Josua Groeger

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Abstract. The classical Wilson loop is the gauge-invariant trace of the parallel transport around a closed path with respect to a connection on a vector bundle over a smooth manifold. We build a precise mathematical model of the super Wilson loop, an extension introduced by Mason-Skinner and Caron-Huot, by endowing the objects occurring with auxiliary Graßmann generators coming from S-points. A key feature of our model is a supergeometric parallel transport, which allows for a natural notion of holonomy on a supermanifold as a Lie group valued functor. Our main results for that theory comprise an Ambrose-Singer theorem as well as a natural analogon of the holonomy principle. Finally, we compare our holonomy functor with the holonomy supergroup introduced by Galaev in the common situation of a topological point. It turns out that both theories are different, yet related in a sense made precise.

1 Introduction

Gluon scattering amplitudes have been known to be dual to Wilson loops along lightlike polygons [1], [2], [7], [11]. While these quantum expectation values, which are formally calculated by means of the path integral, remain problematic from a mathematical point of view, the underlying classical theory has been well understood. In fact, a Wilson line refers to parallel transport with respect to a connection on a vector bundle along a path in the underlying smooth manifold. In the usual context of flat spacetime (Minkowski space) with a single global coordinate chart, the corresponding solution operator can be written in terms of a path-ordered exponential.

Recently, a similar duality (at weak coupling) between the full superamplitude of $\mathcal{N} = 4$ super Yang-Mills theory and two variants of a supersymmetric extension of the Wilson loop has been claimed. The first approach [21] originates in momentum twistor space and translates into the integral over a superconnection in spacetime, while the second [9] attaches to lightlike polygons certain edge and

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vertex operators, whose shape is determined by supersymmetry constraints [15]. Both approaches agree, in the common domain of definition, up to a term depending on the equations of motion [6] and indeed satisfy the conjectured duality upon subtracting an anomalous contribution [5].

The first purpose of the present article is to build a supergeometric model of super Wilson loops that leads to the same characteristic formulas as summarised in Section 2.2 of [6]. The main idea is to give the objects occurring an inner structure through auxiliary Graßmann generators coming from S-points. While the resulting additional degrees of freedom come without physical significance, this approach is well-justified mathematically and has been performed successfully in modelling other aspects of superfield theory. Notably, consider "maps with flesh" as introduced by Hélein in [18] as models for superfields including bosons and fermions. See also [10], [16], [19] for the same concept under different terminology and [14] for their differential calculus.

A key feature of our model is the supergeometric parallel transport introduced in Section 2, which allows for a natural notion of holonomy at an S-point of a supermanifold as a Lie group valued functor. A different notion of holonomy on supermanifolds was introduced by Galaev in [12] by taking a suitable generalisation of the Ambrose-Singer theorem as the definition of a super Lie algebra and endowing this to a Harish-Chandra pair, thus obtaining a super Lie group for every topological point of the manifold. Developing a new holonomy theory by means of our parallel transport, and comparing it to Galaev's, is the second objective of this article.

In Section 3, we establish two main results generalising properties of classical holonomy. The first is an Ambrose-Singer theorem, which describes the holonomy Lie algebra in terms of curvature, while the second formulates a natural analogon of the holonomy principle relating parallel sections to holonomy-invariant vectors.

Our Ambrose-Singer theorem facilitates the comparison of our holonomy functor with Galaev's theory, which is the subject matter of Section 4. Since this functor is, in general, not representable, both theories are different in the common situation of a topological point. Nevertheless, we show that they are related in that the generators of Galaev's holonomy algebra can be extracted as certain coefficients by considering special S-points. This construction is based on the knowledge of the geometric significance of the elements and, in this sense, is not algebraic.

2 Super Wilson Loops and Parallel Transport

The super Wilson loop described in [6] and [21] is constructed as follows. Consider n "superpoints" (x_i, θ_i) in chiral superspace, which are symbolic quantities in that their exact mathematical type is not important, only their calculation rules such as

$$x_i^\mu \cdot x_j^\nu = x_j^\nu \cdot x_i^\mu \, , \qquad \theta_i^{\alpha A} \cdot \theta_j^{\beta B} = -\theta_j^{\beta B} \cdot \theta_i^{\alpha A} \tag{1}$$

These superpoints are connected by "straight lines"

$$x(t_i) = x_i - t_i x_{i,i+1} \, , \qquad \theta(t_i) = \theta_i - t_i \theta_{i,i+1} \tag{2}$$

thus yielding a closed "superpath" γ parametrised by one bosonic variable t, which enters the Wilson loop via

$$W_\gamma = \mathrm{tr}\left(X \mapsto \mathcal{P}\exp\left(\int_0^1 ig\mathcal{B}(t)\,dt\right)[X]\right), \qquad \mathcal{B}(t) = \mathcal{B}_\xi \cdot \dot{\gamma}^\xi(t) \tag{3}$$

where $\mathcal{P}\exp$ denotes a path-ordered exponential, and $\mathcal{B}_\xi$ is a connection one-form in coordinates ξ. This connection has a very specific form due to supersymmetry conditions which, however, is not relevant for our purposes.

As a mathematical model for a more general situation, let M be a supermanifold of dimension $\dim M = (\dim M)_{\bar{0}}|(\dim M)_{\bar{1}}$ (such as chiral superspace), and let S be another supermanifold which should be thought of as auxiliary. Throughout, we employ the definitions of Berezin-Kostant-Leites [20]. A supermanifold M is thus, in particular, a ringed space $M = (M_0, \mathcal{O}_M)$, and a morphism $\varphi\colon M \to N$ consists of two parts $\varphi = (\varphi_0, \varphi^\sharp)$ with $\varphi_0\colon M_0 \to N_0$ a smooth map and $\varphi^\sharp$ a generalised pullback of superfunctions $f \in \mathcal{O}_N$. Modern monographs on the general theory of supermanifolds include [8] and [27].

Definition 1. An S-point of M is a morphism $x = (x_0, x^\sharp)\colon S \to M$. A (smooth) S-path γ connecting S-points x and y is a morphism

$$\gamma = (\gamma_0, \gamma^\sharp)\colon S \times [0,1] \to M \qquad \text{such that} \qquad \mathrm{ev}|_{t=0}\gamma^\sharp = x^\sharp\,, \quad \mathrm{ev}|_{t=1}\gamma^\sharp = y^\sharp$$

which we shall denote, by a slight abuse of notation, by $\gamma\colon x \to y$. It is called closed (or an S-loop) if $x = y$.

In the following, we will exclusively consider superpoints

$$S = \mathbb{R}^{0|L} = \left(\{0\}, \bigwedge\mathbb{R}^L\right), \qquad \bigwedge\mathbb{R}^L = \langle \eta^1, \dots, \eta^L\rangle\,, \qquad L \in \mathbb{N}\,. \tag{4}$$

Although most of our results should continue to hold accordingly for general S, this restriction will turn out to suffice for reproducing the characteristic formulas of super Wilson loops as well as allowing for a powerful holonomy theory. This significance of superpoints does not come unexpected. According to [25], an inner Hom object $\underline{\mathrm{Hom}}(M,N)$ in the category of supermanifolds is determined by its $\bigwedge\mathbb{R}^L$-points

$$\underline{\mathrm{Hom}}(M,N)\left(\bigwedge\mathbb{R}^L\right) \cong \mathrm{Hom}_{\mathrm{SMan}}\left(\mathbb{R}^{0|L} \times M,\, N\right)$$

in the sense of Molotkov-Sachse theory [22], [24]. The morphisms on the right are the aforementioned "maps with flesh" [18].

Definition 2. Let $x, y, z\colon S \to M$ be S-points and $\gamma\colon x \to y$ and $\delta\colon y \to z$ be S-paths. For fixed $t_0 \in [0,1]$, we prescribe

$$\mathrm{ev}|_{t=t_0}(\delta \star \gamma)^\sharp := \begin{cases} \mathrm{ev}|_{t=2t_0}\gamma^\sharp & t_0 \le 1/2\,, \\ \mathrm{ev}|_{t=2\left(t_0-\frac{1}{2}\right)}\delta^\sharp & t_0 \ge 1/2\,. \end{cases}$$

This defines an S-point which coincides with x,y,z for $t_0 = 0, \frac{1}{2}, 1$, respectively. Similarly, we define

$$\mathrm{ev}|_{t=t_0}(\gamma^{-1})^\sharp := \mathrm{ev}|_{t=(1-t_0)}\gamma^\sharp\,.$$

Considering all $t_0 \in [0,1]$ at a time, the previous definition yields S-paths $\delta \star \gamma\colon x \to z$ and $\gamma^{-1}\colon y \to x$, referred to as the concatenation of γ and δ and the inverse of γ, respectively. The concatenation is, however, only piecewise smooth in the sense of the following definition.

Definition 3. Let x, y be S-points. A piecewise smooth S-path $\gamma\colon x \to y$ connecting x with y is a tuple $(\gamma_j\colon S\times[t_j,t_{j+1}] \to M)_{j=0}^{l}$ with $t_0 = 0$, $t_l = 1$ and $t_j < t_{j+1}$ such that $\mathrm{ev}|_{t=t_{j+1}}\gamma_j^\sharp = \mathrm{ev}|_{t=t_{j+1}}\gamma_{j+1}^\sharp$ and $\mathrm{ev}|_{t=0}\gamma_0^\sharp = x^\sharp$ and $\mathrm{ev}|_{t=1}\gamma_l^\sharp = y^\sharp$, and such that $\gamma_j|_{S\times[t_j,t_{j+1}]}$ is a morphism.

The concatenation $(\delta \star \gamma)$ and inverse γ^{-1} of piecewise smooth paths δ and γ are defined analogously. The construction is such that the underlying path $(\delta\star\gamma)_0$ is the classical concatenation of δ_0 and γ_0, and $(\gamma^{-1})_0 = (\gamma_0)^{-1}$.

Example 1. Comparing with the objects in [6], we state the following dictionary. Let $(x^\mu, \theta^{\alpha A})$ denote (global) coordinates on $M \cong \mathbb{R}^{n|m}$ (using spacetime indices μ rather than spinor indices $\dot{\alpha}\alpha$). Then a superpoint is an S-point $\xi = (\xi_0, \xi^\sharp)\colon S \to M$, identified with $(\xi^\sharp(x^\mu), \xi^\sharp(\theta^{\alpha A})) \in (\mathcal{O}_S)^{n|m}$. The latter tuple is then abbreviated $(x,\theta) = (x^\mu, \theta^{\alpha A})$, for which (1) is satisfied. The straight line connecting superpoints (x_i, θ_i) and (x_{i+1}, θ_{i+1}) is the S-path $\xi_{i,i+1}\colon S\times[0,1] \to M$ defined as follows.

$$\begin{aligned}&\big(\xi^\sharp_{i,i+1}(x^\mu), \xi^\sharp_{i,i+1}(\theta^{\alpha A})\big)\\ &\quad := \Big(\xi^\sharp_i(x^\mu) - t\big(\xi^\sharp_i(x^\mu) - \xi^\sharp_{i+1}(x^\mu)\big),\, \xi^\sharp_i(\theta^{\alpha A}) - t\big(\xi^\sharp_i(\theta^{\alpha A}) - \xi^\sharp_{i+1}(\theta^{\alpha A})\big)\Big)\\ &\quad \in (\mathcal{O}_{S\times[0,1]})^{n|m}.\end{aligned}$$

In this sense, we can understand (2). The last line is $\xi_{n,0}$. Concatenation thus yields a loop.

2.1 Super Vector Bundles and Connections

A super vector bundle $\mathcal{E}$ over a supermanifold M is a sheaf of locally free $\mathcal{O}_M$ supermodules on M. We shall denote its even and odd parts by $\mathcal{E}_{\bar{0}}$ and $\mathcal{E}_{\bar{1}}$, respectively. An important example is the super tangent bundle $\mathcal{S}M := \mathrm{Der}(\mathcal{O}_M)$, which is the sheaf of $\mathcal{O}_M$-superderivations. $\mathcal{E}(U)$ is, for $U \subseteq M_0$ sufficiently small, by definition isomorphic to $\mathcal{O}_M(U)^{\mathrm{rk}\,\mathcal{E}}$ with $\mathrm{rk}\,\mathcal{E} = (\mathrm{rk}\,\mathcal{E})_{\bar{0}}|(\mathrm{rk}\,\mathcal{E})_{\bar{1}}$ the rank of $\mathcal{E}$. Let $(T^j)_{j=1}^{\mathrm{rk}\,\mathcal{E}}$ be an adapted local basis such that $X \in \mathcal{E}(U)$ is identified with the tuple $(X^j)_{j=1}^{\mathrm{rk}\,\mathcal{E}}$ of functions $X^j \in \mathcal{O}_M(U)$ with respect to right coefficients $X = T^j \cdot X^j$ (sum convention). In general, it is preferable to consider right coordinates on supermodules over supercommutative superalgebras, for then superlinear maps can be identified with matrices. For example, the matrix of the differential $d\varphi[X] := X \circ \varphi^\sharp$ for

$X = \partial_{\xi^k} \cdot X^k \in \mathcal{S}M$ of a map $\varphi\colon M \to N$ with respect to coordinates (ξ^j) and (ζ^j) is given by

$$d\varphi^i{}_k := (-1)^{(|\xi^k|+|\zeta^i|)\cdot|\zeta^i|} \frac{\partial \varphi^\sharp(\zeta^i)}{\partial \xi^k} \quad \text{s.th.} \quad d\varphi[X] = \sum_{i,k} \left(\varphi^\sharp \circ \frac{\partial}{\partial \zeta^i} \right) \cdot d\varphi^i{}_k \cdot X^k \tag{5}$$

Lemma 1 (Chain Rule). *Let $\varphi\colon M \to N$ and $\psi\colon N \to P$ be morphisms. Then*

$$d(\psi \circ \varphi)[X] = \left(\varphi^\sharp \circ \psi^\sharp \circ \frac{\partial}{\partial \pi^l} \right) \cdot \varphi^\sharp(d\psi^l{}_i) \cdot d\varphi^i{}_k \cdot X^k$$

with (π^l) coordinates on P and indices k, i referring to (unlabelled) coordinates on M and N, respectively.

