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OSCILLATION OF THE THIRD ORDER EULER
DIFFERENTIAL EQUATION WITH DELAY

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Abstract. In the paper we offer criteria for oscillation of the third order Euler differential equation with delay

$$y'''(t) + \frac{k^2}{t^3}y(ct) = 0.$$

We provide detail analysis of the properties of this equation, we fill the gap in the oscillation theory and provide necessary and sufficient conditions for oscillation of equation considered.

Keywords: third-order functional differential equation; Euler equation; oscillation; nonoscillation

MSC 2010: 34K11, 34C10

1. INTRODUCTION

The object of this paper is to present sufficient conditions for the oscillation of the third-order functional differential equation

$$(E_D) \quad y'''(t) + \frac{k^2}{t^3}y(ct) = 0, \quad t \geq t_0 > 0,$$

where $0 < c < 1$ and $k \neq 0$. By a solution of (E_D) we mean a function defined on the initial interval $[ct_0, t_0]$ which satisfies (E_D) for every $t \geq t_0$. A solution of (E_D) is said to be oscillatory if it has arbitrarily large zeros, and otherwise it is called nonoscillatory. Equation (E_D) is said to be oscillatory if all its solutions are oscillatory.

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It follows from the familiar lemma of Kiguradze [5], [6], [10] that the set of all nonoscillatory (say, positive) solutions \mathcal{N} of (E_D) has the decomposition

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2,$$

where

$$\begin{aligned} y(t) \in \mathcal{N}_0 &\iff y(t) > 0, y'(t) < 0, y''(t) > 0, y'''(t) < 0, \\ y(t) \in \mathcal{N}_2 &\iff y(t) > 0, y'(t) > 0, y''(t) > 0, y'''(t) < 0. \end{aligned}$$

Equation (E_D) is a natural generalization of the well-known Euler differential equation

$$(E) \quad y'''(t) + \frac{k^2}{t^3}y(t) = 0$$

and will be called the Euler differential equation with delay. Following [3], [8], [9], we say that equation (E_D) has property (A) if $\mathcal{N} = \mathcal{N}_0$. Property (A) of various third order differential equations has been studied by many authors [5], [6], [8]–[10].

Both the equations (E) and (E_D) play a very important role in the oscillation theory, especially in the comparison theory, where these equations serve as comparative equations from which we deduce properties of more general equations, see e.g. [1]–[10]. Therefore it is desirable to have strong criteria for oscillation and/or property (A) of (E_D) . It is well known, see [4] and [6], that

$$\text{equation (E) has property (A)} \iff k > \frac{2}{3\sqrt{3}}.$$

Applying the existing comparison theorems, see [4] and [8], we can extend this result to equation (E_D) , as follows:

$$\begin{aligned} k^2 > \frac{2}{c^2 3\sqrt{3}} &\implies \text{equation } (E_D) \text{ has property (A),} \\ k^2 \leq \frac{2}{3\sqrt{3}} &\implies \text{equation } (E_D) \text{ has not property (A).} \end{aligned}$$

However, these results do not apply to the remaining case $2/(3\sqrt{3}) < k^2 \leq 2/(c^2 3\sqrt{3})$. On the other hand, these criteria say nothing about oscillation of (E_D) . In this paper, we will fill this gap and get an efficient necessary and sufficient condition for oscillation of (E_D) , and what is more, we provide also an interesting connection between oscillation of (E_D) and the classes \mathcal{N}_0 and \mathcal{N}_2 .

2. MAIN RESULTS

First we transform equation (E_D) into the form of a delay differential equation with constant coefficients and constant delay. We set

$$(2.1) \quad t = e^s, \quad y(t) = x(s), \quad r = -\ln c,$$

with $r > 0$. Then as usual

$$t \frac{dy}{dt} = \frac{dx}{ds}, \quad t^2 \frac{d^2y}{dt^2} = \frac{d^2x}{ds^2} - \frac{dx}{ds}, \quad t^3 \frac{d^3y}{dt^3} = \frac{d^3x}{ds^3} - 3 \frac{d^2x}{ds^2} + 2 \frac{dx}{ds}$$

and (E_D) becomes

$$(E_C) \quad \frac{d^3x}{ds^3} - 3 \frac{d^2x}{ds^2} + 2 \frac{dx}{ds} + k^2 x(s-r) = 0.$$

Obviously equation (E_D) oscillates if and only if equation (E_C) does. With equation (E_D) we associate the characteristic equation

$$(CH) \quad \lambda(\lambda-1)(\lambda-2) + k^2 c^\lambda \equiv \lambda(\lambda-1)(\lambda-2) + k^2 e^{-r\lambda} = 0$$

which is obtained by assuming the solution of (E_D) of the form $y(t) = t^\lambda$ or looking for the solution of (E_C) of the type $x(s) = e^{\lambda s}$.

