Mathematica Bohemica

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Mathematica Bohemica, Vol. 139 (2014), No. 4, 587--595

Persistent URL: `http://dml.cz/dmlcz/144136`

FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH RIEMANN-LIOUVILLE INTEGRALS IN THE NONLINEARITIES

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(Received September 24, 2013)

Abstract. A sufficient condition for the nonexistence of blowing-up solutions to evolution functional-differential equations in Banach spaces with the Riemann-Liouville integrals in their right-hand sides is proved. The linear part of such type of equations is an infinitesimal generator of a strongly continuous semigroup of linear bounded operators. The proof of the main result is based on a desingularization method applied by the author in his papers on integral inequalities with weakly singular kernels. The result is illustrated on an example of a scalar equation with one Riemann-Liouville integral.

Keywords: fractional differential equation; Riemann-Liouville integral; blowing-up solution

MSC 2010: 34K37, 34A08, 34K05, 34G20

1. Introduction

The influence of viscous fluids on vibrating systems is often modeled using the Riemann-Liouville or Caputo fractional derivative. These derivatives play the role of a damping force called fractional damping. The well known Bagley-Torvik equation (see [21])

$$u''(t) + A^c D^{3/2} u(t) = au(t) + \varphi(t), \tag{1.1}$$

modelling the motion of a rigid plate immersing in a viscous liquid, is one of the equations describing the motion with the fractional damping term $A^c D^{3/2} u(t)$ with

This work was supported by the Slovak Research and Development Agency under the contract No. APVV-0134-10 and by the Slovak Grant Agency VEGA No. 1/0507/11 and No. 1/0071/14.

the Caputo fractional derivatives. Solutions of the linear fractionally damped oscillator equation with the Caputo derivative are analysed in the paper [12]. Interesting results concerning boundary value problems for the following generalized Bagley-Torvik equation

$$u''(t) + A^c D^\alpha u(t) = f\bigl(t, u(t), {}^c D^\beta u(t), u'(t)\bigr) \tag{1.2}$$

and for some fractional differential equations are published in the papers [1], [2], [13] and [17]. We were motivated by the paper [16], where an existence and uniqueness result for the initial value problem

$$Au'' + \sum_{k=1}^{N} B_k^c D^{\alpha_k} u(t) = f(t, u), \tag{1.3}$$
$$u(0) = u_0, \quad u'(0) = c_1, \quad 0 < \alpha_k < 2, \; k = 1, 2, \ldots, N$$

is proved. This type of equations can be written as systems of differential equations with the Riemann-Liouville integrals on their right-hand sides (see Section 2). Some existence results for the initial value problems corresponding to these equations are proved in the papers [5] and [7]. In the papers [18]–[20], several existence results for fractional differential equations are proved. For basic definitions of fractional calculus and fundamentals of the theory of fractional differential equations see, e.g., the monograph [15].

This paper is concerned with the following initial value problem

$$\dot{x}(t) = Ax(t) + f\bigl(t, x(t), x_t, (I^{\alpha_1}[g_1 x])(t), \ldots, (I^{\alpha_m}[g_m x])(t)\bigr), \quad t > 0, \tag{1.4}$$

$$x(t) = \Phi(t), \quad t \in [-r, 0], \tag{1.5}$$

where $r > 0$, $\Phi \in C_r := C([-r, 0], X)$, X is a Banach space with the norm $\|v\|$, $v \in X$, $x(t) \in X$, $x_t \in C$, $x_t(\Theta) := x(t + \Theta)$, $t > 0$, $\Theta \in [-r, 0]$, A is the infinitesimal generator of a strongly continuous semigroup $\{S(t)\}_{t \geqslant 0}$, $S(t) := \mathrm{e}^{At}$, $f\colon \mathbb{R}_+ \times X \times C_r \times X^m \to X$, $X^m := X \times \ldots \times X$ (m times) is a continuous map, $\mathbb{R}_+ = [0, \infty)$, $g_i\colon \mathbb{R}_+ \times X \to X$, $(t, x) \mapsto g_i(t, x)$, $i = 1, 2, \ldots, m$ are continuous maps,

$$(I^{\alpha_i}[g_i x])(t) := \frac{1}{\Gamma(\alpha_i)} \int_0^t (t - s)^{\alpha_i - 1} g_i(s, x(s)) \, \mathrm{d}s, \quad 0 < \alpha_i < 1 \tag{1.6}$$

are the Riemann-Liouville fractional integrals of the function $[g_i x](t) := g_i(t, x(t))$ of order α_i.

