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BV SOLUTIONS OF RATE INDEPENDENT DIFFERENTIAL INCLUSIONS

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Abstract. We consider a class of evolution differential inclusions defining the so-called stop operator arising in elastoplasticity, ferromagnetism, and phase transitions. These differential inclusions depend on a constraint which is represented by a convex set that is called the characteristic set. For BV (bounded variation) data we compare different notions of BV solutions and study how the continuity properties of the solution operators are related to the characteristic set. In the finite-dimensional case we also give a geometric characterization of the cases when these kinds of solutions coincide for left continuous inputs.

Keywords: differential inclusion; stop operator; rate independence; convex set

MSC 2010: 34A60, 74C05, 52B99

1. Introduction

Let us assume that $\mathcal{H}$ is a real Hilbert space and $\mathcal{Z} \subseteq \mathcal{H}$ is a closed convex proper subset containing the zero vector: $0 \in \mathcal{Z} \neq \mathcal{H}$. If a final time $T > 0$ is given, together with a function $u \in W^{1,1}(0,T;\mathcal{H})$ and a vector $z_0 \in \mathcal{Z}$, we consider the following evolution differential inclusion for the unknown $x \in W^{1,1}(0,T;\mathcal{H})$:

$$x(t) \in \mathcal{Z} \quad \forall t \in [0,T], \tag{1.1}$$

$$x'(t) + \partial I_{\mathcal{Z}}(x(t)) \ni u'(t) \quad \text{for a.e. } t \in [0,T], \tag{1.2}$$

$$x(0) = z_0. \tag{1.3}$$

Here, $W^{1,1}(0,T;\mathcal{H})$ is the Sobolev space of $\mathcal{H}$-valued absolutely continuous functions

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(cf. [3], Appendix), $I_{\mathcal{Z}}\colon \mathcal{H} \longrightarrow [0,\infty]$ is the indicator function of $\mathcal{Z}$ defined by

$$I_{\mathcal{Z}}(x) := \begin{cases} 0 & \text{if } x \in \mathcal{Z}, \\ \infty & \text{if } x \notin \mathcal{Z}, \end{cases}$$

and $\partial I_{\mathcal{Z}}\colon \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ is the subdifferential of $I_{\mathcal{Z}}$ in the sense of convex analysis (cf. [3], Example 2.1.4, page 21):

$$\partial I_{\mathcal{Z}}(x) = \begin{cases} \{y \in \mathcal{H}\colon \ \langle y, v - x\rangle \leqslant 0 \ \forall v \in \mathcal{Z}\} & \text{if } x \in \mathcal{Z}, \\ \emptyset & \text{if } x \notin \mathcal{Z}. \end{cases} \tag{1.4}$$

Observe that $\partial I_{\mathcal{Z}}(x)$ is a cone, i.e., a subset $\mathcal{K} \subseteq \mathcal{H}$ such that $\lambda x \in \mathcal{K}$ whenever $\lambda > 0$ and $x \in \mathcal{K}$.

This differential inclusion arises in mathematical models of material memory and can be solved by means of the classical methods for evolution equations governed by maximal monotone operators. To be more precise, it is well known that exploiting [3], Proposition 3.4, Remark 3.7, page 69, one can prove the following theorem.

Theorem 1.1. *If $z_0 \in \mathcal{Z}$ and $u \in W^{1,1}(0,T;\mathcal{H})$, then there exists a unique $x \in W^{1,1}(0,T;\mathcal{H})$ such that* (1.1)–(1.3) *hold.*

The solution operator of problem (1.1)–(1.3) is also called the *stop operator*: it is the operator $\mathsf{S} = \mathsf{S}(\cdot, z_0)\colon W^{1,1}(0,T;\mathcal{H}) \longrightarrow W^{1,1}(0,T;\mathcal{H})$ that with each $u \in W^{1,1}(0,T;\mathcal{H})$ associates the solution $\mathsf{S}(u,z_0) := x$ of (1.1)–(1.3). Usually the functions u and $\mathsf{S}(u,z_0)$ are called, respectively, *input* and *output* of the stop operator. The constraint $\mathcal{Z}$ is called the *characteristic set*. The dependence on the fixed initial state z_0 will be omitted in the remainder of this paper, and the notation $\mathsf{S}(u,z_0)$ will be shortened to $\mathsf{S}(u)$.