In the next lemma which is due to Arino and Györi [1] we connect the properties of (E_C) and (CH).

Lemma 2.1. *Equation (E_C) oscillates if and only if (CH) has no real roots.*

Now we will explore the properties of the characteristic equation (CH). Let us denote $f(\lambda) = -\lambda(\lambda-1)(\lambda-2)c^{-\lambda}$, then the characteristic equation can be written in the form

$$(CH) \quad f(\lambda) = k^2.$$

It is easy to verify that for any $c \in (0, 1)$ the function $f(\lambda)$ has three zero points and moreover

$$\begin{aligned} f(\lambda) &\rightarrow -\infty && \text{for } \lambda \rightarrow \infty, \\ f(\lambda) &\rightarrow 0 && \text{for } \lambda \rightarrow -\infty. \end{aligned}$$

Consequently, we can conclude that there exists the maximum f_{\max} of $f(\lambda)$, $\lambda \in \mathbb{R}$. The information obtained permits us to sketch the graph of the function $f(\lambda)$ (see Figures 1 and 2).

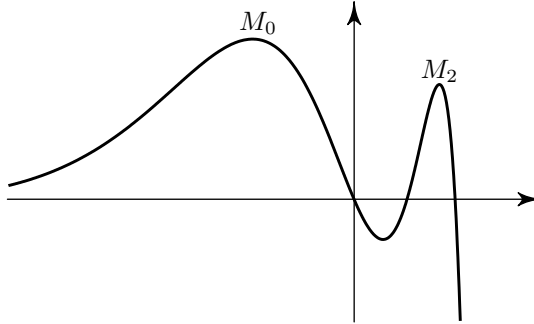


Figure 1. Graph of $f(\lambda)$ with $c = 0.35$.

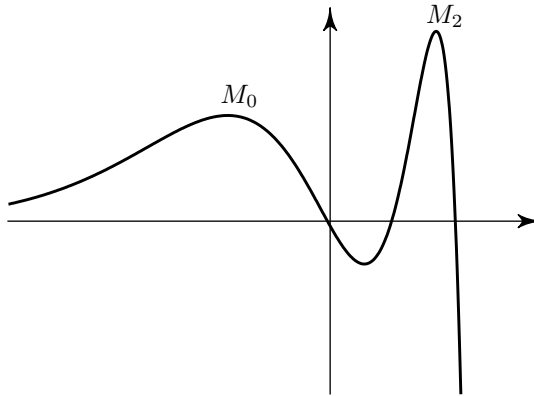


Figure 2. Graph of $f(\lambda)$ with $c = 0.27$.

Obviously, the characteristic equation (CH) has no real root if and only if holds $k^2 > f_{\max}$. Let us denote

$$M_0(c) = \max_{\lambda \in (-\infty, 0)} f(\lambda), \quad M_2(c) = \max_{\lambda \in (1, 2)} f(\lambda),$$

then in view of our previous observation the next result is obvious.

Theorem 2.1. Equation (E_D) is oscillatory iff $k^2 > \max\{M_0(c), M_2(c)\}$.

Remark 2.1. Evidently $M = \max\{M_0(c), M_2(c)\} \in (2/(3\sqrt{3}), 2/(c^2 3\sqrt{3}))$ and due to Theorem 2.1 if $k^2 > M$ equation (E_D) is oscillatory, while for $k^2 < M$ equation (E_D) possesses a nonoscillatory solution. Therefore there is no gap for oscillation of equation (E_D) .

Now we explore which of the local maxima $M_0(c)$ and $M_2(c)$ will be dominated and becomes maximum f_{\max} and how this process depends on value of parameter c . We consider $M_0(c)$ and $M_2(c)$ as functions defined on $(0, 1)$.

Lemma 2.2. *Function $M_0(c)$ is increasing on $(0, 1)$ and moreover,*

$$M_0(0) = \lim_{c \rightarrow 0^+} M_0(c) = 0, \quad M_0(1) = \lim_{c \rightarrow 1^-} M_0(c) = \infty.$$

Lemma 2.3. *Function $M_2(c)$ is decreasing on $(0, 1)$ and moreover,*

$$M_2(0) = \lim_{c \rightarrow 0^+} M_2(c) = \infty, \quad M_2(1) = \lim_{c \rightarrow 1^-} M_2(c) = \frac{2}{3\sqrt{3}}.$$

Proof. The proofs of both lemmas follow immediately from the definition of the functions $M_0(c)$ and $M_2(c)$ and so they can be omitted. \square

Combining the last two results, we obtain the following theoretical result dealing with the relationship between the dominance of $M_0(c)$ and $M_2(c)$.

Theorem 2.2. *There exists a unique $c^* \in (0, 1)$ such that*

- \triangleright for $c \in (0, c^*)$, the maximum is $f_{\max} = M_2(c) > M_0(c)$,
- \triangleright for $c \in (c^*, 1)$, the maximum is $f_{\max} = M_0(c) > M_2(c)$.

