Mathematica Bohemica

Luděk Nechvátal

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Mathematica Bohemica, Vol. 139 (2014), No. 4, 667--675

Persistent URL: `http://dml.cz/dmlcz/144143`

ON ASYMPTOTICS OF DISCRETE MITTAG-LEFFLER FUNCTION

Luděk Nechvátal, Brno

(Received September 30, 2013)

Abstract. The (modified) two-parametric Mittag-Leffler function plays an essential role in solving the so-called fractional differential equations. Its asymptotics is known (at least for a subset of its domain and special choices of the parameters). The aim of the paper is to introduce a discrete analogue of this function as a solution of a certain two-term linear fractional difference equation (involving both the Riemann-Liouville as well as the Caputo fractional h-difference operators) and describe its asymptotics. Here, we shall employ our recent results on stability and asymptotics of solutions to the mentioned equation.

Keywords: discrete Mittag-Leffler function; fractional difference equation; asymptotics; backward h-Laplace transform

MSC 2010: 33E12, 34A08, 39A12

1. Introduction

The classical Mittag-Leffler function is a special function generalizing the exponential function. Its two-parametric variant is given by the power series

$$E_{\alpha,\beta}(t) := \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+\beta)}, \quad t \in \mathbb{R} \text{ (or } \mathbb{C}), \ \alpha, \beta > 0. \tag{1.1}$$

This function plays a key role in the fractional calculus (theory of differentiation and integration of non-integer order). In particular, the modified Mittag-Leffler function

$$E^{\lambda}_{\alpha,\beta}(t) := t^{\beta-1} E_{\alpha,\beta}(\lambda t^{\alpha}) = \sum_{k=1}^{\infty} \lambda^k \frac{t^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)}, \quad \lambda \in \mathbb{C} \tag{1.2}$$

The research has been supported by the Grant P201/11/0768 of the Czech Science Foundation.

solves the fractional differential equation $D^\alpha x(t) = \lambda x(t)$, $t > 0$, $0 < \alpha < 1$, with $\beta = \alpha$ if D^α is the Riemann-Liouville fractional derivative and with $\beta = 1$ if D^α is the Caputo fractional derivative.

Considering any special function of this type, it is always worth to know its asymptotic behaviour as its definition form (usually via series or infinite products) does not provide such an information. In this connection, let us mention, e.g., the well-known case of Stirling's formula for the gamma function growth. Some asymptotic expansions for $E_{\alpha,\beta}$ were obtained by Wright in 1940, see [18], Theorem 1.4. As a consequence of these expansions it can be shown that

$$(1.3) \qquad E^\lambda_{\alpha,\alpha}(t) = O(t^{-1-\alpha}) \quad \text{and} \quad E^\lambda_{\alpha,1}(t) = O(t^{-\alpha}) \quad \text{as } t \to \infty$$

provided $|\arg(\lambda)| > \alpha\pi/2$. This fact was used in [15] and [19] in order to establish stability regions for the fractional equation (system of equations) of the above type.

Moving from the "continuous" world to a "discrete" one, some natural questions arise. How to introduce a discrete analogue of (1.2)? Can such a function retain the same qualitative properties as in (1.3)? The aim of the paper is to answer these questions. More precisely, inspired by (1.2), we introduce the discrete Mittag-Leffler function $E^{\lambda,h}_{\alpha,\beta}(t_n)$ as a solution of the two-term fractional difference equation $\nabla^\alpha_h y(t_n) = \lambda y(t_n)$, $0 < \alpha < 1$, $\lambda \in \mathbb{C}$, considered on a uniform mesh of points t_n (with distance h). The symbol ∇^α_h represents either the Riemann-Liouville or Caputo fractional h-difference operator. Then, we are going to derive an asymptotic description of $E^{\lambda,h}_{\alpha,\beta}$ for $\beta = \alpha$ and $\beta = 1$. To this end we shall exploit some of our recent results [9]. This asymptotic result seems to be new.

The foundations of fractional difference calculus (for the case of $h = 1$) were laid down in the papers by Miller and Ross [16] (for the case of forward sums and differences) and by Gray and Zhang [14] (for the case of backward sums and differences). Since then quite a number of contributions to the topic have appeared including also an analysis of fractional difference equations, see, e.g., [1], [2], [5], [6], [11] and the references therein. The form of the mentioned discrete Mittag-Leffler function also has been observed and analysed several times (for the first time by Nagai in [17], including the case of q-calculus as well). In this connection we refer to [3], [4] and [10].

The paper is organized as follows. To keep the text self-contained, Section 2 recalls basic definitions and notions related to fractional differentiation. It also discusses the backward h-Laplace transform and its properties as a main tool for our analysis. In Section 3 we shall discuss the solvability of the testing two-term equation under a special choice of the initial conditions, while Section 4 is devoted to the discrete Mittag-Leffler function and to the description of its asymptotic. Additional remarks in Section 5 close the paper.