Proof. This is proved by a straightforward calculation in local coordinates. □

Definition 4. For a super vector bundle $\mathcal{E}$, and S as in (4), we define

$$\mathcal{E}_S := \mathcal{E} \otimes_{\mathcal{O}_M} \mathcal{O}_{S \times M} \,.$$

An S-connection on M is an even $\mathbb{R}$-linear sheaf morphism

$$\nabla\colon \mathcal{E}_S \to \mathcal{S}M_S^* \otimes_{\mathcal{O}_{S\times M}} \mathcal{E}_S \,, \quad \nabla(fe) = df \otimes_{\mathcal{O}_{S\times M}} e + f \cdot \nabla e \quad \text{for} \quad f \in \mathcal{O}_{S\times M} \,.$$

In particular, $\mathcal{E}_S$ can be considered as a super vector bundle on $S \times M$ and, in this sense, ∇ is an ordinary superconnection. The local picture is as follows. Let $\xi = (x, \theta)$ be coordinates on M and (T^j) be an $\mathcal{E}$-basis. Then $X \in \mathcal{E}_S$ can be expaned as $X = T^j \cdot X^j$ with $X^j \in \mathcal{O}_{S\times M}(\{0\} \times U)$, and

$$\nabla_{\partial_{\xi^i}} X = (-1)^{|\xi^i||T^j|} T^j \partial_{\xi^i}(X^j) + \Gamma_{\xi^i}[T^j] \cdot X^j \,, \qquad \Gamma_{\xi^i}[T^j] := \nabla_{\partial_{\xi^i}} T^j \tag{6}$$

where $\Gamma_{\xi^i} \in \mathrm{Mat}_{\mathrm{rk}\,\mathcal{E}\times\mathrm{rk}\,\mathcal{E}}(\mathcal{O}_{S\times M}(\{0\} \times U))$, which has an expansion

$$\Gamma_{\xi^i} = \sum_{I=(i_1,\ldots,i_{|I|})} \theta^I \cdot (\Gamma_{\xi^i})_I \,, \qquad (\Gamma_{\xi^i})_I \in \mathrm{Mat}_{\mathrm{rk}\,\mathcal{E}\times\mathrm{rk}\,\mathcal{E}}(\mathcal{O}_{S\times M_0}(\{0\} \times U)) \,.$$

Example 2. Consider the trivial vector bundle $\mathcal{E} := \mathfrak{su}(N) \otimes_{\mathbb{R}} \mathcal{O}_M$ with $N \in \mathbb{N}$ of rank $\mathrm{rk}\,\mathcal{E} = \dim \mathfrak{su}(N)|0$ over flat superspace with global coordinates $\xi = (x^\mu, \theta^{\alpha A})$. Define $\mathcal{A}_\mu := \Gamma_{x^\mu}$ and $\mathcal{F}_{\alpha A} := \Gamma_{\theta^{\alpha A}}$. With this notation, the θ-expansion assumes the form

$$\mathcal{A}_\mu = (\mathcal{A}_\mu)_0 + \theta^{\beta B} (\mathcal{A}_\mu)_{\beta B} + \theta^{\beta B} \theta^{\gamma C} (\mathcal{A}_\mu)_{\beta B\, \gamma C} + \ldots$$
$$\mathcal{F}_{\alpha A} = (\mathcal{F}_{\alpha A})_0 + \theta^{\beta B} (\mathcal{F}_{\alpha A})_{\beta B} + \theta^{\beta B} \theta^{\gamma C} (\mathcal{F}_{\alpha A})_{\beta B\, \gamma C} + \ldots$$

Since ∇ is, by definition, even it follows that $\mathcal{A}_\mu$ and $\mathcal{F}_{\alpha A}$ are even respectively odd. The parity of the θ-coefficients in the expansion is thus alternating. This is the situation considered in [6]. In case of a plain connection on $\mathcal{E}$, the odd coefficients in the $\mathcal{A}_\mu$-expansion would be missing, and analogous for $\mathcal{F}_{\alpha A}$.

Let $\mathcal{E} \to N$ be a super vector bundle over N and $\varphi\colon M \to N$ be a morphism of supermanifolds. The pullback of $\mathcal{E}$ under φ is defined as

$$\varphi^*\mathcal{E}(U) := \mathcal{O}_M(U) \otimes_\varphi (\varphi_0^*\mathcal{E})(U) \,, \qquad U \subseteq M_0 \quad \text{open}\,. \tag{7}$$

Here, $\varphi_0^*\mathcal{E}$ is the pullback of the sheaf $\mathcal{E}$ under the continuous map φ_0 which, in terms of its sheaf space, is the bundle of stalks $\mathcal{E}_{\varphi_0(x)}$ attached to $x \in M_0$. In this context, one can define the pullback $\varphi_0^*X \in \varphi_0^*\mathcal{E}$ of $X \in \mathcal{E}$. (7) indeed yields a super vector bundle on M of rank $\operatorname{rk}\mathcal{E}$. For details, consult [16] and [26].

A local frame (T^k) of $\mathcal{E}$ gives rise to a local frame $(\varphi_0^*T^k)$ of $\varphi_0^*\mathcal{E}$ and a local frame $(\varphi^*T^k := 1 \otimes_\varphi \varphi_0^*T^k)$ of $\varphi^*\mathcal{E}$ such that, locally, every section $X \in \varphi^*\mathcal{E}$ can be written $X = \varphi^*T^k \cdot X^k$ with $X^k \in \mathcal{O}_M(U)$. For $Y = T^kY^k \in \mathcal{E}$, we find

$$\varphi^*Y = \varphi^*(T^kY^k) = \varphi^*T^k \cdot \varphi^\sharp(Y^k)\,.$$

Definition 5. In the following, we shall identify maps $\varphi\colon S \times M \to N$ with maps $\hat{\varphi}\colon S \times M \to S \times N$ by composing φ with the canonical inclusion $N \hookrightarrow S \times N$.

In particular, we will use this identification for S-points $x\colon S \to M$ and S-paths $\gamma\colon S \times [0,1] \to M$. In terms of generators η^j as in (4), the construction is such that $\hat{\varphi}^\sharp(\eta^j) = \eta^j$.

Lemma 2. *Let $\varphi\colon S \times M \to N$ and $\mathcal{E} \to N$ be a super vector bundle. Then $\varphi^*\mathcal{E} \cong \hat{\varphi}^*\mathcal{E}_S$.*

Locally, this isomorphism is such that $X = (\hat{\varphi}^*T^k)\cdot X^k \in \hat{\varphi}^*\mathcal{E}_S$ is identified with $X = \varphi^*T^k \cdot X^k \in \varphi^*\mathcal{E}$. We define the pullback of $X \in \mathcal{E}_S$ under $\varphi\colon S \times M \to N$ by

$$\varphi^*X := \hat{\varphi}^*X \in \hat{\varphi}^*\mathcal{E}_S \cong \varphi^*\mathcal{E}\,. \tag{8}$$

Similarly, an endomorphism $E \in \operatorname{End}_{\mathcal{O}_{S\times N}}(\mathcal{E}_S)$ is pulled back under φ to an endomorphism along φ as follows.

$$E_\varphi \in \operatorname{End}_{\mathcal{O}_{S\times M}}(\hat{\varphi}^*\mathcal{E}_S)\,, \qquad E_\varphi(\varphi^*Y) := \varphi^*E(Y) \tag{9}$$

and analogous for other tensors.

Let ∇ be a connection on $\mathcal{E} \to N$ and $\varphi\colon M \to N$ be a morphism. There are two types of pullback connections. With respect to coordinates (ξ^k) of M, we write $X = (\varphi^*\partial_{\xi^i}) \cdot X^i \in \varphi^*\mathcal{S}N$ and prescribe

$$\begin{aligned}&(\varphi^*\nabla)\colon \varphi_0^*\mathcal{E} \to (\varphi^*\mathcal{S}N)^* \otimes_{\mathcal{O}_M} \varphi^*\mathcal{E}\\ &(\varphi^*\nabla)_{(\varphi^*\partial_i)X^i}(\varphi^*Z) := (-1)^{|X^i||\partial_i|}X^i \cdot \varphi^*(\nabla_{\partial_i}Z)\end{aligned} \tag{10}$$

The local representations glue together to a well-defined object satisfying a Leibniz rule. For the second, more common, pullback note that $X \in \varphi^*\mathcal{S}N$ acts naturally on sections $f \in \mathcal{O}_N$ as the superderivation $X(f) := (-1)^{|X^i||f|}(\varphi^\sharp \circ \partial_{\xi^i})(f) \cdot X^i$ along φ. We define

$$\begin{aligned}&(\varphi^*\nabla)\colon \varphi^*\mathcal{E} \to \mathcal{S}M^* \otimes_{\mathcal{O}_M} \varphi^*\mathcal{E}\\ &(\varphi^*\nabla)_X((\varphi^*T^k)Z^k) := (-1)^{|X||T^k|}(\varphi^*T^k) \cdot X(Z^k) + (\varphi^*\nabla)_{d\varphi[X]}\varphi^*T^k \cdot Z^k\end{aligned} \tag{11}$$

using (5) and (10) for the second summand. Again, this prescription is independent of coordinates and $\mathcal{E}$-bases and yields a connection on $\varphi^*\mathcal{E} \to M$.

Let now ∇ be an S-connection on $\mathcal{E}_S$ over N and $\varphi\colon S \times M \to N$. We may consider ∇ as an ordinary connection over $S \times N$ and apply (10) to obtain

$$(\hat{\varphi}^*\nabla)\colon \hat{\varphi}_0^*\mathcal{E}_S \to (\hat{\varphi}^*\mathcal{S}(S \times N))^* \otimes_{\mathcal{O}_{S\times M}} \hat{\varphi}^*\mathcal{E}_S\,.$$

Concatenating this with the adjoint of the inclusion $\mathcal{S}N_S \subseteq \mathcal{S}(S \times N)$, and using $\hat{\varphi}_0^*\mathcal{E}_S = \varphi_0^*\mathcal{E} \otimes \mathcal{O}_S$ as well as $\hat{\varphi}^*\mathcal{S}N_S = \varphi^*\mathcal{S}N$, we yield the first pullback, denoted

$$(\varphi^*\nabla)\colon \varphi_0^*\mathcal{E} \otimes \mathcal{O}_S \to (\varphi^*\mathcal{S}N)^* \otimes_{\mathcal{O}_{S\times M}} \varphi^*\mathcal{E}\,. \tag{12}$$

The second pullback is the connection

$$(\varphi^*\nabla)\colon \varphi^*\mathcal{E} \to \mathcal{S}M_S^* \otimes_{\mathcal{O}_{S\times M}} \varphi^*\mathcal{E} \tag{13}$$

defined verbatim to (11) by means of (12) The local picture is as follows.

$$(\varphi^*\nabla)_X Z = (-1)^{|X||T^k|}(\varphi^*T^k)X(Z^k) + X(\varphi^*(\xi^l))\hat{\varphi}^*(\nabla_{\partial_{\xi^l}}T^k)\cdot Z^k \tag{14}$$

2.2 Parallel Transport

Definition 6. A section $X \in \gamma^*\mathcal{E}$ is called parallel if $(\gamma^*\nabla)_{\partial_t}X \equiv 0$.

The local form is as follows. As above, we write $X = (\gamma^*T^k)\cdot X^k$, thus identifying X with the t-dependent column vector $X(t) \in (\mathcal{O}_S)^{\mathrm{rk}\,\mathcal{E}}$. We further use the notation $\Gamma_{lk}^m \cdot T^m := \Gamma_{\xi^l}[T^k]$ with Γ_{ξ^l} as in (6). By (14), the parallelness condition in local coordinates reads

$$\partial_t X(t) = -\mathcal{B}(t)\cdot X(t)\,, \qquad \mathcal{B}(t)^m{}_k = (-1)^{|T^m|(|T^k|+1)}\partial_t(\gamma^*(\xi^l))\cdot\hat{\gamma}^*(\Gamma_{lk}^m) \tag{15}$$

with $\mathcal{B}(t) \in \mathrm{End}_{\mathcal{O}_S}(\gamma^*\mathcal{E})_{\bar{0}} \cong \mathrm{Mat}_{\mathrm{rk}\,\mathcal{E}\times\mathrm{rk}\,\mathcal{E}}(\mathcal{O}_S)_{\bar{0}}$.

Example 3. In the situation of Example 2, the matrix $\mathcal{B}(t)$ can be written in the form

$$\mathcal{B}(t) = \dot{x}^\mu(t)\mathcal{A}_\mu + \dot{\theta}^{\alpha A}(t)\mathcal{F}_{\alpha A}\,.$$

This is equation (17) of [6].

The next result follows from standard facts on ODEs applied to (15).