Definition 1.1. A semigroup $\{S(t)\}_{t \geqslant 0}$, $S(t) := \mathrm{e}^{At}$ of bounded linear operators on X is called a strongly continuous semigroup of bounded linear operators if

$$\lim_{t \to 0^+} S(t)x = x \quad \text{for every } x \in X. \tag{1.7}$$

Theorem 1.1 ([14], Theorem 2.2). *If $\{S(t)\}_{t\geqslant 0}$ is a strongly continuous semigroup of bounded linear operators, then there exist constants $\omega \geqslant 0$ and $M \geqslant 1$ such that*

$$\|S(t)\| \leqslant M\mathrm{e}^{\omega t} \quad \text{for all } t \in [0,\infty).$$

Definition 1.2. By a mild solution of the initial value problem (1.4), (1.5) on an interval $[-r,b)$, $b>0$, we mean a continuous mapping $x \in C([-r,b),X)$ satisfying

$$x(t) = \mathrm{e}^{At}\Phi(0) + \int_0^t \mathrm{e}^{A(t-s)} f\big(s, x(s), x_s, (I^{\alpha_1}[g_1x])(s), \ldots, (I^{\alpha_m}[g_mx])(s)\big)\,\mathrm{d}s, \quad t \in (0,b), \tag{1.8}$$

$$x(t) = \Phi(t), \quad t \in [-r,0]. \tag{1.9}$$

Definition 1.3. We say that a mild solution $y(t)$ of the initial value problem (1.4), (1.5) defined on the interval $[0,b)$, $0 < b \leqslant \infty$, blows-up at a finite time $T \in (0,b)$ if $\lim\limits_{t\to T^-} \|y(t)\| = \infty$.

The main aim of this paper is to prove a sufficient condition for the nonexistence of blowing-up solutions to the equation (1.4).

Example 1.1 (for ODEs).

$$u'(t) = u^2(t), \quad u(0) = u_0. \tag{1.10}$$

Solution:

$$u(t) = \frac{u_0}{tu_0 - 1} = -\frac{1}{t - 1/u_0}, \quad \lim_{t\to T^-} |u(t)| = \infty, \quad T = \frac{1}{u_0}. \tag{1.11}$$

Example 1.2 (for delay equations).

$$u'(t) = u(t-2)u(t)^2, \tag{1.12}$$

$$u(t) = -t + 1, \quad t \in [-2,0]. \tag{1.13}$$

Solution:

$$u(t) = \frac{2}{t^2 - 6t + 2}, \quad \lim_{t\to T^-} |u(t)| = \infty, \quad T = 3 - \sqrt{7}. \tag{1.14}$$

Some further examples and results on the existence of blowing-up solutions for delay differential equations can be found in the paper [4].

E x a m p l e 1.3 (for delay fractional equations).

$$u'(t) = u(t-2)u^2(t) + \left(\int_0^t (t-s)^{-1/2} u(s-2)\,\mathrm{d}s\right) u^2(t), \tag{1.15}$$

$$u(t) = 1, \quad t \in [-2,0]. \tag{1.16}$$

Solution:

$$u(t) = -\frac{3}{4t^{3/2}+3t-3}, \quad t \in [0,2], \ \lim_{t\to T^-} |u(t)| = \infty, \tag{1.17}$$

where $T = x^2 \in (0, 1.44)$, $x \in (0, 1.2)$ is the unique positive root of the polynomial $P(x) = x^3 + x^2 - 3$ ($P(0) = -3$, $P(1.2) = 3.168$).

2. Fractionally damped pendulum

Fractionally damped pendulums or oscillators are studied, e.g., in the papers [12], and [16], where also some further papers devoted to this type of equations can be found in the list of references.