One of the main features of the differential inclusion (1.1)–(1.3) is its *rate independence*: If the speed of u changes, then the speed of the resulting output changes in the same way. This is the content of the following proposition, which is easily proved using the positive zero-homogeneity of $\partial I_{\mathcal{Z}}$.

Proposition 1.2. *The operator $\mathsf{S}\colon W^{1,1}(0,T;\mathcal{H}) \longrightarrow W^{1,1}(0,T;\mathcal{H})$ is rate independent, i.e., if $\varphi\colon [0,T] \longrightarrow [0,T]$ is absolutely continuous, increasing and surjective, then*

$$\mathsf{S}(u \circ \varphi) = \mathsf{S}(u) \circ \varphi \quad \forall u \in W^{1,1}(0,T;\mathcal{H}).$$

Rate independent operators have been widely studied, especially when $\mathcal{H}$ is one-dimensional. We refer the reader to the monographs and surveys [4], [7]–[9], [29] and the references therein.

Since the 1970s, some authors have been investigating how to extend the formulation of problem (1.1)–(1.3) for data $u \in \mathrm{BV}(0,T;\mathcal{H})$, the space of functions of bounded variation. For this space of functions we refer, e.g., to [3], [22]. Anyway, in order to fix the terminology, let us just recall that the (*pointwise*) *variation* of a function $u\colon [0,T] \longrightarrow \mathcal{H}$ on $[a,b] \subseteq [0,T]$ is defined as

$$\mathrm{V}(u,[a,b]) := \sup\biggl\{\sum_{j=1}^{m} \|u(t_j) - u(t_{j-1})\|\colon\ m \in \mathbb{N},\ a = t_0 < \ldots < t_m = b\biggr\},$$

where $\|\cdot\|$ is the norm on $\mathcal{H}$ induced by its scalar product $\langle\cdot,\cdot\rangle$ and $\mathbb{N} = \{1,2,\ldots\}$ is the set of positive integers. Hence

$$\mathrm{BV}(0,T;\mathcal{H}) = \{u\colon\ [0,T] \longrightarrow \mathcal{H}\colon\ \mathrm{V}(u,[0,T]) < \infty\}.$$

Let us also recall that $W^{1,1}(0,T;\mathcal{H}) \subseteq \mathrm{BV}(0,T;\mathcal{H}) \subseteq \mathrm{Reg}(0,T;\mathcal{H})$, where we denote by $\mathrm{Reg}(0,T;\mathcal{H})$ the space of regulated functions, i.e., those functions u that admit left and right limits $u(t-)$, $u(t+)$ in $\mathcal{H}$ at any point $t \in [0,T]$, with the convention that $u(0-) = u(0)$ and $u(T+) = u(T)$ (cf. [1], [2], [5]).

The first works about BV solutions of rate independent differential inclusions are due to J. J. Moreau (cf. [18]–[20]), who studied the so-called *sweeping processes*, which contain (1.1)–(1.3) as a particular case. For a survey on this subject we refer, e.g., to [14], [17], [28] and to the references therein.

In [10], a variational approach is adopted instead. Since, thanks to (1.4), the inclusion (1.2) can be rewritten as a variational inequality, by integrating this inequality in time one gets an integral variational formulation of (1.1)–(1.3) that allows to interpret the stop operator when the input u is a BV function. Here the integral has to be considered in the sense of Kurzweil (cf. [15]).

Another method, which strongly relies on the rate independence property, can be found in the paper [22]. There the extension of S is constructed by “filling in” the jumps of the input u with straight segments along which the input moves with infinite speed. In this way, one can reduce the problem to the continuous case and then use the rate independence to define the new output $\mathsf{S}(u)$. This approach can be applied to a large class of rate independent vectorial operators and is also related to the study of continuity properties of S with respect to the BV topology, as we will see in the next sections.

The aim of this paper is to recall and compare these notions of BV solutions. In particular, in the finite-dimensional case, we provide a geometrical characterization, in terms of the set $\mathcal{Z}$, of the cases when all these extensions are equivalent. Detailed proofs can be found in [13]. Here, we give some more details about the geometric description of the sets $\mathcal{Z}$ for which the various extensions coincide: the *non-obtuse polyhedra*.