Lemma 3. *Let $X_x \in x^*\mathcal{E}$ be a section along an S-point $x\colon S \to M$, and γ be a piecewise smooth S-path with $\mathrm{ev}|_{t=0}\gamma^\sharp = x^\sharp$. Then there exists a unique parallel section $X \in \gamma^*\mathcal{E}$ along γ such that $\mathrm{ev}|_{t=0}X = X_x$.*

Definition 7. Let $\gamma\colon x \to y$ be a smooth S-path and let $X_x \in x^*\mathcal{E}$ be a vector field along $x\colon S \to M$. We define the parallel transport

$$P_\gamma\colon x^*\mathcal{E} \to y^*\mathcal{E}\,, \qquad P_\gamma(X_x) := \mathrm{ev}_{t=1}X$$

where $X \in \gamma^*\mathcal{E}$ denotes the parallel vector field such that $\mathrm{ev}_{t=0}X = X_x$.

For smooth S-paths $\gamma\colon x \to y$ and $\delta\colon y \to z$, we define parallel transport of the concatenation by $P_{\delta\star\gamma} := P_\delta \circ P_\gamma$. If $\delta\star\gamma$ happens to be smooth, this definition agrees with the one from Definition 7 by the following lemma.

Lemma 4. *Let $a < b < c$ and $\gamma\colon S \times [a,c] \to M$ be a smooth S-path. Then*

$$P_{\gamma|_{S\times[b,c]}} \circ P_{\gamma|_{S\times[a,b]}} = P_\gamma\,.$$

Proof. Let $X_x \in x^*\mathcal{E}$. We define $X \in \gamma^*\mathcal{E}$ by setting

$$\begin{aligned} X(t) &:= P_{\gamma|_{S\times[a,t]}}[X_x] \quad (t \in [a,b])\,,\\ X(t) &:= P_{\gamma|_{S\times[b,t]}} \circ P_{\gamma|_{S\times[a,b]}}[X_x] \quad (t \in [b,c])\,. \end{aligned}$$

Then $X(t)$ satisfies $(\gamma^*\nabla)_{\partial_t} X = 0$ for every $t \in [a,c]$ and has the initial condition $X(0) = X_x$. By uniqueness of the solution, we thus conclude that $X(t) = P_{\gamma|_{S\times[a,t]}}[X_x]$. □

Lemma 5. *P_γ is even (i.e. parity-preserving), $\mathcal{O}_S$-superlinear and invertible such that $(P_\gamma)^{-1} = P_{\gamma^{-1}}$.*

Proof. This is shown by standard ODE arguments as follows. Parallel transport is even since the matrix $\mathcal{B}(t)$ in (15) is even. From the same equation, $\mathcal{O}_S$-linearity is clear. It is invertible since both P_γ and $P_{\gamma^{-1}}$ satisfy the same equation (15) at t and $1-t$, respectively. □

By restriction, an S-connection ∇ on $\mathcal{E}_S$ induces a connection

$$\nabla^{\mathcal{E}}\colon \mathcal{E} \to \mathcal{S}M^* \otimes_{\mathcal{O}_M} \mathcal{E}\,.$$

By further restriction, we obtain a classical connection

$$\nabla^0\colon \Gamma(E) \to \Gamma(TM_0) \otimes \Gamma(E)$$

on the vector bundle $E := \bigcup_{x\in M_0} \mathcal{E}_x \to M_0$ (denoted $\tilde{\nabla}$ in [12]). Let $P_{\gamma_0}\colon E_{\gamma_0(0)} \to E_{\gamma_0(1)}$ denote parallel transport along a path $\gamma_0\colon [0,1] \to M_0$ (denoted τ_γ in [12]). On the other hand, let $(P_\gamma)^0\colon E_{\gamma_0(0)} \to E_{\gamma_0(1)}$ denote the restriction of ∇-parallel transport along $\gamma\colon S \times [0,1] \to M$.

Lemma 6. *Let $\gamma\colon x \to y$ be an S-path. Then $(P_\gamma)^0 = P_{\gamma_0}$.*

Proof. This follows immediately from (15). Note that $\partial_t(\gamma^*(\xi^l))$ is odd, for ξ^l an odd coordinate, and thus projected to zero, leaving only even indices l in $\gamma^*(\Gamma^m_{lk})$. □

By Lemma 5, P_γ is an isomorphism from $x^*\mathcal{E}$ to $y^*\mathcal{E}$. With respect to local bases (T^k) and $(\tilde{T}^k)$ of $\mathcal{E}$ around $\gamma_0(0)$ and $\gamma_0(1)$, respectively, it can thus be identified with a matrix in $GL_{\mathrm{rk}\,\mathcal{E}}(\mathcal{O}_S)$.

Lemma 7. *The solution to (15) is given by*

$$X(t) = \mathcal{P}\exp\left(-\int_0^t \mathcal{B}(\tau)d\tau\right)[X_x]$$
$$:= \sum_{j=0}^{\infty}(-1)^j \int_0^t d\tau_j \ldots \int_0^{\tau_2} d\tau_1 \mathcal{B}(\tau_j)\cdot \ldots \cdot \mathcal{B}(\tau_1)X_x$$

where $X_x \in x^*\mathcal{E}$ *and* $x^\sharp = \mathrm{ev}|_{t=0}\gamma^\sharp$.

Proof. By assumption, γ_0 takes values in $U_0 \subseteq M_0$ such that both $M|_{U_0}$ and $\mathcal{E}|_{U_0}$ are trivial. We may thus identify (as vector spaces), for every $t \in [0,1]$, $\mathrm{ev}|_t\gamma^*\mathcal{E}$ with $\mathbb{R}^{\mathrm{rk}\,\mathcal{E}} \otimes \bigwedge \mathbb{R}^L \cong \mathbb{R}^M$ for some $M \in \mathbb{N}$. With this identification, the $\mathbb{R}$-linear operator $\mathcal{B}(t)$ becomes a matrix in $\mathrm{Mat}(M \times M, \mathbb{R})$, and $\partial_t X(t) = -\mathcal{B}(t)\cdot X(t)$ can be considered as a classical first order linear ordinary differential equation. It remains to show that the series stated converges absolutely in the Banach space $C^1([0,1], \mathrm{Mat}(M \times M, \mathbb{R}))$. Then, differentiating termwise, it follows that it is indeed the solution operator. These steps are standard. See Lemma 2.6.7 of [4] for a similar treatment. □

Remark 1. Redefining (6) as $\Gamma_{\xi^i}[T^j] := \frac{i}{g}\nabla_{\partial_{\xi^i}}T^j$, we get the parallelness equation $\partial_t X(t) = ig\mathcal{B}(t)\cdot X(t)$, and thus the solution operator

$$X(t) = \mathcal{P}\exp\left(ig\int_0^t \mathcal{B}(\tau)d\tau\right)[X_x]$$
$$:= \sum_{j=0}^{\infty}(ig)^j \int_0^t d\tau_j \ldots \int_0^{\tau_2} d\tau_1 \mathcal{B}(\tau_j)\cdot \ldots \mathcal{B}(\tau_1)X_x$$

as in (3). This convention is more usual in the physical literature.

An important property of the Wilson loop is its gauge-invariance. We close this chapter showing that the trace of parallel transport around an S-loop is gauge--invariant, thus qualifying as a model for the super Wilson loop. We restrict attention to local gauge transformations in a coordinate chart $U \subseteq M$, which is sufficient for the situation $M \cong \mathbb{R}^{n|m}$ considered in [6] and avoids the theory of super principal bundles.

Definition 8. A (local) gauge transformation is a morphism of supermanifolds

$$V\colon S \times U \to GL_{\mathrm{rk}\,\mathcal{E}} \quad \text{identified with} \quad \big(V^\sharp(\zeta^{kl})\big)_{kl} \in GL_{\mathrm{rk}\,\mathcal{E}}\big(\mathcal{O}_{S\times M}(U)\big)$$

where ζ^{kl} denote the global standard coordinates of the super Lie group $GL_{\mathrm{rk}\,\mathcal{E}}$. It acts on sections $\psi \in \mathcal{E}_S(U)$ and connections ∇ via

$$\psi \mapsto V\cdot\psi\,, \qquad \Gamma_{\xi^i} \mapsto V\cdot\Gamma_{\xi^i}\cdot V^{-1} - (\partial_{\xi^i}V)V^{-1}$$

where Γ_{ξ^i} is as in (6).

Consider an S-path $\gamma\colon S\times[0,1]\to U$ and the concatenation

$$V_\gamma := V\circ\hat{\gamma}\colon S\times[0,1]\to GL_{\mathrm{rk}\,\mathcal{E}}\,.$$

Let $\mathcal{B}(t)$ be as in (15) with respect to the original connection ∇ (and γ) and $\tilde{\mathcal{B}}(t)$ be its gauge transformed counterpart. Then

$$\begin{aligned}\tilde{\mathcal{B}}(t) &= \partial_t(\hat{\gamma}^*(\xi^l))\cdot\hat{\gamma}^*\left(V\Gamma_{\xi^l}V^{-1}-(\partial_{\xi^l}V)V^{-1}\right)\\ &= V_\gamma\cdot\mathcal{B}(t)\cdot V_\gamma^{-1}-(\partial_t V_\gamma)\cdot V_\gamma^{-1}\end{aligned}$$

It follows that

$$(\gamma^*\tilde{\nabla})_{\partial_t}(V_\gamma\cdot X)=\left(\partial_t+V_\gamma\cdot\mathcal{B}(t)\cdot V_\gamma^{-1}-(\partial_t V_\gamma)\cdot V_\gamma^{-1}\right)V_\gamma\cdot X = V_\gamma\cdot(\gamma^*\nabla)_{\partial_t}X$$

In particular, $X\in\gamma^*\mathcal{E}$ is ∇-parallel if and only if $V_\gamma\cdot X\in\gamma^*\mathcal{E}$ is $\tilde{\nabla}$-parallel.

Now let $X_x\in x^*\mathcal{E}$, and let $\gamma\colon x\to y$ connect the S-points x and y. Then, $V_x\cdot X_x$ with $V_x:=V\circ\hat{x}$ is moved by $\tilde{\nabla}$-parallel transport to V_y times ∇-parallel transport of X_x. We thus arrive at the following result.

Proposition 1. *Let $\tilde{P}$ denote parallel transport with respect to the gauge transformed connection $\tilde{\nabla}$. Then $\tilde{P}=V_y\cdot P\cdot V_x^{-1}$. In particular, if $\gamma\colon x\to x$ is closed,*

$$\tilde{P}=V_x\cdot P\cdot V_x^{-1}$$

and the trace $\mathrm{tr}\,P=\mathrm{tr}\,\tilde{P}$ is a gauge invariant quantity.

By now, we have achieved the first aim of this article of constructing a mathematical model of super Wilson loops. Superpoints are S-points, and a super Wilson loop is the gauge-invariant trace of parallel transport around an S-loop. The exact choice of $S=\mathbb{R}^{0|L}$ is not important, except that L should be sufficiently large to make calculations consistent. By means of S, the super Wilson loop acquires an (unphysical) inner structure.

3 The Holonomy of an S-Point

Let $\mathcal{E}$ continue to denote a super vector bundle over a supermanifold M and ∇ be an S-connection on $\mathcal{E}_S$ with S a superpoint (4). In this section, we define the holonomy group of an S-point $x\colon S\to M$ and prove an analogon of the Ambrose-Singer theorem. After endowing the holonomy group to a functor, we establish a holonomy principle in this context, whose proof makes use of at least $(\dim M)_{\bar{1}}$ additional Graßmann generators.

Definition 9. A piecewise smooth S-homotopy is a map

$$\Xi\colon S\times[0,1]\setminus\{t_0,\ldots,t_l\}\times[0,1]\to M$$

such that, denoting the real coordinates by t and s, respectively,

(i) the prescription $\Xi^\sharp_{s_0}:=\mathrm{ev}|_{s=s_0}\Xi^\sharp$ yields a piecewise smooth S-path Ξ_{s_0} for every $s_0\in[0,1]$, and

(ii) $\Xi^\sharp(f)$ is smooth in s for every $f \in \mathcal{O}_M$.

Ξ is called proper if $\mathrm{ev}_{s,t=0}\Xi^\sharp = p^\sharp$ and $\mathrm{ev}_{s,t=1}\Xi^\sharp = q^\sharp$ for all $s \in [0,1]$ and S-points p and q.

Definition 10 (S-Holonomy). Let $x\colon S \to M$ be an S-point. We set

$$\mathrm{Hol}_x := \{P_\gamma \mid \gamma\colon x \to x \text{ piecewise smooth}\} \subseteq \mathrm{End}_{\mathcal{O}_S}(x^*\mathcal{E})$$
$$\mathrm{Hol}_x^0 := \{P_\gamma \mid \gamma\colon x \to x \text{ piecewise smooth and contractible}\}$$

with contractible in the sense that there exists a piecewise smooth proper homotopy Ξ such that $\Xi_0 = x$ and $\Xi_1 = \gamma$.

By Lemma 5, Hol_x is a group which can be identified with a subgroup of $GL_{\mathrm{rk}\,\mathcal{E}}(\mathcal{O}_S)$ with respect to a local basis (T^k) of $\mathcal{E}$. By Theorem 1 below, it is indeed a Lie group. For $S = \mathbb{R}^{0|0}$, it follows by Lemma 6 that $\mathrm{Hol}_x = \mathrm{Hol}_{\nabla^0}(x_0(0))$ is the holonomy group with respect to the underlying connection ∇^0.

We call M path-connected if, for any two S-points x, y, there is an S-path $\gamma\colon x \to y$. By the following result this, as well as contractability, is determined by the classical counterparts such that, in particular, Hol_x does not depend on the restriction of M to any connected component of M_0 different from that of $x_0(0)$.

