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BOUNDEDNESS OF SOLUTIONS TO PARABOLIC-ELLIPTIC CHEMOTAXIS-GROWTH SYSTEMS WITH SIGNAL-DEPENDENT SENSITIVITY

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Abstract. This paper deals with parabolic-elliptic chemotaxis systems with the sensitivity function $\chi(v)$ and the growth term $f(u)$ under homogeneous Neumann boundary conditions in a smooth bounded domain. Here it is assumed that $0 < \chi(v) \leq \chi_0/v^k$ ($k \geq 1$, $\chi_0 > 0$) and $\lambda_1 - \mu_1 u \leq f(u) \leq \lambda_2 - \mu_2 u$ ($\lambda_1, \lambda_2, \mu_1, \mu_2 > 0$). It is shown that if $\chi_0$ is sufficiently small, then the system has a unique global-in-time classical solution that is uniformly bounded. This boundedness result is a generalization of a recent result by K. Fujie, M. Winkler, T. Yokota.

Keywords: chemotaxis; global existence; boundedness

MSC 2010: 35B40, 35K60

1. Introduction and main result

In this paper we consider the global existence and boundedness in the parabolic-elliptic chemotaxis-growth system

$$
\begin{align*}
&u_t = \Delta u - \nabla \cdot (u\chi(v)\nabla v) + f(u), \quad x \in \Omega, \ t > 0, \\
&0 = \Delta v - v + u, \quad x \in \Omega, \ t > 0, \\
&\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, \ t > 0, \\
&u(x, 0) = u_0(x), \quad x \in \Omega,
\end{align*}
$$

(1.1)

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ ($n \in \mathbb{N}$) with smooth boundary $\partial \Omega$. We assume

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that the initial data $u_0$ satisfies

\begin{equation}
(1.2) \quad u_0 \in C^0(\Omega), \quad u_0 \geq 0 \quad \text{and} \quad \int_\Omega u_0 > 0.
\end{equation}

As for the chemotactic sensitivity function, we assume that

\begin{equation}
(1.3) \quad \chi \in C^1((0, \infty)) \quad \text{with} \quad \chi > 0.
\end{equation}

Also we assume that $f \in C^1([0, \infty))$ and there exist $\lambda_1, \lambda_2, \mu_1, \mu_2 > 0$ such that

\begin{equation}
(1.4) \quad \lambda_1 - \mu_1 s \leq f(s) \leq \lambda_2 - \mu_2 s \quad \text{for all} \quad s \in [0, \infty).
\end{equation}

This system was introduced by Keller and Segel [6], [7] (see also [4], [14], [15]), and the mathematical study of this system has developed extensively. In this paper we especially focus on the signal-sensitivity function and the growth term. There are some known results related to this system in [1], [2], [8]–[13], [16]–[19]. The present work is devoted to the global existence and boundedness. We remark that the existence of classical solutions to (1.1) is shown by a similar way as in [3]. Since $f(0) \geq \lambda_1 > 0$ by (1.4), the solution to (1.1) is nonnegative.

In order to formulate our main result, given a nonnegative $0 \not\equiv u_0 \in C^0(\Omega)$, let us define a constant $\gamma > 0$ as

\begin{equation}
(1.5) \quad \gamma := \min \left\{ \|u_0\|_{L^1(\Omega)}, \frac{\lambda_1}{\mu_1} |\Omega| \right\} \int_0^\infty \frac{1}{(4\pi t)^{n/2}} e^{-\left(t+(\text{diam} \Omega)^2/(4t)\right)} dt < \infty,
\end{equation}

where $\text{diam} \Omega := \max_{x, y \in \Omega} |x - y|$. We remark that the integrand in (1.5) decays exponentially not only as $t \to \infty$ but also as $t \to 0$, and so $\gamma < \infty$ for all $n \in \mathbb{N}$. The constant $\gamma$ marks an a priori pointwise lower bound on the solution component $v$, as we shall see below. In what follows, when $k = 1$ we regard the value of $k^k/(k - 1)^{k-1}$ as 1.

**Theorem 1.1.** Let $n \in \mathbb{N}$, and suppose that $u_0, \chi$ and $f$ satisfy (1.2), (1.3) and (1.4), respectively. Moreover, assume that $\chi$ satisfies

$$\chi(s) \leq \frac{\chi_0}{s^k} \quad \text{for all} \quad s \in [\gamma, \infty),$$

with some $k \geq 1$ and some $\chi_0 > 0$ fulfilling

$$\chi_0 < \frac{2}{n} \frac{k^k}{(k - 1)^{k-1}} \gamma^{k-1}.$$
Then (1.1) possesses a unique global classical solution \((u, v)\) which satisfies
\[
\|u(\cdot, t)\|_{L^\infty} \leq M_\infty \quad \text{for all} \ t \in [0, \infty)
\]
with some constant \(M_\infty > 0\).