Let us stress the fact that when $\mathcal{H} = \mathbb{R}$, the notions of BV solutions given in [10], [18], [22] coincide. This is well known, but it is also a straightforward consequence of our result.

Of course, the list of notions of BV solutions recalled above is not exhaustive. In particular we would like to mention [23] and the recent work [6], where the procedure of [4] is generalized to Banach spaces. Moreover, we point out that in the one-dimensional case much more has been done, and we mention for instance [4], [11], [16], [24], [25], [27].

In this paper we will limit ourselves to left continuous functions and we will consider the space

$$\mathrm{BV}_L(a, b; \mathcal{H}) := \{u\colon\ [a, b] \longrightarrow \mathcal{H}\colon\ \mathrm{V}(u, [0, T]) < \infty,\ u \text{ is left continuous}\}.$$

This is essentially equivalent to dealing with Lebesgue equivalence classes of functions with a special view on the initial point 0, allowing us to take into account Dirac masses at 0 (see e.g. [13], [22]). The use of left continuous functions is also justified by the fact that the viscous regularizations of rate independent processes converge to a left continuous function when the viscosity coefficient tends to zero (cf. [12], Theorem 2.4).

Our result can be extended to the case where the input function u is left continuous or right continuous at every $t \in (0, T)$, but we do not deal with input functions u such that $u(t) \notin \{u(t-), u(t+)\}$ for some $t \in (0, T)$.

In Sections 2–5 we recall the notions of BV solutions introduced in [10], [18], [22], here reformulated for the stop operator S instead of the dual concept of play operator $\mathsf{P}(u) = u - \mathsf{S}(u)$. Some more details can be found in [13].

2. BV solutions in the sense of sweeping processes

In this section we describe the notion of BV solution introduced by Moreau. As mentioned in the Introduction, Moreau dealt with a more general problem, but we present his procedure for the stop operator only. We first consider the case when u is a left continuous step function, i.e., there exist $m \in \mathbb{N}$, a subdivision $0 = t_0 < \ldots < t_m = T$ of the interval $[0, T]$, and vectors $u^1, \ldots, u^m \in \mathcal{H}$ such that $u(t) = u^k$ for $t \in (t_{k-1}, t_k]$. The idea considered in [18] is to discretize the evolution differential inclusion (1.2) using an implicit Euler scheme. Thus, if $h_k := t_k - t_{k-1}$ for $k \in \{1, \ldots, m\}$, one has to look for vectors $x^1, \ldots, x^m \in \mathcal{H}$ such that if $x^0 := z_0$, $u^0 := u(0)$, then

$$\frac{x^k - x^{k-1}}{h_k} + \partial I_{\mathcal{Z}}(x^k) \ni \frac{u^k - u^{k-1}}{h_k}, \quad k = 1, \ldots, m. \tag{2.1}$$

Since the subdifferential of $I_{\mathcal{Z}}$ is positively zero-homogeneous, the previous inclusion is equivalent to

$$x^k - x^{k-1} + \partial I_{\mathcal{Z}}(x^k) \ni u^k - u^{k-1}, \quad k = 1, \dots, m, \tag{2.2}$$

i.e.,

$$x^{k-1} + u^k - u^{k-1} - x^k \in \partial I_{\mathcal{Z}}(x^k), \quad k = 1, \dots, m, \tag{2.3}$$

which is in turn equivalent to

$$x^k = \mathrm{Proj}_{\mathcal{Z}}(x^{k-1} + u^k - u^{k-1}), \quad k = 1, \dots, m, \tag{2.4}$$

where $\mathrm{Proj}_{\mathcal{Z}}$ denotes the orthogonal projection on the convex set $\mathcal{Z}$ characterized by the relation

$$x = \mathrm{Proj}_{\mathcal{Z}}(y) \iff \langle y - x, x - v \rangle \geqslant 0 \quad \forall v \in \mathcal{Z},$$

cf. (1.4). This suggests to define (cf. [18], page 353) the stop operator for a left continuous step input function u as