Lemma 8. *M is path-connected if and only if M_0 is. Moreover, a piecewise smooth S-loop $\gamma\colon x \to x$ is contractible to x if and only if γ_0 is contractible to x_0.*

Proof. It is clear that path-connectedness of M implies that of M_0. Conversely, let $x, y\colon S \to M$ and $\gamma_0\colon x_0(0) \to y_0(0)$ be a connecting classical path. Let $t_j \in [0,1]$ be such that $\gamma_0|_{[t_j,t_{j+1}]}$ is smooth and its image is contained in the open set U_0 for a coordinate chart $U \subseteq M$ with coordinates (ξ^k). Any (smooth) morphism $\gamma^j\colon S \times [t_j, t_{j+1}] \to U$ can be identified with $(\dim M)_{\bar{0}} + (\dim M)_{\bar{1}}$ smooth maps $\gamma^{j\sharp}(\xi^k)\colon [t_j, t_{j+1}] \to \bigwedge \mathbb{R}^L$ or, equivalently, with a single smooth map $\tilde{\gamma}^j\colon [t_j, t_{j+1}] \to \mathbb{R}^M$ for some $M \in \mathbb{N}$. An S-path $\gamma\colon x \to y$ with underlying path γ_0 can then be constructed by glueing together suitable maps $\tilde{\gamma}^j$. The details are standard and thus omitted. The proof of the second statement is similar.

□

3.1 An Ambrose-Singer Theorem

The classical Ambrose-Singer theorem characterises the holonomy Lie algebra in terms of the curvature of the connection considered. In this section, we show that this theorem continues to hold in the more general situation of S-holonomy in the sense of Definition 10. Our proof is modelled on a classical proof due to Levi-Civita as presented in [3]. We define the curvature of ∇ as usual by

$$R(X, Y)\,Z := \nabla_X \nabla_Y Z - (-1)^{|X||Y|} \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for $X, Y \in \mathcal{S}M_S$ and $Z \in \mathcal{E}_S$, where $[X, Y] := XY - (-1)^{|X||Y|} YX$ is the supercommutator. This definition is such that

$$R \in \mathrm{Hom}_{\mathcal{O}_{S\times M}}\left(\mathcal{S}M_S \otimes_{\mathcal{O}_{S\times M}} \mathcal{S}M_S \otimes_{\mathcal{O}_{S\times M}} \mathcal{E}_S, \mathcal{E}_S\right)_{\bar{0}} .$$

The curvature is skew-symmetric

$$R(Y, X) = -(-1)^{|X||Y|} R(X, Y)$$

which is inherited to the pullback. Let $\varphi\colon S \times N \to M$ be a supermanifold morphism. Then

$$R_\varphi(A, B) = -(-1)^{|A||B|} R_\varphi(B, A) \tag{16}$$

for $A, B \in \varphi^* \mathcal{S}M$. This is shown by a straightforward calculation in coordinates, writing $A = (\varphi^* \partial_{\xi^k}) \cdot A^k$ etc.

Definition 11. Let $x\colon S \to M$ be an S-point. Let $\mathfrak{g}_x$ denote the Lie subalgebra of $(\mathfrak{gl}_{\mathrm{rk}\,\mathcal{E}}(\bigwedge \mathbb{R}^L))_{\bar{0}}$ which is generated by the following set of endomorphisms.

$$\{P_\gamma^{-1} \circ R_y(u, v) \circ P_\gamma \;\big|\; y\colon S \to M\,,\ \gamma\colon x \to y \text{ piecewise smooth}\,,\ u, v \in (y^* \mathcal{S}M)_{\bar{0}}\}$$

Hol_x is contained in $GL_{\mathrm{rk}\,\mathcal{E}}(\bigwedge \mathbb{R}^L)$. By the following lemma, this is a Lie group. In general, every Lie subalgebra of the Lie algebra of a Lie group is the Lie algebra of a unique immersed connected Lie subgroup (see Chapter 2 of [13]). Let $G_x \subseteq GL_{\mathrm{rk}\,\mathcal{E}}(\bigwedge \mathbb{R}^L)$ denote this Lie subgroup corresponding to $\mathfrak{g}_x \subseteq (\mathfrak{gl}_{\mathrm{rk}\,\mathcal{E}}(\bigwedge \mathbb{R}^L))_{\bar{0}}$.

Lemma 9. *$GL_{n|m}(\bigwedge \mathbb{R}^L)$ is a real Lie group with Lie algebra $(\mathfrak{gl}_{n|m}(\bigwedge \mathbb{R}^L))_{\bar{0}}$.*

Proof. $M \in (\mathfrak{gl}_{n|m}(\bigwedge \mathbb{R}^L))_0$ is invertible if and only if its image under the canonical projection to $\mathfrak{gl}_{n|m}(\mathbb{R})$ is (Lemma 3.6.1 in [27]). Therefore

$$GL_{n|m}\left(\bigwedge \mathbb{R}^L\right) = (GL_n(\mathbb{R}) \times GL_m(\mathbb{R})) \oplus \left(\mathfrak{gl}_{n|m}\left(\bigwedge (\mathbb{R}^L)_{\mathrm{nilpotent}}\right)\right)_{\bar{0}}$$

which is open in $(\mathfrak{gl}_{n|m}(\bigwedge \mathbb{R}^L))_{\bar{0}}$ and as such a submanifold with a group structure such that the tangent space at 1 can be identified with $(\mathfrak{gl}_{n|m}(\bigwedge \mathbb{R}^L))_{\bar{0}}$. Writing the matrix entries of a product $M \cdot L$ in terms of real coefficients of odd generators, it is clear that multiplication is smooth, and similar for inversion. One further shows that the Lie algebra commutator coincides with the commutator $[X, Y] = XY - YX$. □

Theorem 1 (Ambrose-Singer Theorem). *The Lie groups $G_x = \mathrm{Hol}_x^0$ coincide. In particular, Hol_x is a Lie group with identity component Hol_x^0 and Lie algebra $\mathrm{hol}_x = \mathfrak{g}_x$.*

We defer the proof of the theorem to the end of the present section. It is based on Proposition 2 and Proposition 3 below. The following two lemmas are needed in the proof of the first proposition.

Lemma 10. *Let $f\colon S \times [a, b] \times [b, c] \to M$ be a morphism and $X \in f^* \mathcal{E}$ be a section along f. Then*

$$(f^*\nabla)_{\partial_s} (f^*\nabla)_{\partial_t} X - (f^*\nabla)_{\partial_t} (f^*\nabla)_{\partial_s} X = R_f\left(df[\partial_s], df[\partial_t]\right) X$$

where (s, t) denote the standard coordinates on $[a, b] \times [b, c]$.

Proof. This is shown by a direct calculation in local coordinates (ξ^k) of M and a trivialisation (T^k) of $\mathcal{E}$, writing $X = (\varphi^* T^l) \cdot X^l$ with $X^l \in \mathcal{O}(S \times [a,b] \times [b,c])$. $\square$

Let $x, y\colon S \to M$ and $\gamma\colon x \to y$. A tuple $(e^1, \dots, e^k)$ of sections $e^j \in \gamma^*\mathcal{E}$ is a basis of $\gamma^*\mathcal{E}$ if and only if $(\mathrm{ev}|_{t=t_0} e^1, \dots, \mathrm{ev}|_{t=t_0} e^k)$ is a basis of $\mathrm{ev}|_{t=t_0}\gamma^*\mathcal{E}$ for every $t_0 \in [0,1]$. It is called parallel if all e^i are parallel. Such a basis is determined by its evaluation at $t = 0$ via $\mathrm{ev}|_{t=t_0} e^j = P_{\gamma|_{S\times[0,t_0]}}(\mathrm{ev}|_{t=0} e^j)$. In particular, a parallel basis, as used in the proof of the following lemma, exists.

Lemma 11. *Let $X \in \gamma^*\mathcal{E}$ be a section along γ. Let $P_t := P^{-1}_{\gamma|_{S\times[0,t]}}$ be the parallel displacement from $\mathrm{ev}|_t\gamma^\sharp$ to $x^\sharp = \mathrm{ev}|_{t=0}\gamma^\sharp$. Then*

$$P_t \mathrm{ev}|_t (\gamma^*\nabla)_{\partial_t} X = \partial_t P_t(\mathrm{ev}|_t X) \in x^*\mathcal{E}$$

Proof. Let (e^j) be a parallel basis along γ. Writing $X = e^i \cdot X^i$ with $X^i \in \mathcal{O}_{S\times[0,1]}$, it follows that $P_t(\mathrm{ev}|_t X) = \mathrm{ev}_{t=0} e^i \cdot \mathrm{ev}|_t X^i$, and

$$\partial_t P_t(\mathrm{ev}|_t X) = \mathrm{ev}|_{t=0} e^i \cdot \mathrm{ev}|_t(\partial_t X^i)$$

On the other hand, $(\gamma^*\nabla)_{\partial_t} X = e^i \cdot \partial_t(X^i)$ implies

$$P_t \mathrm{ev}|_t (\gamma^*\nabla)_{\partial_t} X = P_t(\mathrm{ev}|_t e^i \cdot \mathrm{ev}|_t(\partial_t X^i)) = \mathrm{ev}|_{t=0} e^i \cdot \mathrm{ev}|_t(\partial_t X^i)$$

such that both sides agree. $\square$

For the following proposition note that, for a proper S-homotopy Ξ, we may identify $\mathrm{ev}|_{t=0}\Xi^\sharp$ and $\mathrm{ev}|_{t=1}\Xi^\sharp$ with single S-points $x, y\colon S \to M$, respectively.

Proposition 2. *Let Ξ be a proper S-homotopy, and let $P_{s,t} := P_{\Xi_s|_{S\times[t,1]}}$ denote parallel transport along the restriction of the S-path Ξ_s to $S \times [t,1]$. Then*

$$\partial_s P_{s,0} = \left(\int_0^1 R_{s,t}\, dt\right) P_{s,0} \in \mathrm{Hom}_{\mathcal{O}_{S\times[0,1]}}(x^*\mathcal{E},\, y^*\mathcal{E})$$

with $R_{s,t} := P_{s,t}\mathrm{ev}|_{s,t} R_\Xi\left(d\Xi[\partial_t],\, d\Xi[\partial_s]\right) P_{s,t}^{-1}$

Proof. Let $Z \in \Xi^*\mathcal{E}$. For Ξ proper, the term $\partial_s(\Xi_s^*(\xi^l))$ in (15) vanishes for $t = 0$ as well as $t = 1$, such that

$$\mathrm{ev}|_{s,t=0}(\Xi^*\nabla)_{\partial_s} Z = \partial_s \mathrm{ev}|_{s,t=0} Z\ , \qquad \mathrm{ev}|_{s,t=1}(\Xi^*\nabla)_{\partial_s} Z = \partial_s \mathrm{ev}|_{s,t=1} Z$$

Consider Z such that the first term vanishes and, moreover, $(\Xi^*\nabla)_{\partial_t} Z \equiv 0$. By Lemma 11 and Lemma 10, we yield

$$\begin{aligned}
\partial_t P_{s,t}\mathrm{ev}|_{s,t}(\Xi^*\nabla)_{\partial_s} Z &= P_{s,t}\mathrm{ev}_{s,t}(\Xi^*\nabla)_{\partial_t}(\Xi^*\nabla)_{\partial_s} Z \\
&= P_{s,t}\mathrm{ev}|_{s,t} R_\Xi\left(d\Xi[\partial_t],\, d\Xi[\partial_s]\right) Z \\
&= R_{s,t}\mathrm{ev}_{s,t=1} Z
\end{aligned}$$

since $\mathrm{ev}_{s,t}Z = P_{s,t}^{-1}\mathrm{ev}_{s,t=1}Z$ by assumption. This, together with the assumptions on Z and $P_{s,1} = \mathrm{id}$, implies the following.

$$\begin{aligned}\partial_s P_{s,0}\mathrm{ev}|_{s,t=0}Z &= \partial_s \mathrm{ev}|_{s,t=1}Z \\ &= P_{s,1}\mathrm{ev}|_{s,t=1}(\Xi^*\nabla)_{\partial_s}Z - P_{s,0}\mathrm{ev}|_{s,t=0}(\Xi^*\nabla)_{\partial_s}Z \\ &= \int_0^1 \partial_t \left(P_{s,t}\mathrm{ev}_{s,t}(\Xi^*\nabla)_{\partial_s}Z\right) dt \\ &= \left(\int_0^1 R_{s,t}dt\right) P_{s,0}\mathrm{ev}|_{s,t=0}Z\end{aligned}$$

Let $Z_x \in x^*\mathcal{E}$ .Then, setting $Z(s,t) := P_{\Xi_s|_{S\times[0,t]}}Z_x$, defines a section $Z \in \Xi^*\mathcal{E}$ that satisfies the assumptions made in the beginning of the proof as well as $\mathrm{ev}|_{s,t=0}Z = Z_x$, such that the equation to be proved holds applied to Z_x. Since Z_x was arbitrary, it holds in general. □

