_{d|f} \chi_1(a) a,
\]

where \( \mu(\cdot) \) is the Möbius function.

Therefore, putting

\[
S(\chi_1) = \sum_{a=1}^{f} \chi_1(a) a,
\]

we have by the arithmetic class number formula for \( h^* \)

\[
\frac{(-2f)^{n/2} g^* h^*}{Q w} = \prod_{\chi_1} S(\chi_1)
\]

and hence our task is to show that the product of the right-hand side is \( \pm c^{-1} \det U \).

Recall that \( \chi_1^* \) is a fixed odd character of \( K \). For an even character \( \chi_0 \) of \( K \) let

\[
H_{\chi_0} = \{ x \mod f \in (\mathbb{Z}/f\mathbb{Z})^\times : \chi_0(x) = 1 \}.
\]

Choose a system of representatives \( s \mod f \) of \( (\mathbb{Z}/f\mathbb{Z})^\times /H_{\chi_0} \). Then, for an odd character \( \chi_1 = \chi_0 \chi_1^* \) of \( K \) we have

\[
S(\chi_1) = S(\chi_0 \chi_1^*) = \sum_{s \mod H_{\chi_0}} \chi_0(s) u_{\chi_0}(s),
\]
A determinant formula for the relative class number

where

\[ u_{\chi_0}(s) = \chi_1^*(s) \sum_{x=1}^{f} \chi_1^*(x) R_f(sx) \, . \]

Therefore we have

\[ \prod_{\chi_1} S(\chi_1) = \prod_{\chi_0 \mod H_{\chi_0}} \chi_0(s) u_{\chi_0}(s) \, , \]

where the product \( \prod_{\chi_0} \) is taken over the even characters \( \chi_0 \) of \( K \).

Here we use Lemmas 1 and 2 by letting \( G \) be the group \((\mathbb{Z}/f\mathbb{Z})^* / H_0\) and by replacing \( n \) by \( n/2 \), \( \chi \) by \( \chi_0 \), \( U_{\mathfrak{p}} \) by \( U \), \( c_{\mathfrak{p}} \) by \( c \), and \( u_\psi(s) \) by \( u_{\psi_0}(s) \).

To use Lemma 1, we need to check the \( u_{\chi_0}(s) \) for meeting the conditions (i) and (ii) in \( \S 3 \). First let \( \nu \) be an integer relatively prime to the order of \( \chi_0 \). Then

\[ \chi_{\nu \chi_0}(x) = 1 \text{ if and only if } \chi_0(x) = 1 \, . \]

Hence

\[ u_{\chi_0 \nu}(s) = \chi_1^*(s) \sum_{x=1}^{f} \chi_1^*(x) R_f(sx) \]

\[ = u_{\chi_0}(s) \, . \]

Secondly let \( s, s' \) be integers relatively prime to \( f \) satisfying \( \chi_0(s) = \chi_0(s') \). Hence

\[ u_{\chi_0}(s') = \chi_1^*(s') \sum_{x=1}^{f} \chi_1^*(x) R_f(s'x) \]

\[ = \chi_1^*(s') \sum_{x=1}^{f} \chi_1^*(s(s')^{-1}x) R_f(s' \cdot s(s')^{-1}x) \]

\[ = \chi_1^*(s') \sum_{x=1}^{f} \chi_1^*(s) \chi_1^*(s')^{-1} \chi_1^*(x) R_f(sx) \]

\[ = \chi_1^*(s) \sum_{x=1}^{f} \chi_1^*(x) R_f(sx) \]

\[ = u_{\chi_0}(s) \, . \]

Here \( (s')^{-1} \mod f \) is the inverse of \( s' \mod f \). Therefore we have checked the conditions.
Consequently, by Lemma 1 we obtain
\[
\frac{(-2f)^{n/2}g^*h^*}{Qw} = \prod_{\chi_1} S(\chi_1) = \frac{1}{\pm c} \det U,
\]
that is,
\[
\det U = \pm \frac{(2f)^{n/2}c g^*}{Qw} h^*
\]
and by Lemma 2 we immediately obtain the expression of \(c\). This completes the proof.

Corollaries 1 and 2 are directly obtained by Theorem 1, because for the cyclotomic fields \(K\) of prime power conductors we have \(g^* = 1\) by definition, \(c = 1\) by Lemma 2 and \(Q = 1\) by [2 Satz 27].

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References


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