$$\mathsf{S}_P(u; z_0)(t) := \begin{cases} x^0 := z_0 & \text{if } t = 0, \\ x^k := \mathrm{Proj}_{\mathcal{Z}}(x^{k-1} + u^k - u^{k-1}) & \text{if } t \in (t_{k-1}, t_k],\ k = 1, \dots, m, \end{cases} \tag{2.5}$$

where the index P stands for "Projection". Since every left continuous $u \in \mathrm{BV}(0,T;\mathcal{H})$ can be approximated in the topology of uniform convergence by a sequence of left continuous step functions u_n, we can consider the resulting sequence $x_n := \mathsf{S}_P(u_n, z_0)$. One of the main results in [18] is that x_n converges uniformly to a BV function. Here is the precise statement ([18], Proposition 2 a, page 355, Proposition 3 c, page 373).

Theorem 2.1. *Assume that $u \in \mathrm{BV}_L(0,T;\mathcal{H})$ and u_n is a sequence of left continuous step functions such that $u_n \to u$ uniformly on $[0,T]$. If $x_n := \mathsf{S}_P(u_n, z_0)$, the operator S_P being defined for step functions as in (2.5), then there exists $x \in \mathrm{BV}([0,T];\mathcal{H})$ such that $x_n \to x$ uniformly on $[0,T]$. If $\mathsf{S}_P(u) := x$, the resulting operator $\mathsf{S}_P\colon \mathrm{BV}_L(0,T;\mathcal{H}) \longrightarrow \mathrm{BV}_L(0,T;\mathcal{H})$ is continuous with respect to the topology of uniform convergence. Moreover, if $u \in W^{1,1}(0,T;\mathcal{H})$, then $\mathsf{S}_P(u) = \mathsf{S}(u)$.*

The previous theorem provides the desired extension of the stop operator to BV. We mention the fact that $\mathsf{S}_P(u)$ can also be characterized as the solution of a differential inclusion involving the so-called differential measures, we refer to [18], Section 3 a, for details.

3. BV solutions in the Kurzweil integral sense

In order to describe the formalism proposed in [10], let us recall that using (1.4), the differential inclusion (1.2) can be rewritten as the following variational inequality:

$$\langle x(t) - z(t), u'(t) - x'(t)\rangle \geqslant 0 \quad \forall z \in \mathcal{Z}, \text{ for a.e. } t \in [0,T]. \tag{3.1}$$

Another way to define the generalized BV solutions is thus to integrate inequality (3.1) in time. When u is not absolutely continuous, one has to interpret such integration in a proper way, and a possible choice is to adopt the Kurzweil integral (cf. [15]), or the Young integral as its special case. This is the procedure followed in [10], Lemma 2.1, Theorem 2.3, where the following theorem is proved.

Theorem 3.1. *If* $u \in \mathrm{BV}_L(0,T;\mathcal{H})$*, then there exists a unique* $\mathsf{S}_K(u) := x \in \mathrm{BV}_L(0,T;\mathcal{H})$ *such that*

$$x(t) \in \mathcal{Z} \quad \forall t \in [0,T], \tag{3.2}$$

$$\int_0^T \langle x(s+) - z(s),\, \mathrm{d}(u-x)(s)\rangle \geqslant 0 \quad \forall z \in \mathrm{Reg}([0,T];\mathcal{H}),\ z([0,T]) \subseteq \mathcal{Z}, \tag{3.3}$$

$$x(0) = z_0, \tag{3.4}$$

where the integral in (3.3) *is meant in the sense of Kurzweil. Moreover, if* $u \in W^{1,1}(0,T;\mathcal{H})$ *then,* $\mathsf{S}_K(u) = \mathsf{S}(u)$.

We also refer to [12] for further extensions. Notice that in (3.3), the test functions belong to the space of regulated functions, and this allows to extend the definition of the stop operators even to $\mathrm{Reg}([0,T];\mathcal{H})$, provided that the interior of $\mathcal{Z}$ is nonempty. This is shown in [10], following an idea of A. Vladimirov which is published in [7], Section 19.