Let $a\colon S \to M$ be an S-point and $u, v \in (a^*\mathcal{S}M)_{\overline{0}}$. With respect to local coordinates (ξ^i) on $U \subseteq M$ around $a_0(0)$, we write $u = (a^*\partial_{\xi^i}) \cdot u^i$ with $u^i \in \mathcal{O}_S$ and likewise for v. Let (x, y) denote standard coordinates of $\mathbb{R}^2$. Then the map

$$f\colon S \times \mathbb{R}^2 \to U\ , \qquad f^\sharp(\xi^i) := a^\sharp(\xi^i) + (-1)^{|\xi^i|}u^i \cdot x + (-1)^{|\xi^i|}v^i \cdot y$$

is such that

$$\mathrm{ev}|_{(x,y)=(0,0)}f^\sharp = a^\sharp\ , \qquad \mathrm{ev}|_{(0,0)}df[\partial_x] = u\ , \qquad \mathrm{ev}|_{(0,0)}df[\partial_y] = v \tag{17}$$

Consider also the following piecewise smooth homotopy $g\colon S \times [0,1] \times [0,1] \to \mathbb{R}^2$.

$$g_0(s,t) := \begin{cases} (4st, 0) & 0 \le t \le 1/4 \\ (s, s(4t-1)) & 1/4 \le t \le 1/2 \\ (s(3-4t), s) & 1/2 \le t \le 3/4 \\ (0, 4s(1-t)) & 3/4 \le t \le 1 \end{cases}\ , \qquad \begin{aligned} g^\sharp(x) &:= g_0^*(x) \\ g^\sharp(y) &:= g_0^*(y) \end{aligned}$$

Proposition 3. *Let $a\colon S \to M$ be an S-point and $u, v \in (a^*\mathcal{S}M)_{\overline{0}}$. Let f be such that* (17)*, and let P_s denote parallel translation along Ξ_s for*

$$\Xi := f \circ \hat{g}\colon S \times [0,1] \times [0,1] \to U \subseteq M$$

Then

$$\mathrm{ev}|_{s=0}\partial_s P_s = 0\ , \qquad \mathrm{ev}|_{s=0}\partial_s\partial_s P_s = 2R_a\left(v,\, u\right)$$

Proof. By Lemma 1, we have $d\Xi[\partial_t] = (\Xi^*\partial_l)g^\sharp(df^l{}_i)\partial_t g^\sharp(x^i)$ where x^i runs over x and y. For $t \le 1/4$,

$$R_\Xi\left(d\Xi[\partial_t],\, d\Xi[\partial_s]\right) = R_\Xi\left((\Xi^*\partial_l)g^\sharp(df^l{}_x)4s,\, (\Xi^*\partial_m)g^\sharp(df^m{}_x)4t\right) = 0$$

vanishes by skew-symmetry (16), and analogous for $t \ge 3/4$. For $1/4 \le t \le 3/4$, we find

$$R_\Xi\left(d\Xi[\partial_t],\, d\Xi[\partial_s]\right) = -R_\Xi\left((\Xi^*\partial_l)g^\sharp(df^l{}_x),\, (\Xi^*\partial_m)g^\sharp(df^m{}_y)\right)\cdot 4s\,.$$

Using (17), we further calculate, for $1/4 \le t \le 3/4$,

$$\begin{aligned} R_{s,t} &= P_{s,t}\mathrm{ev}|_{s,t}R_\Xi\left(d\Xi[\partial_t],\, d\Xi[\partial_s]\right)P_{s,t}^{-1} \\ &= 4s\cdot P_{s,t}\mathrm{ev}|_{s,t}R_\Xi\left((\Xi^*\partial_l)g^\sharp(df^l{}_y),\,(\Xi^*\partial_m)g^\sharp(df^m{}_x)\right)P_{s,t}^{-1} \\ &= 4s\cdot P_{s,t}R_a\left((a^*\partial_l)v^l,\,(a^*\partial_m)u^m\right)P_{s,t}^{-1}\,. \end{aligned}$$

Proposition 2 now yields

$$\partial_s P_s = 4s\left(\int_{\frac{1}{4}}^{\frac{3}{4}} P_{s,t}R_a\,(v,\,u)\,P_{s,t}^{-1}dt\right)P_s$$

which vanishes for $s \to 0$. Likewise

$$\mathrm{ev}_{s=0}\partial_s(\partial_s P_s) = \lim_{s\to 0}\left(\frac{1}{s}4s\left(\int_{\frac{1}{4}}^{\frac{3}{4}} P_{s,t}R_a\,(v,\,u)\,P_{s,t}^{-1}dt\right)P_s\right) = 2R_a\,(v,\,u)\,. \quad \square$$

Proof. [Proof of Theorem 1] Let $\gamma\colon x \to x$ be piecewise smooth and contractible. We choose a piecewise smooth proper homotopy Ξ such that $\Xi_0 = x$ and $\Xi_1 = \gamma$, and let $P_s := P_{\Xi_s} \in GL_{\mathrm{rk}\,\mathcal{E}}(\bigwedge\mathbb{R}^L)$ denote parallel translation along Ξ_s. By Proposition 2, it satisfies the differential equation

$$\partial_s P_s = g(s)\cdot P_s\,, \qquad g(s) := \left(\int_a^b R_{s,t}dt\right) \in \mathfrak{g}_x$$

By standard Lie group theory (cf. Chapter 2 of [13]), we conclude that $P_s \in G_x$ and, in particular, $P_\gamma = P_1 \in G_x$. Therefore, $\mathrm{Hol}_x^0 \subseteq G_x$ is a path-connected subgroup. By a theorem of Yamabe [28], it is a Lie subgroup.

Let a be an S-point, $\gamma\colon x \to a$ and $u, v \in (a^*\mathcal{S}M)_{\overline{0}}$. Let Ξ be as in Proposition 3, and let $P_s \in \mathrm{Hol}_x^0$ denote parallel translation along $\hat{\Xi}_s := \gamma \star \Xi_s \star \gamma^{-1}$. Then

$$\begin{aligned} \partial_s P_s|_{s=0} &= P_\gamma \circ \partial_s P_{\Xi_s}|_{s=0} \circ P_\gamma^{-1} = 0 \\ \partial_s\partial_s P_s|_0 &= P_\gamma \circ \partial_s\partial_s P_{\Xi_s}|_0 \circ P_\gamma^{-1} = 2P_\gamma \circ R\,(v,\,u) \circ P_\gamma^{-1} \end{aligned}$$

by Proposition 3. Hol_x^0 can be identified with a submanifold of some $\mathbb{R}^M$. By the vanishing of the first derivative we can thus conclude that $\partial_s\partial_s P_s|_0 \in \mathrm{hol}_x = T_e(\mathrm{Hol}_x^0)$. Therefore, all generators of $\mathfrak{g}_x$ are contained in hol_x. It follows that $\mathfrak{g}_x = \mathrm{hol}_x$ and $\mathrm{Hol}_x^0 = G_x$. $\square$

3.2 The Holonomy Group Functor

So far, we have considered a fixed superpoint $S = \mathbb{R}^{0|L}$ along with an S-connection ∇ on an S-bundle $\mathcal{E}_S$. In Section 2, it was argued that having S-connections (compared to plain connections in $\mathcal{E}$) is necessary to model superconnections as in [6], whereas the exact value of L cannot have any physical significance. But also for purely mathematical reasons, it is desirable to allow for extending the number of auxiliary Graßmann generators, as will become clear in the proof of the holonomy principle (Theorem 2) below. This extension results in a categorical theory to be described next.

Let ∇ be an S-connection on $\mathcal{E}_S$ with respect to $S=\mathbb{R}^{0|L}$, and let $T=\mathbb{R}^{0|L'}$ be another superpoint. By $\bigwedge\mathbb{R}^{L'}$-linear extension, ∇ can be considered as an $S\times T$-connection on $\mathcal{E}_{S\times T}$. Similarly, an S-point $x\colon S\to M$ canonically induces an $S\times T$-point $x_T\colon S\times T\to M$ by composing x with the canonical projection $S\times T\to S$. For the next proposition, note that a morphism $\varphi\colon T\to T'$ can be identified with a Graßmann algebra morphism φ^* and as such acts naturally on $GL_{\mathrm{rk}\,\mathcal{E}}(\mathcal{O}_{T'})$.

Proposition 4. *The assignment*

$$T\mapsto \mathrm{Hol}_x(T):=\mathrm{Hol}_{x_T}\,,\qquad (\varphi\colon T\to T')\mapsto \big(L\mapsto \varphi^*(L),\,\mathrm{Hol}_{x_{T'}}\to\mathrm{Hol}_{x_T}\big)$$

defines a group-valued functor.

In the following, we will denote both the holonomy with respect to x and the induced holonomy functor by Hol_x. We will also use the notation $\mathrm{hol}_x(T):=\mathrm{hol}_{x_T}$.

Proof. Let $L\in\mathrm{Hol}_{x_{T'}}$. We must show that the pullback $\varphi^*(L)$ is indeed contained in Hol_{x_T}. Then the induced map $\mathrm{Hol}_{x_{T'}}\to\mathrm{Hol}_{x_T}$ is clearly a group homomorphism.

Let $\gamma\colon x_{T'}\to x_{T'}$ be such that $L=P_\gamma$, and prescribe

$$x_\varphi := x_{T'}\circ(\mathrm{id}_S\times\varphi)\colon S\times T\to M\,,$$
$$\gamma_\varphi:=\gamma\circ\varphi:=\gamma\circ(\mathrm{id}_S\times\varphi\times\mathrm{id}_{[0,1]})\colon x_\varphi\to x_\varphi\,.$$

It is clear that $x_\varphi=x_T$ independent of φ. Let $\mathcal{B}(t)$ be as in (15) with respect to γ. It follows that the local parallelness condition with respect to γ_φ reads

$$\partial_t X(t) = -\big(\varphi^*\mathcal{B}(t)\big)\cdot X(t)\,.$$

We can, therefore, conclude that $X\in\gamma^*\mathcal{E}$ parallel along γ implies that $\varphi^*X\in\gamma_\varphi^*\mathcal{E}$ is parallel along γ_φ. Therefore

$$\varphi^*\left(P_\gamma[X_{x_{T'}}]\right)=P_{\gamma_\varphi}\left[\varphi^*(X_{x_{T'}})\right]\quad\text{for all }\ X_{x_{T'}}\in x_{T'}^*\mathcal{E}$$

and $\varphi^*(L)=\varphi^*P_\gamma=P_{\gamma_\varphi}\in\mathrm{Hol}_{x_T}$. □

The Molotkov-Sachse theory defines a supermanifold to be a certain functor from the category Gr of Graßmann algebras to that of smooth manifolds [22], [24] such that, in the finite-dimensional case, the resulting category is equivalent to that of Berezin-Kostant-Leites supermanifolds. It is thus natural to conjecture that Hol_x is representable in that it defines such a supermanifold. If this was true, a neighbourhood of 1 in $\mathrm{Hol}_x(T)$ would be isomorphic to $(V\otimes\bigwedge\mathbb{R}^{L'})_{\overline{0}}$ for a fixed finite-dimensional super vector space V. It would follow that

$$\mathrm{hol}_x(T)\cong T_e(\mathrm{Hol}_x(T))\cong(V\otimes\bigwedge\mathbb{R}^{L'})_{\overline{0}}$$

such that, in particular, $\mathrm{hol}_x(\bigwedge\mathbb{R}^0)=V_{\overline{0}}$. The following example shows that the holonomy functor is, in general, not representable.

Example 4. Consider $S := \mathbb{R}^{0|0}$ and $M := \mathbb{R}^{0|1}$ with the (S-)connection defined by $\nabla_{\partial_\theta}\partial_\theta = \theta\partial_\theta$ on $\mathcal{E}_S := \mathcal{S}M_S = \mathcal{S}M$ such that $R(\partial_\theta, \partial_\theta)\,\partial_\theta = 2\partial_\theta$. Let 0 denote the unique S-point corresponding to $0 \in \mathbb{R}^0$. By Theorem 1, $\mathrm{hol}_0(T)$ is generated by $P_\gamma^{-1} \circ R_y(u, v) \circ P_\gamma$ for $y\colon T \to M$, $\gamma\colon x \to y$ and $u, v \in (y^*\mathcal{S}M)_{\bar{0}}$. We write $u = (y^*\partial_\theta)\cdot u^\theta$ with $u^\theta \in (\mathcal{O}_T)_{\bar{1}}$ and analogous for v. Let $w \in y^*\mathcal{S}M$. Then a short calculation yields

$$P_\gamma^{-1} \circ R_y(u, v)\, P_\gamma[w] = -2u^\theta v^\theta \cdot w\,.$$

For $T = \mathbb{R}^{0|0}$, u^θ and v^θ vanish, such that $\mathrm{hol}_0 = \{0\}$ is trivial, while

$$\mathrm{hol}_0(T) = \mathfrak{gl}(0|1) \otimes ((\mathcal{O}_T)_{\bar{1}})^2 \subseteq \mathfrak{gl}(0|1) \otimes (\mathcal{O}_T)_{\bar{0}} = (\mathfrak{gl}(0|1) \otimes \mathcal{O}_T)_{\bar{0}}$$

for $T = \mathbb{R}^{0|L'}$, $L' \geq 2$. By the preceding paragraph, the functor $\mathrm{Hol}_0(T)$ is thus not representable.

By the holonomy principle, to be established next, a parallel section $X \in \mathcal{E}_S$ is uniquely determined by its $\mathrm{Hol}_x(T)$-invariant pullback $x^*X \in x^*\mathcal{E}$ as defined in (8), where the number L' of additional generators must be sufficiently large.