2. Preliminaries

We begin with the following lemma shown in [3]. This lemma is key to deriving a uniform-in-time estimate for \(v\).

**Lemma 2.1.** Let \(w \in C^0(\overline{\Omega})\) be a nonnegative function such that \(\int_\Omega w > 0\). If \(z\) is a weak solution to
\[
\begin{cases}
-\Delta z + z = w, & x \in \Omega, \\
\frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega,
\end{cases}
\]
then
\[
z \geq \left( \int_0^\infty \frac{1}{(4\pi t)^{n/2}} e^{-(t+(\text{diam} \Omega)^2/(4t))} \, dt \right) \int_\Omega w > 0 \quad \text{in} \ \Omega.
\]

Here we give an a priori pointwise lower bound on the solution component \(v\). The first equation in (1.1) and the condition (1.4) imply
\[
\frac{d}{dt} \int_\Omega u = \int_\Omega f(u) \geq \lambda_1 |\Omega| - \mu_1 \int_\Omega u.
\]
Integrating this inequality, we have
\[
\int_\Omega u \geq \frac{\lambda_1}{\mu_1} |\Omega| + e^{-\mu_1 t} \left( \|u_0\|_{L^1(\Omega)} - \frac{\lambda_1}{\mu_1} |\Omega| \right) \quad \text{for all} \ t \in (0, \infty),
\]
and then
\[
\int_\Omega u \geq \min \left\{ \|u_0\|_{L^1(\Omega)}, \frac{\lambda_1}{\mu_1} |\Omega| \right\}.
\]
By virtue of Lemma 2.1 we can thereby estimate \(v\) from below as follows:
\[
(2.1) \quad v(x, t) \geq \gamma
\]
for all \(x \in \Omega\) and \(t \in (0, T)\), whenever \((u, v)\) solves (1.1) in \(\Omega \times (0, T)\) for some \(T > 0\). Here \(\gamma > 0\) is a constant defined as (1.5).

**Remark 2.1.** The maximum principle yields the lower pointwise estimate for \(v(\cdot, t)\) for fixed \(t > 0\). On the other hand, Lemma 2.1 and the uniform-in-time estimate for mass imply the uniform estimate (2.1).

We next collect some known facts concerning the Neumann Laplacian in \(\Omega\). For the proof of (iii) see [5], Lemma 2.1.
Lemma 2.2. For \( r \in (1, \infty) \), let \( \Delta \) denote the realization of the Laplacian in \( L^r(\Omega) \) with domain \( \{ w \in W^{2,r}(\Omega) ; \partial w/\partial n = 0 \text{ on } \partial \Omega \} \). Then the operator \( -\Delta + 1 \) is sectorial and possesses closed fractional powers \( (-\Delta + 1)^\theta, \theta \in (0,1) \), with dense domain \( D((-\Delta + 1)^\theta) \). Moreover, the following statements hold:

(i) If \( m \in \{0,1\}, p \in [1,\infty] \) and \( q \in (1,\infty) \), then there exists a constant \( c_{m,p} > 0 \) such that for all \( w \in D((-\Delta + 1)^\theta) \),

\[
\|w\|_{W^{m,p}(\Omega)} \leq c_{m,p} \|(-\Delta + 1)^\theta w\|_{L^q(\Omega)},
\]

provided that \( m < 2\theta \) and \( m - n/p < 2\theta - n/q \).

(ii) Let \( p \in (1,\infty) \). Then there exist \( c > 0 \) and \( \nu_1 > 0 \) such that for all \( u \in L^p(\Omega) \) and any \( t > 0 \),

\[
\|(-\Delta + 1)^\theta e^{t(\Delta-1)}u\|_{L^p(\Omega)} \leq ct^{-\theta}e^{-\nu_1 t}\|u\|_{L^p(\Omega)}.
\]

(iii) Let \( p \in (1,\infty) \). Then there exists \( \nu_2 > 1 \) such that for \( \varepsilon > 0 \) there exists \( c_\varepsilon > 0 \) such that for all \( \mathbb{R}^n \)-valued \( z \in C_0^\infty(\Omega) \),

\[
\|(-\Delta + 1)^\theta e^{t(\Delta-1)}\nabla \cdot z\|_{L^p(\Omega)} \leq c_\varepsilon t^{-\theta - 1/2 - \varepsilon} e^{-\nu_2 t}\|z\|_{L^p(\Omega)}, \quad t > 0.
\]

Accordingly, for all \( t > 0 \) the operator \( (-\Delta + 1)^\theta e^{t\Delta} \nabla \cdot \) admits a unique extension to all of \( L^p(\Omega) \) which, again denoted by \( (-\Delta + 1)^\theta e^{t\Delta} \nabla \cdot \), satisfies the above estimate for all \( \mathbb{R}^n \)-valued \( z \in L^p(\Omega) \).