4. BV-continuous solutions

In this section we show how to define a notion of BV solution by means of a continuity method. In dealing with extensions from $W^{1,1}$ to BV, the natural topology is the one induced by the *strict metric on* BV, defined by:

$$d_s(u,v) := \|u - v\|_{L^1(0,T;\mathcal{H})} + |\mathrm{V}(u,[0,T]) - \mathrm{V}(v,[0,T])|, \quad u,v \in \mathrm{BV}_L(0,T;\mathcal{H}).$$

We say that $u_n \to u$ *strictly on* $[0,T]$ if $d_s(u_n,u) \to 0$ as $n \to \infty$. Indeed, any $u \in \mathrm{BV}_L(0,T;\mathcal{H})$ can be approximated in the strict metric by a sequence in $C^\infty([0,T];\mathcal{H})$ (cf. [22], Proposition A.2, for the vector case).

Thus if $u_n \in C^\infty([0,T];\mathcal{H})$ and $u_n \to u$ strictly on $[0,T]$, it makes sense to consider the sequence $\mathsf{S}(u_n)$ of the stop operator applied to the sequence u_n of regular inputs. The question is if the outputs $\mathsf{S}(u_n)$ converge to some function $x\colon [0,T] \longrightarrow \mathcal{H}$ in a reasonable topology and, in this case, the limit x is another candidate for the definition of the stop operator for BV data. First we observe that the classical stop operator enjoys an even stronger continuity property proved in [22], Theorem 3.7.

Theorem 4.1. *The operator* $\mathsf{S}\colon W^{1,1}(0,T;\mathcal{H}) \longrightarrow W^{1,1}(0,T;\mathcal{H})$ *is continuous if* $W^{1,1}(0,T;\mathcal{H})$ *is endowed with the strict metric.*

The stop $\mathsf{S}\colon W^{1,1}(0,T;\mathcal{H}) \longrightarrow W^{1,1}(0,T;\mathcal{H})$ is always continuous with respect to the Sobolev norm in $W^{1,1}(0,T;\mathcal{H})$, see [9], Theorem I.3.12. It is interesting to note that the continuity with respect to the strict metric is a consequence of this fact for general rate independent operators. It was first proved in [26] for the scalar case and then in [22] for the vectorial case. Now we state the result obtained by this continuity method.

Theorem 4.2. *The stop operator* $\mathsf{S}\colon W^{1,1}(0,T;\mathcal{H}) \longrightarrow W^{1,1}(0,T;\mathcal{H})$ *admits a unique continuous extension to* $\mathrm{BV}_L(0,T;\mathcal{H})$ *in the following sense. If* $u \in \mathrm{BV}_L(0,T;\mathcal{H})$*,* $u_n \in W^{1,1}(0,T;\mathcal{H})$ *and* $u_n \to u$ *strictly on* $[0,T]$*, then there exists* $\mathsf{S}_V(u) := x \in \mathrm{BV}(0,T;\mathcal{H})$ *such that* $\mathsf{S}(u_n) \to x$ *in* $L^1(0,T;\mathcal{H})$*.*

The previous theorem can be found in [22], Theorem 3.2, Proposition 3.3, Theorem 3.7, and together with (4.1) enables to define an operator $\mathsf{S}_V\colon \mathrm{BV}_L(0,T;\mathcal{H}) \longrightarrow \mathrm{BV}_L(0,T;\mathcal{H})$ which extends $\mathsf{S}\colon W^{1,1}(0,T;\mathcal{H}) \longrightarrow W^{1,1}(0,T;\mathcal{H})$. Here the index V stands for “Variation”. By a diagonalization procedure it turns out that S_V is continuous when its domain is endowed with the strict metric and its codomain is endowed with the L^1-topology. Let us mention the fact that a similar continuity method can be exploited for continuous sweeping processes (see [21]).

5. BV solutions by filling in the jumps

The property of rate independence suggests a very natural way to extend the stop operator to BV functions by “filling in” the discontinuities with segments traversed at “infinite speed”. In order to do this, if $u \in \mathrm{BV}_L(0,T;\mathcal{H})$ is nonconstant, the following normalized arc length function $l_u\colon [0,T] \longrightarrow [0,T]$ is considered:

$$l_u(t) := \frac{T}{\mathrm{V}(u,[0,T])}\mathrm{V}(u,[0,t]), \quad t \in [0,T]. \tag{5.1}$$