Theorem 2 (Holonomy Principle). *Let M be connected. Let ∇ be an S-connection on $\mathcal{E}_S$, $x\colon S \to M$ be an S-point and $T = \mathbb{R}^{0|L'}$ with $L' \geq (\dim M)_{\bar{1}}$. Then the following holds true.*

(i) Let $X \in \mathcal{E}_S$ be a parallel section $\nabla X \equiv 0$ and define $X_x := x^*X \in x^*\mathcal{E}$. Then, for all $y\colon S \times T \to M$ and $\gamma\colon x \to y$, it holds $y^*X = P_\gamma[X_x]$, where X_x is identified with a section of $x_T^*\mathcal{E}$. In particular, X_x is holonomy invariant $\mathrm{Hol}_x(T) \cdot X_x = X_x$.

(ii) Conversely, let $X_x \in x^*\mathcal{E}$ be a section such that $\mathrm{Hol}_x(T) \cdot X_x = X_x$. Then there exists a unique section $X \in \mathcal{E}_S$ with $x^*X = X_x$, which is parallel $\nabla X \equiv 0$.

Proof. Let $\gamma\colon x \to y$ be a piecewise smooth S-path. The assumption $\nabla X \equiv 0$ implies $\nabla_{\partial_t}(\gamma^*X) = 0$. Parallel transport along γ is thus

$$P_\gamma[X_x] = \mathrm{ev}|_{t=1}\gamma^*X = y^*X$$

which proves the first assertion.

Conversely, let $X_x \in x^*\mathcal{E}$ be such that $\mathrm{Hol}_x(T) \cdot X_x = X_x$. For a superpoint $y\colon S \times T \to M$, we define $X_y := P_\gamma[X_x]$ where $\gamma\colon x \to y$ is an $S \times T$-path. Since X_x is $\mathrm{Hol}_x(T)$-invariant, X_y is well-defined independent of the choice of γ. We aim at constructing X out of the set of X_y inductively over the degree of $\mathcal{O}_S$-monomials. Without loss of generality, we may assume that $M \cong \mathbb{R}^{n|m}$ has global coordinates $\xi = (x, \theta)$. For, assume that the statement is true for M replaced by a neighbourhood $U \subseteq M$ of $x_0(0)$, thus resulting in a parallel section $X \in \mathcal{E}_S(U)$. Then, by the first part of the theorem, X satisfies $\mathrm{Hol}_y(T) \cdot X_y = X_y$ for all $y\colon S \times T \to U$. Repeating the local construction in a neighbourhood $V \subseteq M$

of $y_0(0)$ yields a parallel section $\tilde{X} \in \mathcal{E}_S(V)$ which, by uniqueness, agrees with X on the intersection $U_0 \cap V_0$. Without loss of generality, we may further assume that $\mathcal{E}$ is trivial with a global adapted basis (T^j). We expand

$$X_y = X_y|_{\eta^I} \cdot \eta^I = T^j \cdot X_y^j|_{\eta^I} \cdot \eta^I\ , \qquad X = X|_{\eta^I} \cdot \eta^I = T^j \cdot X^j|_{\eta^I} \cdot \eta^I$$

for multiindices $I = (i_1, \ldots, i_{|I|})$ with $1 \leq i_j \leq L$, such that $X_y^j|_{\eta^I} \in \mathbb{R}$ and $X^j|_{\eta^I} \in \mathcal{O}_M$ and $X|_{\eta^I} \in \mathcal{E}$. Similarly, ∇ is characterised by $\Gamma_{ij}^k = \left(\Gamma_{ij}^k\right)|_{\eta^I} \cdot \eta^I$.

In the first step, we construct $X^0 \in \mathcal{E}$. Letting $q := y_0(0)$, we define its value at q by $X^0(q) := X_y|_{\eta^0} = (P_\gamma[X_x])|_{\eta^0}$. By Lemma 6, it arises by classical parallel transport along γ_0. It is thus independent of y such that $q = y_0(0)$, and $X^0(q)$ depends smoothly on q. By (16) of [12] applied to the induced connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$, $X^0(q)$ extends to a section $X^0 \in \mathcal{E}$ such that $0 = \nabla^{\mathcal{E}}_{\partial_{\theta^j}} X^0 = (\nabla_{\partial_{\theta^j}} X^0)|_{\eta^0}$. By construction, X^0 satisfies $(y^* X^0)|_{\eta^0} = X^0(q) = (P_\gamma[X_x])|_{\eta^0}$. Again by Lemma 6, we further note that $(\nabla X^0)|_{\theta^0\eta^0} \equiv 0$.

In the second step, we consider multiindices $I = (i_1, \ldots, i_{|I|})$ with $1 \leq i_j \leq L + (\dim M)_{\bar{1}}$, such that $\eta^I \in \mathcal{O}_{S\times T}$. Assume, by induction, that we have constructed $X^N \in \mathcal{E}_S$ for $N \in \mathbb{N}$ such that

0_N X^N has an expansion $X^N = \sum_{|I|\leq N} X|_{\eta^I} \cdot \eta^I$ such that $X|_{\eta^I} = 0$ whenever there is $i_j \in I$ with $i_j \geq L+1$.

1_N $(y^* X^N)|_{\eta^I} = (P_\gamma[X_x])|_{\eta^I} = X_y|_{\eta^I}$ for every $y\colon S \times T \to M$, $\gamma\colon x \to y$ and $|I| \leq N$.

2_N $(\nabla_{\partial_{\theta^j}} X^N)|_{\eta^I} \equiv 0$ for all $|I| \leq N$.

3_N $(\nabla_{\partial_{x^j}} X^N)|_{\theta^A\eta^B} \equiv 0$ for all A,B such that $|A| + |B| \leq N$, where $A = (a_1, \ldots, a_{|A|})$ with $1 \leq a_j \leq (\dim M)_{\bar{1}}$.

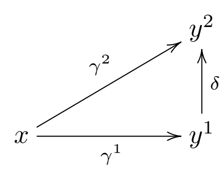
Condition 1_{N+1} is equivalent to 1_N together with

$$X_y|_{\eta^J} \stackrel{!}{=} (y^* X^{N+1})|_{\eta^J} = y_0^*(X|_{\eta^J}) + (y^* X^N)|_{\eta^J} \qquad \text{for } |J| = N+1$$

We are thus led to define the value of $X|_{\eta^J}$ at q by

$$X|_{\eta^J}(q) := X_y|_{\eta^J} - (y^* X^N)|_{\eta^J} \qquad \text{for } |J| = N+1 \tag{18}$$

This prescription is independent of $y\colon S \times T \to M$ such that $y_0(0) = q$. Indeed, let y^1,y^2 be two such $S \times T$-points and $\gamma^{1,2}\colon x \to y^{1,2}$ be connecting S-paths. Moreover, let $\delta\colon y^1 \to y^2$ be such that $\delta_0(t) \equiv q$.



Since X_x is holonomy invariant, we have $X_{y^2} = P_\delta[X_{y^1}]$. We calculate, using (15),

$$\begin{aligned}\partial_t\left(X_\delta|_{\eta^J} - (\delta^* X^N)|_{\eta^J}\right) &= \left(\partial_t P_\delta|_{[0,t]}[X_{y^1}] - \partial_t \delta^* X^N\right)\Big|_{\eta^J}\\ &= \Big(-(-1)^{|T^m|(|T^n|+1)}(\delta^* T^m)\partial_t(\delta^*(\xi^l))\cdot\hat{\delta}^*(\Gamma^m_{ln})\cdot P_\delta|_{[0,t]}[X_{y^1}]^n\\ &\qquad - \partial_t\delta^*(\xi^l)(\delta^*\circ\partial_{\xi^l})(X^N)\Big)\Big|_{\eta^J}\end{aligned}$$

By assumption, the term $\partial_t(\delta^*(\xi^l))$ is nilpotent such that, using induction assumption 1_N, we may replace $P_\delta|_{[0,t]}[X_{y^1}]^n$ by $(\delta^* X^N)^n = \delta^* X^{N\,n}$. Therefore, the right hand side equals

$$\begin{aligned}&\Big((\delta^* T^m)\partial_t(\delta^*(\xi^l))\hat{\delta}^*\left(-(-1)^{|T^m|(|T^n|+1)}\Gamma^m_{ln}X^{N\,n} - \partial_{\xi^l}X^{N\,m}\right)\Big)\Big|_{\eta^J}\\ &\qquad = -\left((\delta^* T^m)\partial_t(\delta^*(\xi^l))\hat{\delta}^*(\nabla_{\partial_{\xi^l}}X^N)^m\right)\Big|_{\eta^J}\end{aligned}$$

By 2_N and 3_N (and nilpotency of $\partial_t(\delta^*(\xi^l))$), this expression vanishes, thus showing that $X_\delta|_{\eta^J} - (\delta^* X^N)|_{\eta^J}$ is constant, which proves that (18) is well-defined.

We next endow $X|_{\eta^J}(q)$ to a section $X|_{\eta^J} \in \mathcal{E}$ such that

$$X^{N+1} := \sum_{|J|\le N+1} X|_{\eta^J}\cdot\eta^J$$

satisfies 2_{N+1}. 2_N implies that $(\nabla_{\partial_{\theta^m}}X^{N+1})|_{\eta^I} = 0$ with $|I| \le N$ for any such X^{N+1}. Under this induction hypothesis, 2_{N+1} is thus equivalent to $(\nabla_{\partial_{\theta^m}}X^{N+1})|_{\eta^J} = 0$ for $|J| = N+1$ which, in turn, is equivalent to

$$\left(\partial_{\theta^r}\ldots\partial_{\theta^1}\partial_{\theta^m}X^j|_{\eta^J}\right)\big|_{\theta^0} = -(-1)^{|T^j|(|T^i|+1)}\left(\partial_{\theta^r}\ldots\partial_{\theta^1}\left(\Gamma^j_{mi}X^{i\,N+1}\right)\right)\Big|_{\eta^J\theta^0}$$

for all $r \le (\dim M)_{\bar{1}}$. Similar to the construction of $X^0 \in \mathcal{E}$ above, these equations uniquely determine $X|_{\eta^J}$, for $|J| = N+1$, by X^N and $X|_{\eta^J}(q)$, such that 2_{N+1} holds. If any index $l_j \in J$ satisfies $l_j > L$, the right hand side of (18) vanishes upon considering $y\colon S \to M$, such that 0_{N+1} is satisfied. By construction, also 1_{N+1} holds.

We show that X^{N+1} further satisfies 3_{N+1}. 1_{N+1} implies that $(z^* X^{N+1})|_{\eta^I} = P_\delta[X_y]|_{\eta^I}$ for all z and $\delta\colon y \to z$ and $|I| \le N+1$. In particular, we let $q \in M_0$ and define y and δ as follows.

$$\begin{aligned} y^\sharp(x^k) &:= q^*(x^k) = q^k\,, & y^\sharp(\theta^k) &:= \eta^{L+k}(\in \mathcal{O}_T)\,,\\ \delta^\sharp(x^k) &:= q^k + t\delta^{kk_0}\,, & \delta^\sharp(\theta^k) &:= \eta^{L+k}\end{aligned}$$

This is such that $\mathrm{ev}|_{t=0}\delta^\sharp = y^\sharp$. We thus yield

$$0 = \left((\delta^*\nabla)_{\partial_t}(\delta^* X^{N+1})\right)|_{\eta^I} = \hat{\delta}^*(\nabla_{\partial_{x^{k_0}}}X^{N+1})|_{\eta^I}$$

Writing $\nabla_{\partial_{x^{k_0}}}X^{N+1} =: N^{AB}\theta^A\eta^B$ with $\eta^B \in \mathcal{O}_S$, we conclude that

$$0 = \hat{y}^*(\nabla_{\partial_{x^{k_0}}}X^{N+1})|_{\eta^I} = \left(N^{AB}(q)\cdot\eta^{A_L}\eta^B\right)|_{\eta^I}$$

with A_L arising from the multiindex A by shifting all indices by L, such that $\eta^{A_L} \in \mathcal{O}_T$. For $|A|+|B| = |A_L|+|B| = |I| \leq N+1$, this implies that $N^{AB}(q) = 0$.

Proceeding inductively yields a section $X := X^L = X^{L+(\dim M)_{\bar{1}}} \in \mathcal{E}_S$ such that the induction hypotheses hold with respect to $L + (\dim M)_{\bar{1}}$. X is, therefore, parallel. Concerning uniqueness, assume that $\tilde{X} \in \mathcal{E}_S$ is a second such section. Then $y^*(X - \tilde{X}) = 0$ for all $y\colon S \times T \to M$ such that $X - \tilde{X} = 0$ by an argument analogous to that in the previous proof of 3_{N+1}. □

4 Comparison with Galaev's Holonomy Theory

Considering $S = \mathbb{R}^{0|0}$, let ∇ be a connection on a super vector bundle $\mathcal{E} \to M$ and $x \in M_0$ be a (topological) point. In this chapter, we will compare the functor Hol_x with Galaev's holonomy super Lie group $\mathrm{Hol}_x^{\mathrm{Gal}}$, which was introduced in [12] by means of a certain Harish-Chandra pair built around the super Lie algebra $\mathrm{hol}_x^{\mathrm{Gal}}$ generated by endomorphisms

$$P_{\gamma_0}^{-1} \circ \left(\overline{\nabla}^r_{Y_r,\ldots,Y_1} R\right)_y (Y,\, Z) \circ P_{\gamma_0} : x^*\mathcal{E} \to x^*\mathcal{E}$$

with $y \in M_0$, $\gamma_0\colon x \to y$, $r \geq 0$ and $Y_1, \ldots, Y_r, Y, Z \in y^*\mathcal{S}M$, and where $\overline{\nabla}^r_{Y_r,\ldots,Y_1} R$ denotes the r-fold covariant derivative of the curvature R with respect to ∇ and some auxiliary connection $\overline{\nabla}$ on $\mathcal{S}M$ in a neighbourhood of y. This derivative is defined analogous to the classical (non-super) case with appropriate signs. For $r = 1, 2$, it reads as follows.