The equation

$$x''(t) + \lambda_1^c D^{\beta_1} x(t) + \ldots + \lambda_m^c D^{\beta_m} x(t) + \lambda x'(t) + \omega^2 x(t) = g(x_t), \quad t > 0,$$

is a fractional perturbation of the ordinary damped pendulum equation

$$x''(t) + \lambda x'(t) + \omega^2 x(t) = g(x_t), \quad t > 0,$$

with the damping term $\lambda x'(t)$, where

$${}^c D^{\beta_i} x(t) = \frac{1}{\Gamma(1-\beta_i)} \int_0^t (t-s)^{-\beta_i} x'(s)\,\mathrm{d}s$$

is the Caputo derivative of the function $x(t)$ of order $\beta_i \in (0,1)$ and fractional damping terms

$$\lambda_1^c D^{\beta_1} x(t), \ \ldots, \ \lambda_m^c D^{\beta_m} x(t).$$

The external force can be, e.g., the delay feedback $g(x_t) = h(x_t(-r)) = h(x(t-r))$, or the functional feedback of the form $g(x_t) = \int_{-r}^0 h(x_t(\Theta))\,\mathrm{d}\Theta = \int_{-r}^0 h(x(t+\Theta))\,\mathrm{d}\Theta$, or a more general functional feedback.

We can write this equation as the system

$$\begin{aligned} z'(t) = Az(t) &- \lambda_1 B I^{\alpha_1}[z](t) - \ldots - \lambda_m B I^{\alpha_m}[z](t) \\ &- \lambda B z(t) + F(z_t), \quad \alpha_i = 1 - \beta_i, \ i = 1, 2, \ldots, m, \end{aligned}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad z(t) = \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix}, \quad F(z_t) = \begin{pmatrix} 0 \\ g(x_t) \end{pmatrix}.$$

This equation is of the form (1.4).

3. Main result

A sufficient condition for the nonexistence of blowing-up solutions of the initial value problem

$$\text{(3.1)} \qquad u'(t) = Ax(t) + f\big(t, u(t), u_t, (I^\beta g(\cdot, u(\cdot), u.))(t)\big), \quad t > 0,$$

$$\text{(3.2)} \qquad u(t) = \Phi(t), \quad t \in [-r, 0],$$

where

$$\text{(3.3)} \qquad \big(I^\beta g(\cdot, u(\cdot), u.)\big)(t) := \int_0^t (t-s)^{\beta-1} g(s, x(s), x_s)\,\mathrm{d}s, \quad \beta \in (0,1),$$

is proved in the paper [6]. The equation (3.1) contains one fractional integral only. In the papers [5] and [7], existence results for some abstract evolution equations with fractional derivatives in the nonlinearities are proved.

We will study the problem for the fractional equation with several fractional integrals.

We assume that the mapping $f(t, u, v, v_1, \ldots, v_m)$ satisfies the following condition:

(H1) There exist continuous nonnegative functions $F_1(t), F_2(t), F_3(t), \ldots, F_{m+2}(t)$, $t \geqslant 0$ and continuous nonnegative nondecreasing functions $\omega_1(x)$, $\omega_2(x)$, $x \geqslant 0$, positive for $t > 0$, such that

$$\|f(t, u, v, v_1, \ldots, v_m)\| \leqslant F_1(t)\omega_1(\|u\|) + F_2(t)\omega_2(\|v\|_C) + F_3(t)\|v_1\| + \ldots + F_{m+2}(t)\|v_m\|$$

for all $t \geqslant 0$, $u, v_1, \ldots, v_m \in X$, $v \in C_r$, $\|\cdot\|$ is the norm on X and $\|v\|_{C_r} := \sup_{-r \leqslant \Theta} \|v(\Theta)\|$ is the norm on C_r.

We assume the condition:

(H2) There exist continuous nonnegative functions $G_3(t), G_4(t), \ldots, G_{m+2}(t)$, $t \geqslant 0$ and continuous nonnegative and nondecreasing functions $\omega_3(y), \omega_4(y), \ldots, \omega_{m+2}(y)$, $y \geqslant 0$, positive for $y > 0$, such that

$$\|g_j(t, y)\| \leqslant G_j(t)\omega_j(\|y\|) \quad \text{for all } t \geqslant 0,\ x \in X,\ j = 3, 4, \ldots, m+2.$$

Theorem 3.1. *Let the conditions* (H1) *and* (H2) *be satisfied. Let* $p_1 > 1$, $p_2 > 1, \ldots, p_m > 1$ *be such that* $1+p_1(\alpha_1-1) > 0$, $1+p_2(\alpha_2-1) > 0, \ldots, 1+p_m(\alpha_m-1) > 0$, $\alpha_i \in (0,1)$, $i = 1,2,\ldots,m$ *and let the integral condition*

$$\text{(H3)} \qquad \int_{v_0}^{\infty} \frac{\sigma^{q-1}\,\mathrm{d}\sigma}{\omega_1(\sigma)^q + \omega_2(\sigma)^q + \ldots + \omega_{m+2}(\sigma)^q} = \infty,$$

be satisfied, where $q = q_1 q_2 \ldots q_m$, $q_i = p_i/(p_i-1)$, $i = 1,2,\ldots,m$. *Then the initial value problem* (1.4) *and* (1.5) *does not have blowing-up mild solutions for any initial value* $\Phi \in C_r$.