Thus one can obtain a Lipschitz reparametrization $U\colon l_u([0,T]) \longrightarrow \mathbb{R}$ by this arc length, i.e., the unique U such that $u = U \circ l_u$. The classical stop operator S cannot be applied to U that is not defined on the whole $[0,T]$, thus we have to fill in the discontinuities in such a way that the extension of U remains of Lipschitz class. In one dimension this method is adopted in [24], [27] for a quite general class of rate independent operators (see also [4], where the jumps are filled in a different way). In higher dimensions this procedure turns out to be trajectory-dependent, since one has several choices to connect the jump points. A canonical way to do this is to fill in the jumps with segments which represent the shortest way. This analysis is performed in [22], [23]: The reparametrization of u obtained by such an extension of U is the unique Lipschitz function $\tilde{u}\colon [0,T] \longrightarrow \mathcal{H}$ such that $\|\tilde{u}'\|_\infty \leqslant \mathrm{V}(u,[0,T])/T$ and

$$u = \tilde{u} \circ l_u, \tag{5.2}$$

$$\tilde{u} \text{ is affine on } [l_u(t), l_u(t+)] \ \forall t \in [0,T]. \tag{5.3}$$

Hence a natural definition of the extension of the stop operator for a BV function is $\mathsf{S}_R\colon \mathrm{BV}_L(0,T;\mathcal{H}) \longrightarrow \mathrm{BV}_L(0,T;\mathcal{H})$ defined by

$$\mathsf{S}_R(u) := \mathsf{S}(\tilde{u}) \circ l_u, \quad u \in \mathrm{BV}_L([0,T];\mathcal{H}). \tag{5.4}$$

Here the notation S_R is used to remind the use of "Reparametrizations". It is easily seen that if $u \in W^{1,1}(0,T;\mathcal{H})$, then $\mathsf{S}_R(u) = \mathsf{S}(u)$.

6. Comparison of BV solutions

In this section we compare the various kinds of BV solutions described in the previous sections. In fact they reduce to two notions of a solution. Indeed, concerning S_P and S_K, in [10] it is shown that they are exactly the same operator.

Theorem 6.1. $\mathsf{S}_P = \mathsf{S}_K$.

Furthermore, in [22] it is proved that the other two operators are equal.

Theorem 6.2. $\mathsf{S}_V = \mathsf{S}_R$.

So, it remains to compare the two extensions S_K and S_R. A counterexample in [22] with $\mathcal{Z}$ a disc in $\mathbb{R}^2$ shows that in general $\mathsf{S}_K \neq \mathsf{S}_R$. Hence a very natural mathematical issue is to find necessary and sufficient conditions on $\mathcal{Z}$ such that these operators coincide. For the finite-dimensional case this analysis is performed in [13], where such a characterization is provided in terms of the shape of the nonempty convex constraint $\mathcal{Z}$. We can indeed reduce our analysis to the case that $0 \in \mathcal{Z}$. In order to state this result we first recall the following concept.

Definition 6.3. A set $\mathcal{P} \subseteq \mathcal{H}$ containing 0 is called a *(closed convex) polyhedron* if there exist $n_1, \ldots, n_p \in \partial B_1(0)$ and $c_1, \ldots, c_p \in [0, \infty)$ such that

$$\mathcal{P} = \{x \in \mathcal{H} \colon \langle n_j, x \rangle \leqslant c_j,\ j = 1, \ldots, p\}. \tag{6.1}$$

A polyhedron $\mathcal{P}$ of the form (6.1) is called *non-obtuse* if $\langle n_i, n_j \rangle \leqslant 0$ whenever $1 \leqslant i < j \leqslant p$.

In other words, a polyhedron is non-obtuse if all inner angles formed by its faces are smaller than or equal to $\pi/2$.

Here is the main result of [13], which we state in the following form.

Theorem 6.4. *Let* $\dim(\mathcal{H}) < \infty$. *Then* $\mathsf{S}_K = \mathsf{S}_R$ *if and only if* $\mathcal{Z}$ *is a non-obtuse polyhedron.*

Now we give a more geometric explicit description of non-obtuse polyhedra. For any subset $\mathcal{S} \subseteq \mathcal{H}$ we denote by $L(\mathcal{S})$ the linear space generated by $\mathcal{S}$, and by $\mathcal{S}^\perp := \{x \in \mathcal{H} \colon \langle x, s \rangle = 0\ \forall s \in \mathcal{S}\}$ its orthogonal complement. The following definition is important for the classification of non-obtuse polyhedra.