Now, we reformulate the previous results in terms of oscillation of equation (E_D) .

Theorem 2.3. *Let $c^* \in (0, 1)$ be such as in Theorem 2.2.*

- \triangleright If $c \in (0, c^*)$ and $k^2 > M_2(c)$, then equation (E_D) is oscillatory.
- \triangleright If $c \in (c^*, 1)$ and $k^2 > M_0(c)$, then equation (E_D) is oscillatory.
- \triangleright If $k^2 \leq M_2(c)$, then the class \mathcal{N}_2 is nonempty for equation (E_D) .
- \triangleright If $k^2 \leq M_0(c)$, then the class \mathcal{N}_0 is nonempty for equation (E_D) .

Proof. The first two assertions are obvious. We shall prove the third. If $k^2 \leq M_2(c) \leq f_{\max}$, then there exists a root λ_* of the characteristic equation such that $\lambda_* \in (1, 2)$. Then $y_*(t) = t^{\lambda_*}$ is the corresponding solution of equation (E_D) and it is easy to verify that $y_* \in \mathcal{N}_2$. The last assertion can be verified similarly. \square

It is desirable to evaluate the value of c^* . However, actual computation leads to a cubic equation whose roots are formalized as the third root of a complex number. Instead of this, using Matlab, we find out that $c^* = 0.32049$ with the corresponding $f_{\max} = M_0(c^*) = M_2(c^*) = 2.4735$.

We illustrate all our results in the following examples.

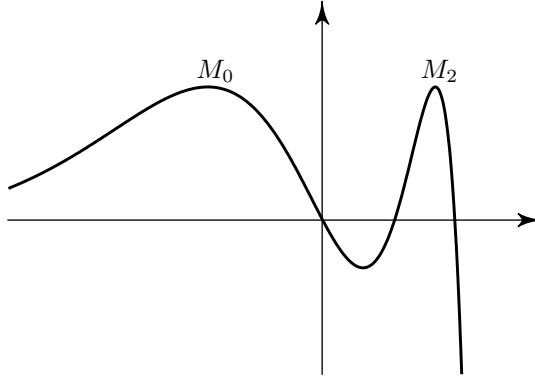


Figure 3. Graph of $f(\lambda)$ with $c^* = 0.32049$.

Example 2.1. Consider the third order Euler delay equation

$$(2.2) \quad y'''(t) + \frac{k^2}{t^3}y(0.4t) = 0, \quad t \geq t_0 > 0.$$

Since $0.4 > c^* = 0.32049$, then (using e.g. Matlab) we evaluate the corresponding values $M_0 = 3.9849$ and $M_2 = 1.7056$. Theorem 2.3 implies that

- ▷ equation (2.2) is oscillatory if and only if $k^2 > 3.9849$;
- ▷ if $k^2 \leq 3.9849$, then the class \mathcal{N}_0 is nonempty for equation (2.2);
- ▷ if $k^2 \leq 1.7056$, then the class \mathcal{N}_2 is nonempty for equation (2.2).

Note that for $k^2 = 3.84$ the solution $y(t) = t^{-2}$ belongs to the class \mathcal{N}_0 of equation (2.2).

Example 2.2. Consider the third order Euler delay equation

$$(2.3) \quad y'''(t) + \frac{k^2}{t^3}y(0.2t) = 0, \quad t \geq t_0 > 0.$$

Since $0.2 < c^* = 0.32049$, then we find out the corresponding values $M_2 = 5.5239$ and $M_0 = 1.2246$ and it follows from Theorem 2.3 that

- ▷ equation (2.3) is oscillatory if and only if $k^2 > 5.5239$;
- ▷ if $k^2 \leq 5.5239$, then the class \mathcal{N}_2 is nonempty for equation (2.3);
- ▷ if $k^2 \leq 1.2246$, then the class \mathcal{N}_0 is nonempty for equation (2.3).

Note that for $k^2 = 4.1926$ we have the solution $y(t) = t^{1.5}$ which belongs to the class \mathcal{N}_2 of equation (2.3), while for $k^2 = 1.2$ we have the solution $y_1(t) = t^{-1}$ which belongs to the class \mathcal{N}_0 . Moreover, there are additional three nonoscillatory solutions. One solution $y_2(t) \approx t^{-1.4009839}$ from the class \mathcal{N}_0 and two solutions $y_3(t) \approx t^{1.1846025}$, $y_4(t) \approx t^{1.973965}$ that belong to the class \mathcal{N}_2 .

3. SUMMARY

In this paper we have presented a new necessary and sufficient condition for oscillation of the third order Euler differential equation with delay. Since the Euler differential equation is often used in comparison results as a reference equation, the results obtained in this paper are useful and important for future investigation of asymptotic properties of differential equations.

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