Proof. Let $x\colon [0,T) \to X$ be a mild solution of the initial value problem (1.4), (1.5) with $0 < T < \infty$, $\lim_{t\to T^-}\|x(t)\| = \infty$. Using the inequality $\|\mathrm{e}^{At}\| \leqslant M\mathrm{e}^{\omega t}$, $t \geqslant 0$ from Theorem 1.1 and the conditions (H1) and (H2) we obtain for $t \in [0,T)$

$$\begin{aligned}\|x(t)\| \leqslant a(T) + K(T)\int_0^t F_1(s)\omega_2(\|x(s)\|_{C_r})\,\mathrm{d}s \\ + L(T)\sum_{i=3}^{m+2}\int_0^t\int_0^s (s-\tau)^{\alpha_i-1}G_i(\tau)\omega_i(\|x(\tau)\|)\,\mathrm{d}\tau\,\mathrm{d}s\end{aligned}$$

for some positive constants $a(T)$, $K(T)$, $L(T)$ depending on the fixed T. We may assume without loss of generality that the constant $a(T)$ is so large that $M\|\Phi(0)\|\mathrm{e}^{\omega T} \leqslant M\|\Phi\|_{C_r}\mathrm{e}^{\omega T} \leqslant a(T)$. Now we apply the following desingularization method suggested in the paper [11] (this method is also applied in the papers [8]–[10]):

$$\begin{aligned}&\int_0^s (s-\tau)^{\alpha_i-1}G_i(\tau)\omega_i(\|x(\tau)\|)\,\mathrm{d}\tau \\ &\quad\leqslant \Big(\int_0^s (s-\tau)^{p_i(\alpha_i-1)}\mathrm{e}^{p_i\tau}\,\mathrm{d}\tau\Big)^{1/p_i}\Big(\int_0^s \mathrm{e}^{-q_i\tau}G_i(\tau)^{q_i}\omega_i(\|x(\tau)\|)^{q_i}\,\mathrm{d}\tau\Big)^{1/q_i} \\ &\quad\leqslant Q_i\mathrm{e}^{p_i s}\Big(\int_0^s \mathrm{e}^{-q_i\tau}G_i(\tau)^{q_i}\omega_i(\|x(\tau)\|)^{q_i}\,\mathrm{d}\tau\Big)^{1/q_i},\end{aligned}$$

where

$$Q_i = \frac{\Gamma(1+p_i(\alpha_i-1))}{p^{1+p_i(\alpha_i-1)}}, \quad i = 1,2,\ldots,m.$$

We have used there the inequality

$$\int_0^s (s-\tau)^{p_i(\alpha_i-1)}\mathrm{e}^{p_i\tau}\,\mathrm{d}\tau \leqslant Q_i\mathrm{e}^{p_i s}, \quad s > 0,$$

proved in the paper [11].

Therefore we obtain the following estimate:

$$\|x(t)\| \leqslant a(T) + L_1 \int_0^t \omega_1(\|x(\tau)\|)\,\mathrm{d}\tau + L_2 \int_0^t \omega_2(\|x_s\|_{C_r})\,\mathrm{d}s + \sum_{i=3}^{m+2} L_i \Big(\int_0^t G_i(\tau)^{q_i} \omega_i(\|x(\tau)\|)^{q_i}\,\mathrm{d}\tau \Big)^{1/q_i},$$

where $M = M(T)$, $L_j = L_j(T)$, $j = 1, 2, \ldots, m+2$ are some positive constants. If $q = q_1 q_2 \ldots q_m$, then