Definition 6.5. Let $\mathcal{T} \subseteq \mathcal{H}$ be a polyhedron of the form (6.1). We say that $\mathcal{T}$ is a *simplex in* $\mathcal{H}$ if $\dim(L(\{n_1, \ldots n_p\})) = p - 1$, and there exist $a_j > 0$ for every $j \in \{1, \ldots, p\}$, such that $\sum\limits_{j=1}^{p} a_j n_j = 0$.

From [13], Proposition 6.1, we infer the following theorem.

Theorem 6.6. *Let* $\mathcal{P} \subseteq \mathcal{H}$ *be a non-obtuse polyhedron given by* (6.1) *with* $p \geqslant 1$. *If the interior of* $\mathcal{P}$ *is nonempty, then one of the following two cases occurs:*

(a) *If* $\mathcal{H}_p := L(\{n_1, \ldots n_p\}) = \mathcal{H}$, *then there exist* $m \in \mathbb{N}$ *and* $\mathcal{H}_1, \ldots, \mathcal{H}_m$ *vector subspaces of* $\mathcal{H}$ *such that*

$$\mathcal{H} = \mathcal{H}_1 \times \ldots \times \mathcal{H}_m, \quad \mathcal{H}_i \text{ is orthogonal to } \mathcal{H}_j \text{ for } i \neq j,$$

and there exist polyhedra $\mathcal{P}_j \subseteq \mathcal{H}_j$ *such that, possibly after a permutation of indices, we have*

$$\begin{aligned}
&\mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_m,\\
&\mathcal{P}_m \text{ is a non-obtuse simplex if } \mathcal{P} \text{ is bounded},\\
&\mathcal{P}_m \text{ is a non-obtuse cone if } \mathcal{P} \text{ is unbounded},\\
&\mathcal{P}_1, \ldots, \mathcal{P}_{m-1} \text{ are non-obtuse simplices in } \mathcal{H}_1, \ldots, \mathcal{H}_{m-1},
\end{aligned}$$

this last condition holding for $m > 1$. *Here* $\mathcal{H}_j \neq \{0\}$ *and* $\mathcal{P}_j$ *has nonempty interior in* $\mathcal{H}_j$ *for each* j.

(b) *If* $\mathcal{H}_p := L(\{n_1, \dots n_p\}) \neq \mathcal{H}$, *then* $\mathcal{P}$ *is the "cylinder"* $\mathcal{P} = (\mathcal{P} \cap \mathcal{H}_p) + \mathcal{H}_p^{\perp}$ *and* $(\mathcal{P} \cap \mathcal{H}_p)$ *is a non-obtuse polyhedron with nonempty interior in the space* $\mathcal{H}_p$.

If $\mathcal{P}$ *has empty interior, then* $\mathcal{P}$ *is contained in a lower-dimensional subspace* $\widetilde{\mathcal{H}}$ *and either* $\mathcal{P} = \{0\}$ *or it is described by one of the cases* (a)–(b) *in the Hilbert space* $\widetilde{\mathcal{H}}$.

Simplices and cones are not decomposable. Now we present some pictures illustrating possible decompositions in $\mathcal{H} = \mathbb{R}^3$, classified according to the relation between the number p of faces and $d = \dim(\mathcal{H}) = 3$. We only show the pictures of non-obtuse polyhedra with nonempty interior in the situation (a), and we limit ourselves to some comments for the remaining cases, since these can be easily reduced to the case (a).

Let us start with Figure 1 showing a cone as the only case in (a) with $p = 3$.

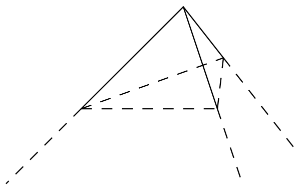

Figure 1. Cone.

For $p = 4$, we either have a non-decomposable simplex (tetrahedron) depicted in Figure 2, or a semi-infinite prism in Figure 3 as the product of a two-dimensional simplex (a triangle) and a one-dimensional cone (a half-line), or an infinite "cake segment" in Figure 4 as the product of a one-dimensional simplex (a segment) and a two-dimensional cone (an angle). Further orthogonal decomposition is possible into the product of a segment and two half-lines if the angle between the two half-lines is $\pi/2$, but this does not represent any new structure.