3. Proof of main result

We first deduce \( L^p \)-boundedness of solutions to (1.1). Next let us show that \( L^p \)-boundedness with sufficiently large \( p \) implies \( L^\infty \)-boundedness. Combining these results will prove our main theorem.

Lemma 3.1. Let \( p > 1 \), and suppose that \( (u,v) \) is a classical solution to (1.1) in \( \Omega \times (0,T) \) for some \( T > 0 \). Then there exist \( C_1, C_2 > 0 \) such that

\[
\frac{1}{p} \int_{\Omega} u^p \leq \frac{p(p-1)}{2} \int_{\Omega} u^{p-2} |\nabla u|^2 + \frac{p(p-1)}{2} \int_{\Omega} u^p \chi^2(v) |\nabla v|^2 
+ C_1 \int_{\Omega} u^p + C_2 \quad \text{for all } t \in (0,T).
\]
The condition (1.4) yields
\[ \phi_\Omega \]
\[ \left( s \phi C_0 \right. \]
\[ \text{some constants} \]
\[ \lambda_2 \int_\Omega u^{p-1} - \mu_2 \int_\Omega u^p \leq C_1 \int_\Omega u^p + C_2 \]
for some constants \( C_1, C_2 > 0 \), and hence we obtain the desired inequality. \( \square \)

The next lemma is obtained in [3]. For convenience we give the sketch of the proof.

**Lemma 3.2.** Let \( p > 1 \), and suppose that \((u, v)\) is a classical solution to (1.1) in \( \Omega \times (0, T) \) for some \( T > 0 \). Moreover, for \( \gamma > 0 \) given by (1.5) (see also (2.1)), let \( \varphi \in C^1([\gamma, \infty)) \) such that \( \varphi > 0 \) and there exists a constant \( M > 0 \) satisfying \( s \varphi(s) \leq M \) for all \( s \geq \gamma \). Let \( A \) and \( B \) be positive constants such that \( AB = p \).

Then
\[ \int_\Omega u^p \left( -\varphi'(v) - \frac{B^2}{2} \varphi^2(v) \right) |\nabla v|^2 \leq \frac{A^2}{2} \int_\Omega u^{p-2} |\nabla u|^2 + M \int_\Omega u^p \quad \text{for all } t \in (0, T). \]

**Sketch of the proof.** Multiplying the second equation in (1.1) by \( u^p \varphi(v) \) and using integration by parts, we see that
\[ - \int_\Omega u^p \varphi'(v) |\nabla v|^2 = p \int_\Omega u^{p-1} \varphi(v) \nabla u \cdot \nabla v + \int_\Omega u^p \varphi(v) v - \int_\Omega u^{p+1} \varphi(v). \]
Applying Young’s inequality completes the proof. \( \square \)

Now we give \( L^p \)-boundedness of solutions to (1.1).

**Proposition 3.3.** Suppose that \( n \in \mathbb{N} \), and that \( u_0, \chi \) and \( f \) satisfy (1.2), (1.3) and (1.4), respectively. Let \((u, v)\) be a classical solution to (1.1) in \( \Omega \times (0, T) \) for some \( T > 0 \). Moreover, let \( \gamma > 0 \) be as in (1.5) and (2.1). Suppose that there exist \( k \geq 1 \) and \( \chi_0 > 0 \) such that \( \chi(s) \leq \chi_0/s^k \) for all \( s \geq \gamma \). Then for any \( p \in [1, \chi_0^{-1} [k^k/(k-1)^{k-1}] \gamma^{k-1}] \) there exists a constant \( M_p > 0 \) fulfilling
\[ \| u(\cdot, t) \|_{L^p} \leq M_p \quad \text{for all } t \in [0, T). \]

**Proof.** Taking any \( p \in [1, \chi_0^{-1} [k^k/(k-1)^{k-1}] \gamma^{k-1}] \), we have \( \chi_0 < p^{-1} [k^k/(k-1)^{k-1}] \gamma^{k-1} \). Now we take \( \varepsilon > 0 \) and \( L > 0 \) such that
\[ \varepsilon < p(p-1), \quad L < \frac{k}{k-1} L \quad \text{and} \quad \chi_0 \leq \frac{1}{p} \sqrt{\frac{p(p-1) - \varepsilon}{p(p-1)}} \frac{k^k}{(k-1)^{k-1}} L^{k-1}. \]