$$\|x(t)\|^q \leqslant \widetilde{K} + \widetilde{L}_1 \Big(\int_0^t \omega_1(\|x(\tau)\|)\,\mathrm{d}\tau \Big)^q + \widetilde{L}_2 \Big(\int_0^t \omega_2(\|x_s\|_{C_r})\,\mathrm{d}s \Big)^q + \sum_{i-3}^{m+2} \widetilde{L}_i \Big(\int_0^t G_i(\tau)^{q_i} \omega_i(\|x(\tau)\|)^{q_i}\,\mathrm{d}\tau \Big)^{\widehat{q_i}},$$

where $\widehat{q_i} = q_1 q_2 \ldots q_{i-1} q_{i+1} \ldots q_m$. Using the integral mean value theorem we obtain

$$\|x(t)\|^q \leqslant a + a_1 \int_0^t \omega_1(\|x(\tau)\|)^q\,\mathrm{d}\tau + a_2 \int_0^t \omega_2(\|x_s\|_{C_r})^q\,\mathrm{d}s + \sum_{i-3}^{m+2} a_i \int_0^t \omega_i(\|x(\tau)\|)^q\,\mathrm{d}\tau$$

for some constants a, a_j, $j = 1, 2, \ldots, m+2$. Let a be so large that $\|\Phi\|_{C_r} \leqslant a$.

Let $g(t)$ be the right-hand side of the above inequality. Then

$$\|x_t\| := \max_{-r \leqslant \Theta \leqslant 0} \|x(t+\Theta)\| \leqslant \max\Big\{ \|\Phi\|_{C_r}, \sup_{0 \leqslant \tau \leqslant t} \|x(\tau)\| \Big\} \leqslant g(t)^{1/q}.$$

This yields

$$\|x(t)\|^q \leqslant g(t) \leqslant a + A \int_0^t \omega(g(s))\,\mathrm{d}s, \quad t \geqslant 0,$$

where

$$\omega(v) = \sum_{i=1}^{m+2} \omega_i(v^{1/q})^q.$$

Let

$$\Omega(v) = \int_{v_0}^{v} \frac{\mathrm{d}\eta}{\omega(\eta)}, \quad v_0 > 0.$$

From the Bihari inequality it follows that

$$\Omega(\|x(t)\|^q) \leqslant \Omega(g(t)) \leqslant \Omega(a) + At, \quad t \geqslant 0.$$

Thus we have

$$\lim_{t\to T^-}\Omega(\|x(t)\|^q)=\lim_{t\to T^-}\int_{v_0}^{\|x(t)\|^q}\frac{\mathrm{d}\tau}{\omega(\tau)}=\int_{v_0}^{\infty}\frac{\mathrm{d}\tau}{\omega(\tau)}$$
$$=q\int_{w_0}^{\infty}\frac{\sigma^{q-1}\,\mathrm{d}\sigma}{\sum_{i=1}^{m+2}\omega_i(\sigma)^q}=\infty,\quad w_0={v_0}^{1/q}.$$

However $\lim\limits_{t\to T^-}[\Omega(a)+At]=\Omega(a)+AT<\infty$ and this is a contradiction. □

Example 3.1. Let $m=1$, $q_1=q>1$, $\alpha_1=\alpha\in(0,1)$, $p(\alpha-1)+1>0$, $1/p+1/q=1$, $\omega_1(v)\equiv(1/2)^{1/q}v$, $\omega_2(v)\equiv(1/2)^{1/q}v$, $\omega_3(v)=[\ln(v^q+2)]^{1/q}$. Then

$$\int_0^{\infty}\frac{\sigma^{q-1}\,\mathrm{d}\sigma}{\omega_1(\sigma)^q+\omega_2(\sigma)^q+\omega_3(\sigma)^q}=\int_0^{\infty}\frac{\sigma^{q-1}\,\mathrm{d}\sigma}{\sigma^q+\ln(\sigma^q+2)}$$
$$=\frac{1}{q}\int_0^{\infty}\frac{\mathrm{d}s}{s+\ln(s+2)}=\infty.$$

This equality follows from the following lemma by A. Constantin [3].

Lemma 3.1. *If $w\in C(\mathbb{R}_+,[0,\infty))$ is a continuous positive nondecreasing function and*

$$\int_0^{\infty}\frac{\mathrm{d}s}{w(s)}=\infty,$$

then

$$\int_0^{\infty}\frac{\mathrm{d}s}{s+w(s)}=\infty.$$

Therefore the conditions (H1), (H2) and (H3) are satisfied and by Theorem 3.1 the initial value problem (1.4), (1.5) does not have blowing-up solutions for any $\Phi\in C_r$.

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