Definition 12. Let

$$R \in \mathrm{Hom}_{\mathcal{O}_{S\times M}}\left(\mathcal{S}M_S \otimes_{\mathcal{O}_{S\times M}} \mathcal{S}M_S \otimes_{\mathcal{O}_{S\times M}} \mathcal{E}_S,\, \mathcal{E}_S\right)\,,$$

and $u, v \in \mathcal{S}M_S$. For $X, Y \in \mathcal{S}M_S$, we define

$$\begin{aligned}
\overline{\nabla}_X R\,(u,\, v) &:= \nabla_X \circ R\,(u,\, v) - (-1)^{|R||X|} R\left(\overline{\nabla}_X u,\, v\right) \\
&\quad - (-1)^{|X|(|R|+|u|)} R\left(u,\, \overline{\nabla}_X v\right) \\
&\quad - (-1)^{|X|(|R|+|u|+|v|)} R\,(u,\, v) \circ \nabla_X \\
\overline{\nabla}^2_{X,Y} R\,(u,\, v) &:= \overline{\nabla}_X \left(\overline{\nabla}_Y R\right)(u,\, v) - \nabla_{\overline{\nabla}_X Y} \circ R\,(u,\, v) \\
&\quad + (-1)^{(|X|+|Y|)|R|} R\left(\overline{\nabla}_{\overline{\nabla}_X Y} u,\, v\right) \\
&\quad + (-1)^{(|X|+|Y|)(|R|+|u|)} R\left(u,\, \overline{\nabla}_{\overline{\nabla}_X Y} v\right) \\
&\quad + (-1)^{(|X|+|Y|)(|R|+|u|+|v|)} R\,(u,\, v) \circ \nabla_{\overline{\nabla}_X Y}
\end{aligned}$$

According to Example 4, the functor Hol_x is, in general, not representable such that Galaev's holonomy theory is a priori different from ours. Nevertheless, we will show that the generators of $\mathrm{hol}_x^{\mathrm{Gal}}$ can be extracted in a geometric way, in a sense to be made precise.

4.1 Parallel Transport and Covariant Derivatives

The aforementioned extraction of generators of $\mathrm{hol}_x^{\mathrm{Gal}}$ is based on the following observation. Consider again the more general situation of an S-connection ∇ on $\mathcal{E}_S$ for $S = \mathbb{R}^{0|L}$ and $x\colon S \to M$ an S-point. As shown next, the pullback connection $x^*\nabla$ – along with its induced connections on tensors as well as higher covariant derivatives – arises by means of infinitesimal parallel transport. We will not treat the most general situation here but content ourselves with the following. First, we consider only even vector fields to be differentiated along. The general case is expected to work along the lines of the flow of vector bundles developed in [23]. Second, we consider tensors of the following type: sections, endomorphisms and curvature-type. The general case should be analogous. Third, we consider covariant derivatives up to second order. Analogous results for higher order derivatives are expected to be obtainable by an inductive proof.

For $X \in \mathcal{S}S$ and $Z \in x^*\mathcal{E}$, the pullback $(x^*\nabla)_X Z \in x^*\mathcal{E}$ was defined in (13). Let also $Y \in \mathcal{S}S$ and $\overline{\nabla}$ be an S-connection on $\mathcal{S}M_S$. We define the second covariant derivative of Z, with respect to ∇ and $\overline{\nabla}$, as follows.

$$(x^*\overline{\nabla}^2)_{X,Y} Z := (x^*\nabla)_X (x^*\nabla)_Y Z - (x^*\nabla)_{(x^*\overline{\nabla})_X [dx[Y]]} Z$$

The corresponding first and second covariant derivatives of endomorphisms and tensors of curvature type are defined likewise.

Definition 13. Let $E \in \mathrm{End}_{\mathcal{O}_{S\times M}}(\mathcal{E}_S)$ be an endomorphism and E_x its pullback under x as in (9). For $X, Y \in \mathcal{S}S$, we define

$$\begin{aligned}
(x^*\nabla)_X E_x &:= (x^*\nabla)_X \circ E_x - (-1)^{|X||E|} E_x \circ (x^*\nabla)_X \in \mathrm{End}_{\mathcal{O}_S}(x^*\mathcal{E}) \\
(x^*\overline{\nabla}^2)_{X,Y} E_x &:= (x^*\nabla)_X \left((x^*\nabla)_Y E_x\right) \\
&\quad - (x^*\nabla)_{(x^*\overline{\nabla})_X [dx[Y]]} \circ E_x \\
&\quad + (-1)^{|E|(|X|+|Y|)} E_x \circ (x^*\nabla)_{(x^*\overline{\nabla})_X [dx[Y]]}
\end{aligned}$$

Definition 14. Let

$$R \in \mathrm{Hom}_{\mathcal{O}_{S\times M}}\left(\mathcal{S}M_S \otimes_{\mathcal{O}_{S\times M}} \mathcal{S}M_S \otimes_{\mathcal{O}_{S\times M}} \mathcal{E}_S,\, \mathcal{E}_S\right)\,,$$

and $u, v \in x^*\mathcal{S}M$. For $X, Y \in \mathcal{S}S$, we define

$$\begin{aligned}
(x^*\overline{\nabla})_X R_x(u,\, v) &:= (x^*\nabla)_X \circ R_x(u,\, v) - (-1)^{|R||X|} R_x\left((x^*\overline{\nabla})_X(u),\, v\right) \\
&\quad - (-1)^{|X|(|R|+|u|)} R_x\left(u,\, (x^*\overline{\nabla})_X(v)\right) \\
&\quad - (-1)^{|X|(|R|+|u|+|v|)} R_x(u,\, v) \circ (x^*\nabla)_X \\
(x^*\overline{\nabla}^2)_{X,Y} R_x(u,\, v) &:= (x^*\overline{\nabla})_X \left((x^*\overline{\nabla})_Y R_x\right)(u,\, v) - (x^*\nabla)_{(x^*\overline{\nabla})_X [dx[Y]]} \circ R_x(u,\, v) \\
&\quad + (-1)^{(|X|+|Y|)|R|} R_x\left((x^*\overline{\nabla})_{(x^*\overline{\nabla})_X [dx[Y]]}(u),\, v\right) \\
&\quad + (-1)^{(|X|+|Y|)(|R|+|u|)} R_x\left(u,\, (x^*\overline{\nabla})_{(x^*\overline{\nabla})_X [dx[Y]]}(v)\right) \\
&\quad + (-1)^{(|X|+|Y|)(|R|+|u|+|v|)} R_x(u,\, v) \circ (x^*\nabla)_{(x^*\overline{\nabla})_X [dx[Y]]}
\end{aligned}$$

Our next lemma ensures existence of an S-path as occurring in the subsequent proposition concerning first covariant derivatives.

Lemma 12. *Let $x\colon S \to M$ be an S-point and $\xi \in (x^*\mathcal{S}M)_{\overline{0}}$. We write (in coordinates around x_0) $\xi = (x^*\partial_i)\cdot\xi^i$ and assume that every $\xi^i \in \mathcal{O}_S$ is nilpotent. Then there is an S-path γ (connecting x to some other S-point y) such that $\mathrm{ev}|_0\partial_t\circ\gamma^\sharp = \xi$.*

Proof. Through Definition 5, and setting $x^\sharp(t) := t$, we extend x to a map $x\colon S\times\mathbb{R} \to S\times M\times\mathbb{R}$. In this sense, we define

$$\gamma^\sharp := x^\sharp \circ \sum_{n=0}^{\infty} \frac{\left(\sum_i (t\xi^i\partial_i)\right)^n}{n!} .$$

Every $\xi^i\partial_i$ is, by assumption, even and nilpotent such that there are no ordering problems and the sum is finite. Such γ is indeed a morphism by the derivation property of $\sum_i(t\xi^i\partial_i)$ as shown analogous as in the proof of Lemma 1.1 in [17]. A straightforward calculation shows, moreover, that γ indeed satisfies the required initial condition. □

Proposition 5. *Let $x\colon S\to M$ be an S-point, $Y \in \mathcal{E}_S$ and $\xi \in (x^*\mathcal{S}M)_{\overline{0}}$. Let γ be an S-path (connecting x to some y) such that $\mathrm{ev}|_0\partial_t\circ\gamma^\sharp = \xi$. Then*

$$\frac{d}{dt}\Big|_0 \left(P_\gamma|_{[0,t]}^{-1}(\gamma^*Y)\right) = (x^*\nabla)_\xi(x^*Y)$$

In particular, for $\xi = X\circ x^\sharp = dx[X]$ with $X \in (\mathcal{S}S)_{\overline{0}}$, we find

$$\frac{d}{dt}\Big|_0 \left(P_\gamma|_{[0,t]}^{-1}(\gamma^*Y)\right) = (x^*\nabla)_X(x^*Y)$$

Similarly, the first covariant derivatives of E_x and R_x, with E and R as in Definition 13 and Definition 14, arise from parallel transport as

$$\frac{d}{dt}\Big|_0 \left(P_\gamma|_{[0,t]}^{-1}\circ E_\gamma\circ P_\gamma|_{[0,t]}\right) = (x^*\nabla)_X E_x$$

$$\frac{d}{dt}\Big|_0 \left(P_\gamma|_{[0,t]}^{-1}\circ R_\gamma\left(\overline{P}_\gamma|_{[0,t]}(u),\, \overline{P}_\gamma|_{[0,t]}(v)\right)\circ P_\gamma|_{[0,t]}\right) = (x^*\overline{\nabla})_X R_x\,(u,\, v)$$

Proof. Let (T^j) be an $\mathcal{E}$-basis in a neighbourhood of $x_0(0)\in M_0$. For t sufficiently small, we identify $P_{\gamma|_{[0,t]}}$ and its inverse with a matrix with respect to bases (x^*T^j) and $(\gamma_t^*T^j)$. By (15), we find that

$$\mathrm{ev}|_{t=0}P_{\gamma|_{[0,t]}} = \mathrm{id}\ , \qquad \partial_t|_0 P_{\gamma|_{[0,t]}} = -\mathcal{B}(0)\ , \qquad \partial_t|_0 P^{-1}_{\gamma|_{[0,t]}} = \mathcal{B}(0)$$

where the sign in the last equation is due to replacing t by $1-t$ in γ^{-1} within the definition of $\mathcal{B}(t)$. The first statement is shown by the following calculation, writing $Y = T^kY^k$.

$$\frac{d}{dt}\Big|_0 \left(P_\gamma|_{[0,t]}^{-1}(\gamma^*Y)\right) = \mathcal{B}(0)\cdot(x^*Y) + (x^*T^k)\partial_t|_0\gamma^*Y^k = (x^*\nabla)_\xi(x^*Y)$$

For the second statement note that, by (9), the matrix of E_γ is the pullback under γ of the matrix of E. For $Y \in x^*\mathcal{E}$, we thus yield

$$\begin{aligned}\frac{d}{dt}\Big|_0 \left(P_\gamma|_{[0,t]}^{-1} \circ E_\gamma \circ P_\gamma|_{[0,t]}\right)(Y) &= \left(\mathcal{B}(0)E_x + \partial_t|_0 E_\gamma - E_x\mathcal{B}(0)\right)(Y) \\ &= \mathcal{B}(0)E_x[Y] + X(E_xY) - E_x[X(Y)] - E_x\mathcal{B}(0)[Y] \\ &= \left((x^*\nabla)_X \circ E_x - E_x \circ (x^*\nabla)_X\right)(Y)\end{aligned}$$

Finally, the third statement is established by an analogous calculation. □

We now come to second covariant derivatives. Let $X, Y \in (\mathcal{S}S)_{\overline{0}}$ and consider a map $\gamma\colon S \times [0,1] \times [0,1] \to M$ such that

$$\mathrm{ev}|_{(0,0)}\partial_t \circ \gamma^* = X \circ x^* \,, \qquad \mathrm{ev}|_{s=0}\partial_s \circ \gamma^\sharp = \overline{P}_{\gamma_{s=0}|_{[0,t]}}(Y \circ x^\sharp) =: Y_t \tag{19}$$

such that $Y_0 = Y \circ x^\sharp$. Such a homotopy indeed exists. First, by Lemma 12, there is $\tilde{\gamma}\colon S \times [0,1] \to M$ (parameter t) such that the first condition in (19) is satisfied. Now fix t. For this t, there is, by the same lemma, an S-path $\gamma_t\colon S \times [0,1] \to M$ (parameter s) such that also the second condition holds true with parallel transport $\overline{P}_{\tilde{\gamma}}|_{[0,t]}$ on the right hand side. By construction, γ_t depends smoothly on t and s, thus yielding γ as required.