643
Applying Lemma 3.2 to \( \varphi(s) := 1/(B^2(s - L)) \), \( A := \sqrt{p(p - 1) - \varepsilon} \) and \( B := p/\sqrt{p(p - 1) - \varepsilon} \), we infer that

\[
(3.1) \quad \int_{\Omega} u^p \left( -\varphi'(v) - \frac{B^2}{2} \varphi^2(v) \right) |\nabla v|^2 \leq \frac{p(p - 1) - \varepsilon}{2} \int_{\Omega} u^{p - 2} |\nabla u|^2 + M \int_{\Omega} u^p
\]

and

\[
(3.2) \quad \frac{p(p - 1)}{2} \chi^2(s) \leq -\varphi'(s) - \frac{B^2}{2} \varphi^2(s) \quad \text{for all } s \geq \gamma.
\]

Now by (3.2), we can combine (3.1) with Lemma 3.1 to see that

\[
(3.3) \quad \frac{d}{dt} \int_{\Omega} u^p \leq - \frac{p(p - 1)}{2} \int_{\Omega} u^{p - 2} |\nabla u|^2 + \frac{p(p - 1) - \varepsilon}{2} \int_{\Omega} u^{p - 2} |\nabla u|^2
\]

\[
= - \frac{\varepsilon}{2} \int_{\Omega} u^{p - 2} |\nabla u|^2 + (M + C_1) \int_{\Omega} u^p + C_2
\]

for all \( t \in (0, T) \). Since the first equation in (1.1) and the condition (1.4) yield

\[
\frac{d}{dt} \int_{\Omega} u = \int_{\Omega} f(u) \leq \lambda_2 |\Omega| - \mu_2 \int_{\Omega} u,
\]

we see that for all \( t \in (0, \infty) \),

\[
\int_{\Omega} u \leq \frac{\lambda_2}{\mu_2} |\Omega| + e^{-\mu_2 t} \left( \|u_0\|_{L^1(\Omega)} - \frac{\lambda_2}{\mu_2} |\Omega| \right) \leq \max \left\{ \|u_0\|_{L^1(\Omega)}, \frac{\lambda_2}{\mu_2} |\Omega| \right\}.
\]

By virtue of this estimate, proceeding similarly as in [3], Proposition 4.3, we can complete the proof from (3.3). \( \square \)

Next, assuming \( L^p \)-boundedness, we derive \( L^\infty \)-boundedness.

\textbf{Proposition 3.4.} Let \( n \in \mathbb{N} \), and assume that \( u_0, \chi \) and \( f \) satisfy (1.2), (1.3) and (1.4), respectively. Let \((u, v)\) be the classical solution to (1.1) in \( \Omega \times (0, T) \), and assume further that \( \chi \in L^\infty((\gamma, \infty)) \) with \( \gamma > 0 \) given by (1.5) (see also (2.1)). Then if there exist \( p > n/2 \) and a constant \( M_p > 0 \) such that \( \|u(\cdot, t)\|_{L^p} \leq M_p \) for all \( t \in (0, T) \), then there exists a constant \( M_\infty > 0 \) independent of \( T \) such that

\[
\|u(\cdot, t)\|_{L^\infty} \leq M_\infty \quad \text{for all } t \in (0, T).
\]
Proof. Let $p > n/2$. We may assume that $p < n$. We see from (1.4) that $f(s) + s \leq C(1 + s)$ for some $C > 0$. We can take $q > n$ so that $q > p$. Then we have

\begin{equation}
\|f(u) + u\|_{L^q(\Omega)} \leq C\|1 + u\|_{L^p(\Omega)}^{p/q} \|1 + u\|_{L^\infty(\Omega)}^{1-p/q} \\
\leq C_p\|1 + u\|_{L^\infty(\Omega)}^{1-p/q} \\
\leq C''_p + C''_p\|u\|_{L^\infty(\Omega)}^{1-p/q},
\end{equation}

where $C'_p, C''_p$ are some positive constants. Recalling the choice of $q$, we see that $1-p/q \in (0, 1)$. Moreover, we choose $q > n$ satisfying further that $1-(n-p)q/(np) > 0$, which enables us to pick $\lambda \in (1, \infty)$ fulfilling $1/\lambda < 1 - (n-p)q/(np)$. The elliptic regularity (\(\|\nabla v\|_{L^n/p/\infty(\Omega)} \leq k_p\|u\|_{L^p(\Omega)}\)) and Hölder’s inequality yield