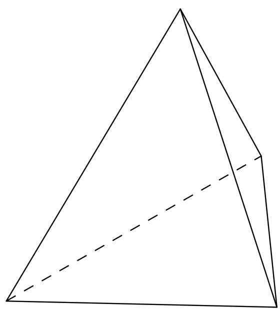

Figure 2. Simplex.

Now we deal with the case $p = 5$. In Figure 5 we have a prism, the product of a triangle and a segment.

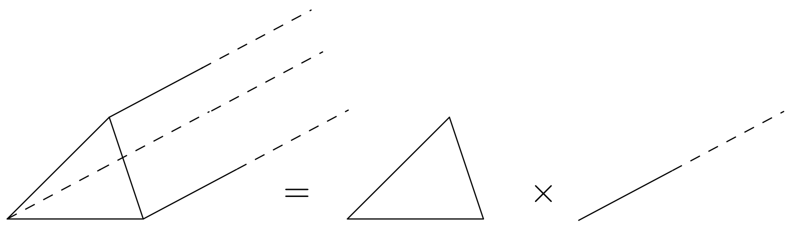

Figure 3. Semi-infinite prism.

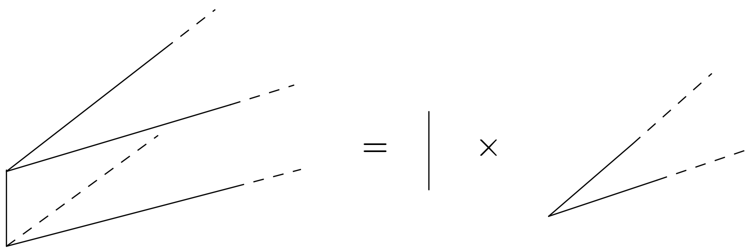

Figure 4. Infinite cake segment.

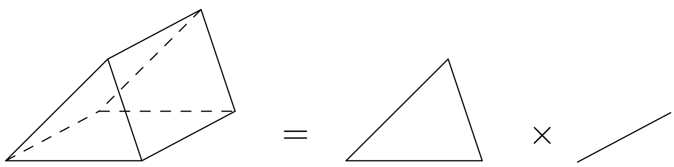

Figure 5. Prism.

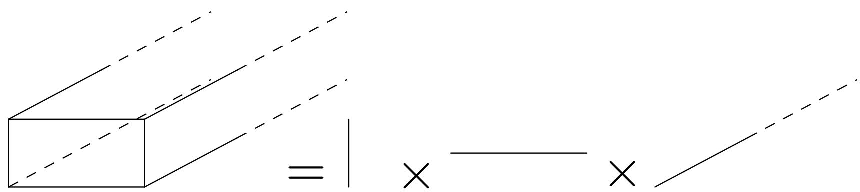

Figure 6. Infinite parallelepiped.

In Figure 6, we have a sort of semi-infinite parallelepiped, which is the product of two segments and a half-line, that is, two simplices in $\mathbb{R}$ and a one-dimensional cone.

Finally, for the case $p = 6$, we only have a parallepiped, which is the Cartesian product of three segments (see Figure 7).

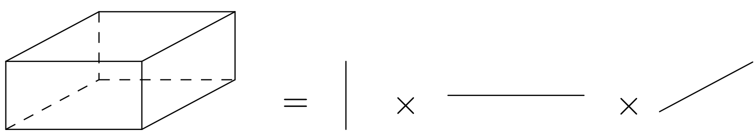

Figure 7. Parallelepiped.

The situations in (b) can be obtained by adding orthogonal dimensions to a lower-dimensional non-obtuse polyhedron. For instance, we have an infinite prism which is the product of a triangle and a line, and an infinite parallelepiped, being the product of a rectangle and a line. The remaining cases of (b) can be obtained in a similar way.

If the interior of $\mathcal{P}$ is empty, then we can reduce to a lower dimension by considering the affine hull of $\mathcal{P}$.

In more than three dimensions there are many more alternatives, for instance, the most special case for $\dim(\mathcal{H}) = 4$ is the orthogonal decomposition into two triangles.

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