Proposition 6. *Let $Z \in \mathcal{E}_S$ and $E \in \mathrm{End}_{\mathcal{O}_N}(\mathcal{E})$. Then*

$$\begin{aligned}\frac{d}{dt}\Big|_0 \frac{d}{ds}\Big|_0 (P^2_{s,t})^{-1}(\gamma^* Z) &= (x^*\overline{\nabla}^2)_{X,Y}(x^*Z) \\ \frac{d}{dt}\Big|_0 \frac{d}{ds}\Big|_0 \left((P^2_{s,t})^{-1} \circ E_\gamma \circ P^2_{s,t}\right) &= (x^*\overline{\nabla}^2)_{X,Y}E_x \\ \frac{d}{dt}\Big|_0 \frac{d}{ds}\Big|_0 \left((P^2_{s,t})^{-1} \circ R_\gamma\left(\overline{P}^2_{s,t}(u), \overline{P}^2_{s,t}(v)\right) \circ P^2_{s,t}\right) &= (x^*\overline{\nabla}^2)_{X,Y}R_x(u,v)\end{aligned}$$

with

$$P^2_{s,t} := P_{\gamma_t|_{[0,s]}} \circ P_{\gamma_{s=0}|_{[0,t]}} \,, \qquad \overline{P}^2_{s,t} := \overline{P}_{\gamma_t|_{[0,s]}} \circ \overline{P}_{\gamma_{s=0}|_{[0,t]}}$$

Proof. Using Proposition 5 and $|Y_t^l| = |dx[Y]^l| = |Y\left(x^*(\xi^l)\right)| = |\xi^l|$, we calculate

$$\begin{aligned}&\partial_s|_0\partial_t|_0(P_{\gamma_{s=0}|_{[0,t]}})^{-1}(P_{\gamma_t|_{[0,s]}})^{-1}(\gamma^*Z) \\ &\quad= \partial_t|_0(P_{\gamma_{s=0}|_{[0,t]}})^{-1}\partial_s|_0(P_{\gamma_t|_{[0,s]}})^{-1}(\gamma^*Z) \\ &\quad= \partial_t|_0(P_{\gamma_{s=0}|_{[0,t]}})^{-1}\left((\gamma_t^*\nabla)_{Y_t}(\gamma_t^*Z)\right) \\ &\quad= (-1)^{|\xi^l||Z|}\partial_t|_0(P_{\gamma_{s=0}|_{[0,t]}})^{-1}\gamma_t^*(\nabla_{\partial_l}Z)\cdot dx[Y]^l + (-1)^{|\xi^l||Z|}x^*(\nabla_{\partial_l}Z)\partial_t|_0Y_t^l\end{aligned}$$

Now we use $\partial_t|_0(P_{\gamma_{s=0}|_{[0,t]}})^{-1}\gamma_t^*(\nabla_{\partial_l}Z) = (x^*\nabla)_X(x^*\nabla_{\partial_l}Z)$ and

$$\partial_t|_0Y_t = -\mathcal{B}^X(s=0)dx[Y] = -(-1)^{|n|}dx[Y]^n dx[X](\xi^l)x^*(\overline{\nabla}_{\partial_{\xi^l}}\partial_{\xi^n})$$

to yield the first statement after a straightforward calculation.

The left hand side of the second equation is treated as follows.

$$\begin{aligned}\mathrm{LHS} &= \frac{d}{dt}\Big|_0 \left(P_\gamma|_{s=0,[0,t]}^{-1}\partial_s|_0\left(P_\gamma|_{t,[0,s]}^{-1}\circ E_\gamma\circ P_\gamma|_{t,[0,s]}\right)\circ P_\gamma|_{s=0,[0,t]}\right)\\ &= \frac{d}{dt}\Big|_0\left(P_\gamma|_{s=0,[0,t]}^{-1}\left((\gamma_t^*\nabla)_{Y_t}\circ E_{\gamma_t}-E_{\gamma_t}\circ(\gamma_t^*\nabla)_{Y_t}\right)\circ P_\gamma|_{s=0,[0,t]}\right)\\ &= (x^*\nabla)_X\left((x^*\nabla)_Y E_x\right)-(x^*\nabla)_{(x^*\overline{\nabla})_X[dx[Y]]}\circ E_x+E_x\circ(x^*\nabla)_{(x^*\overline{\nabla})_X[dx[Y]]}\end{aligned}$$

Here, the second equation follows from Proposition 5 applied to the second condition in (19). For the third equation, we use again Proposition 5 to obtain the first term and find, in addition, two derivative terms with respect to $(\gamma_t^*\nabla)_{Y_t}$ which are obtained as in the previous calculation.

Similarly we yield, for the left hand side of the last equation to be shown,

$$\begin{aligned}\mathrm{LHS} = \frac{d}{dt}\Big|_0 P_\gamma|_{s=0,[0,t]}^{-1}\Big(&(\gamma_t^*\nabla)_{Y_t}\circ R_{\gamma_t}\left(\overline{P}_\gamma|_{s=0,[0,t]}(u),\,\overline{P}_\gamma|_{s=0,[0,t]}(v)\right)\\ &-R_{\gamma_t}\left((\gamma_t^*\overline{\nabla})_{Y_t}(\overline{P}_\gamma|_{s=0,[0,t]}(u)),\,\overline{P}_\gamma|_{s=0,[0,t]}(v)\right)\\ &-R_{\gamma_t}\left(\overline{P}_\gamma|_{s=0,[0,t]}(u),\,(\gamma_t^*\overline{\nabla})_{Y_t}(\overline{P}_\gamma|_{s=0,[0,t]}(v))\right)\\ &-R_{\gamma_t}\left(\overline{P}_\gamma|_{s=0,[0,t]}(u),\,\overline{P}_\gamma|_{s=0,[0,t]}(v)\right)\circ(\gamma_t^*\nabla)_{Y_t}\Big)P_\gamma|_{s=0,[0,t]}\end{aligned}$$

Analogously to the previous calculation for the second statement, Proposition 5 together with derivative terms from the first calculation yields the right hand side as claimed. □

4.2 Reconstruction of Galaev's Holonomy Algebra

By means of the previously established relation between covariant derivatives and parallel transport, we will now make contact with Galaev's holonomy algebra $\mathrm{hol}_x^{\mathrm{Gal}}$. Let $S=\mathbb{R}^{0|0}$, ∇ be a connection on $\mathcal{E}\to M$, and $x\in M_0$ be a topological point identified with an S-point. We aim at gaining generating elements of $\mathrm{hol}_x^{\mathrm{Gal}}$ as coefficients of special elements of $\mathrm{hol}_x(T)$ for $T=\mathbb{R}^{0|L'}$ with $L'\geq(\dim M)_{\overline{1}}$. Let $q\in M_0$, and define the $(S\times)T$-point y by prescribing

$$y^\sharp(x^k):=q^*(x^k)=q^k\ ,\qquad y^\sharp(\theta^i):=\eta^i \tag{20}$$

with respect to coordinates $\xi=(x,\theta)$ around q. Then, a straightforward calculation using (14) shows that

$$\begin{aligned}(y^*\nabla)_{\partial_{\eta^j}}(y^*Z) &= \hat{y}^*(\nabla_{\partial_{\theta^j}}Z)\\ (y^*\nabla)_{(y^*\overline{\nabla})_{\partial_{\eta^j}}[dy[\partial_{\eta^k}]]}(y^*Z) &= \hat{y}^*\left(\nabla_{\overline{\nabla}_{\partial_{\theta^j}}\partial_{\theta^k}}Z\right)\end{aligned}$$

For the curvature terms, it follows that

$$\begin{aligned}R_y\left(y^*\partial_{\xi^i},\,y^*\partial_{\xi^j}\right) &= \hat{y}^*\left(R\left(\partial_{\xi^i},\,\partial_{\xi^j}\right)\right)\\ \left((y^*\overline{\nabla})_{\partial_{\eta^l}}R_y\right)\left(y^*\partial_{\xi^i},\,y^*\partial_{\xi^j}\right) &= \hat{y}^*\left((\overline{\nabla}_{\partial_{\theta^l}}R)\left(\partial_{\xi^i},\,\partial_{\xi^j}\right)\right)\\ \left((y^*\overline{\nabla}^2)_{\partial_{\eta^l},\partial_{\eta^m}}R_y\right)\left(y^*\partial_{\xi^i},\,y^*\partial_{\xi^j}\right) &= \hat{y}^*\left((\overline{\nabla}^2_{\partial_{\theta^l},\partial_{\theta^m}}R)\left(\partial_{\xi^i},\,\partial_{\xi^j}\right)\right)\end{aligned} \tag{21}$$

Lemma 13. *Let y be the T-point (20), $\gamma\colon x \to y$ be a connecting T-path and I_k denote a multiindex of parity $|\xi^k|$ such that $\eta^{I_k} \in \mathcal{O}_T$. Then*

$$\eta^{I_{k_1}}\eta^{I_{k_2}} \cdot P_\gamma^{-1} \circ y^* \left(R\left(\partial_{\xi^{k_2}}, \partial_{\xi^{k_1}}\right)\right) \circ P_\gamma \in \mathrm{hol}_x(T)$$
$$\eta^{I_{k_1}}\eta^{I_{k_2}}\eta^{I_{k_3}} \cdot P_\gamma^{-1} \circ y^* \left((\nabla_{\partial_{\theta^{k_3}}} R)\left(\partial_{\xi^{k_2}}, \partial_{\xi^{k_1}}\right)\right) \circ P_\gamma \in \mathrm{hol}_x(T)$$
$$\eta^{I_{k_1}}\eta^{I_{k_2}}\eta^{I_{k_3}}\eta^{I_{k_4}} \cdot P_\gamma^{-1} \circ y^* \left((\nabla^2_{\partial_{\theta^{k_4}}, \partial_{\theta^{k_3}}} R)\left(\partial_{\xi^{k_2}}, \partial_{\xi^{k_1}}\right)\right) \circ P_\gamma \in \mathrm{hol}_x(T)$$

Proof. By Theorem 1, the first term

$$\eta^{I_{k_1}}\eta^{I_{k_2}} \cdot P_\gamma^{-1} \circ y^* \left(R\left(\partial_{k_2}, \partial_{k_1}\right)\right) \circ P_\gamma = P_\gamma^{-1} \circ R_y\left(\eta^{I_{k_2}} \cdot (y^* \circ \partial_{k_2}),\, \eta^{I_{k_1}} \cdot (y^* \circ \partial_{k_1})\right) \circ P_\gamma$$

is clearly contained in $\mathrm{hol}_x(T)$. For the second, let δ be an S-path connecting y to some S-point z such that $\mathrm{ev}|_0 \partial_t \circ \delta^\sharp = \xi := dy\left[\eta^{I_{k_3}} \cdot \partial_{\eta^{k_3}}\right]$. Using (21), followed by Proposition 5 applied to y, ξ, δ as well as $u := \eta^{I_{k_2}} \cdot (y^* \circ \partial_{k_2})$ and $v := \eta^{I_{k_1}} \cdot (y^* \circ \partial_{k_1})$, we yield

$$\begin{aligned}
&\eta^{I_{k_1}}\eta^{I_{k_2}}\eta^{I_{k_3}} \cdot P_\gamma^{-1} \circ y^* \left((\nabla_{\theta^{k_3}} R)\left(\partial_{k_2}, \partial_{k_1}\right)\right) \circ P_\gamma \\
&\quad = P_\gamma^{-1} \circ \left((y^*\nabla)_{\eta^{I_{k_3}} \cdot \partial_{\eta^{k_3}}} R_y\right)\left(\eta^{I_{k_2}} \cdot (y^* \circ \partial_{k_2}),\, \eta^{I_{k_1}} \cdot (y^* \circ \partial_{k_1})\right) \circ P_\gamma \\
&\quad = P_\gamma^{-1} \circ \partial_t|_0 \left(P_\delta|_{[0,t]}^{-1} \circ R_\delta\left(\overline{P}_\delta|_{[0,t]}(u),\, \overline{P}_\delta|_{[0,t]}(v)\right) \circ P_\delta|_{[0,t]}\right) \circ P_\gamma \\
&\quad = \partial_t|_0 \left(P_\gamma^{-1} \circ P_\delta|_{[0,t]}^{-1} \circ R_\delta\left(\overline{P}_\delta|_{[0,t]}(u),\, \overline{P}_\delta|_{[0,t]}(v)\right) \circ P_\delta|_{[0,t]} \circ P_\gamma\right)
\end{aligned}$$

By Theorem 1, the term in parentheses lies, for every $t \in [0,1]$, in $\mathrm{hol}_x(T)$, which is a vector space. Therefore, the differential is also contained in $\mathrm{hol}_x(T)$.

The second covariant derivative term is treated analogous. □

Consider the zero-derivative term in Lemma 13. For generic choice of $\eta^{I_{k_1}}$ and $\eta^{I_{k_2}}$, we find that

$$\begin{aligned}
&\left(\partial_{\eta^{I_{k_1}}}\partial_{\eta^{I_{k_2}}}\left(\eta^{I_{k_1}}\eta^{I_{k_2}} P_\gamma^{-1} \circ y^* \left(R\left(\partial_{\xi^{k_2}}, \partial_{\xi^{k_1}}\right)\right) \circ P_\gamma\right)\right)_0 \\
&\quad = P_{\gamma_0}^{-1} \circ R_{y_0}\left(\partial_{\xi^{k_2}}, \partial_{\xi^{k_1}}\right) \circ P_{\gamma_0} \in \mathrm{hol}_x^{\mathrm{Gal}}
\end{aligned}$$

and analogous for the first and second derivative terms and, by conjecture, for all higher derivative terms. The generating elements of $\mathrm{hol}_x^{\mathrm{Gal}}$ can thus be extracted out of $\mathrm{hol}_x(T)$ as certain coefficients of special elements in the way made precise by Lemma 13. This construction is based on the knowledge of the geometric significance of the elements. It remains an open question whether $\mathrm{hol}_x^{\mathrm{Gal}}$ can be obtained from $\mathrm{hol}_x(T)$ in a purely algebraic way.

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Author's address:
Universität zu Köln, Institut für Theoretische Physik, Zülpicher Str. 77, 50937 Köln, Germany

E-mail: `groegerj@thp.uni-koeln.de`

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