\begin{equation}
\|u\chi(v)\nabla v\|_{L^n(\Omega)} \leq \|\chi\|_{L^\infty((\gamma, \infty))}\|\nabla v\|_{L^n/(\gamma/(\gamma-1))(\Omega)}\|u\|_{L^n/(\gamma/(\gamma-1))(\Omega)} \\
\leq \|\chi\|_{L^\infty((\gamma, \infty))}\|\nabla v\|_{L^n/(\gamma/(\gamma-1))(\Omega)}\|u\|_{L^n/(\gamma/(\gamma-1))(\Omega)} \\
\leq \|\chi\|_{L^\infty((\gamma, \infty))}\|\nabla v\|_{L^n/(\gamma/(\gamma-1))(\Omega)}^{1-\beta}k_pM_p\|u\|_{L^\infty(\Omega)}^{1-\beta}\|u\|_{L^\infty(\Omega)}^\beta \\
\leq K_p\|u\|_{L^n(\Omega)},
\end{equation}

where $\lambda' := \lambda/(\lambda - 1)$, for some $\beta \in (0, 1)$ and $K_p > 0$. Now let $t \in (0, T)$. Then we have

\[ u(\cdot, t) = e^{t(\Delta-1)}u_0 - \int_0^t e^{(t-s)(\Delta-1)}(\nabla \cdot (u(s)\chi(v(s)))\nabla v(s)) + (f(u(s)) + u(s)) \, ds. \]

Let $\theta \in (n/(2q), 1/2)$ and $\varepsilon \in (0, 1/2 - \theta)$. Using Lemma 2.2, we see that

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} + c_{0, \infty}c_0\varepsilon \int_0^t (t-s)^{-\theta} e^{-\nu_1(t-s)}\|f(u(s)) + u(s)\|_{L^q(\Omega)} \, ds \\
+ c_{0, \infty}c_0\varepsilon \int_0^t (t-s)^{-\theta-1/2-\varepsilon} e^{-\nu_2(t-s)}\|u(s)\chi(v(s))\nabla v(s)\|_{L^q(\Omega)} \, ds.
\]

Combining (3.4) and (3.5) with the above inequality implies the uniform estimate:

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq K_0 + K_1 \left( \sup_{t \in [0, T]} \|u(\cdot, t)\|_{L^\infty(\Omega)} \right)^\beta + K_2 \left( \sup_{t \in [0, T]} \|u(\cdot, t)\|_{L^\infty(\Omega)} \right)^{1-p/q}
\]

for some $K_0, K_1, K_2 > 0$. Since $\beta, 1-p/q \in (0, 1)$, we obtain the desired inequality.

\[ \square \]

We are now in a position to prove the main result.
Proof of Theorem 1.1. As stated in Section 1, by a similar way as in [3] we can show that there exist \( T_{\text{max}} \leq \infty \) (depending only on \( \|u_0\|_{L^\infty(\Omega)} \)) and exactly one pair \((u, v)\) of nonnegative functions \( u \in C^{2,1}(\Omega \times (0, T_{\text{max}})) \cap C^0((0, T_{\text{max}}); C^0(\Omega)) \), and \( v \in C^{2,0}(\Omega \times (0, T_{\text{max}})) \cap C^0((0, T_{\text{max}}); C^0(\Omega)) \) that solves (1.1) in the classical sense. According to the condition for \( k \) and \( \chi_0 \), by Proposition 3.3 we can find some \( p > n/2 \) and \( M_p > 0 \) such that \( \|u(\cdot, t)\|_{L^p} \leq M_p \) for all \( t \in (0, T_{\text{max}}) \). Therefore Proposition 3.4 completes the proof. □

Remark 3.1. The local-in-time existence of classical solutions to (1.1) can be provided under the only lower condition: \( \lambda_1 - \mu_1 s \leq f(s) \). Moreover, if the growth term \( f \) satisfies the relaxed condition: \( \lambda_1 - \mu_1 s \leq f(s) \leq \lambda_2 + \mu_2 s \), then we have the upper mass estimate depending on time \( t \) similarly, and so the global existence of solutions without uniform boundedness is proved.

References

[14] H. G. Othmer, A. Stevens: Aggregation, blowup, and collapse: The ABC’s of taxis in

[15] B. D. Sleeman, H. A. Levine: Partial differential equations of chemotaxis and angiogen-


[17] M. Winkler: Global solutions in a fully parabolic chemotaxis system with singular sen-

[18] M. Winkler: Absence of collapse in a parabolic chemotaxis system with signal-dependent

[19] M. Winkler: Boundedness in the higher-dimensional parabolic-parabolic chemotaxis sys-

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