2. Preliminaries

A basic domain for our considerations is formed by the equidistant points $t_n := hn$, $n = 1, 2, \ldots$, $h > 0$. This set (time scale) is denoted by $h\mathbb{N}$ (we shall also need the set $h\mathbb{N}_0 := h\mathbb{N} \cup \{0\}$). Considering a function $f\colon h\mathbb{N} \to \mathbb{C}$, the backward fractional h-sum is introduced as

$$\nabla_h^{-\mu} f(t_n) := \frac{h}{\Gamma_h(\mu)} \sum_{k=1}^{n} (t_{n-k+1})_h^{(\mu-1)} f(t_k), \quad n = 1, 2, \ldots, \ \mu > 0, \tag{2.1}$$

where $(t_n)_h^{(\mu)} := \Gamma_h(t_n + \mu h)/\Gamma_h(t_n)$, $\mu \in \mathbb{R}$, is a discrete analogue of the power function t^μ and Γ_h denotes the h-gamma function defined by $\Gamma_h(x) := \lim\limits_{k\to\infty} k! h^k (kh)^{x/h-1} [x(x+h)\ldots(x+(k-1)h)]^{-1}$, $x \in \mathbb{Z} \setminus \{0, -h, -2h, \ldots\}$. The key property of the standard Euler gamma function $\Gamma(x+1) = x\Gamma(x)$ is replaced by $\Gamma_h(x+h) = x\Gamma_h(x)$ and we have $\Gamma_h(x) = h^{x-1}\Gamma(x/h)$, for details see, e.g., [12].

The definition (2.1) originates from an extension of the right-hand side of the backward h-Cauchy formula for repeated summation (for details, see [10], Proposition 2.1). Note also that this definition is not quite identical with that originally introduced in [14] (for $h = 1$), where also the point $t_0 = 0$ has been involved in the summation. However, we prefer to stay with (2.1) as it conforms well to the general theory of nabla integration on time scales, see [8], Section 8.4. Here we also mention that lately the backward fractional difference calculus has started to be preferred over the forward one as it does not "shift" the domain of differentiated functions, for details see, e.g., [1] and [11].

Then, analogously to the continuous case, the Riemann-Liouville backward fractional h-difference of order $0 < \alpha < 1$ is defined by ${}_{RL}\nabla_h^\alpha f(t_n) := \nabla_h \nabla_h^{-(1-\alpha)} f(t_n)$, $n = 2, 3, \ldots$ and the Caputo backward fractional h-difference of order $0 < \alpha < 1$ is defined by ${}_C\nabla_h^\alpha f(t_n) := \nabla_h^{-(1-\alpha)} \nabla_h f(t_n)$, $n = 1, 2, \ldots$ provided f is defined on $h\mathbb{N}_0$. Here, ∇_h is the standard backward h-difference, i.e. $\nabla_h f(t_n) := h^{-1}(f(t_n) - f(t_{n-1}))$. Expanding these definitions we can easily verify that both definitions are related as ${}_C\nabla_h^\alpha f(t_n) = {}_{RL}\nabla_h^\alpha f(t_n) - f(0)(t_n)_h^{(-\alpha)}/\Gamma(1-\alpha)$, $n = 2, 3, \ldots$ For a discussion on properties and relations between the Riemann-Liouville and Caputo fractional differences we refer to [1].

For convenience, we introduce the function $M_h^{(\mu)}(t_n) := (t_n)_h^{(\mu)}/\Gamma(\mu+1)$, $n = 1, 2, \ldots$, as an analogue to the Taylor monomial $t^\mu/\Gamma(\mu+1)$.

The convolution product of functions $f, g\colon h\mathbb{N} \to \mathbb{C}$ is given by $(f * g)(t_n) := h \sum\limits_{k=1}^{n} f(t_{n-k+1}) g(t_k)$, $n = 1, 2, \ldots$

Following the approach presented in [7], the backward h-Laplace transform of a function $f\colon h\mathbb{N} \to \mathbb{C}$ is introduced as $\mathcal{L}_h\{f\}(s) := h \sum\limits_{k=1}^{\infty} f(t_k)(1 - hs)^{k-1}$ for all the points $s \in \mathbb{C}$, where the series is convergent. The backward h-Laplace transform is given by a power series with centre at $s_0 = h^{-1}$, i.e., if the series converges at $s \neq h^{-1}$, then there exists $r > 0$ such that the series converges locally uniformly and absolutely in the open disk $D(h^{-1}, r) := \{s \in \mathbb{C} \colon |s - h^{-1}| < r\}$. Moreover, the series expansion is determined uniquely (i.e. two distinct functions on $h\mathbb{N}$ cannot have the same Laplace image) and $\mathcal{L}_h\{f\}(s)$ is an analytic function on $D(h^{-1}, r)$. We shall use the following properties (for a guide to the proofs, see [9]).

Lemma 2.1. *Let $f, g \colon h\mathbb{N} \to \mathbb{C}$ be functions such that $\mathcal{L}_h\{f\}(s)$ and $\mathcal{L}_h\{g\}(s)$ converge on $D(h^{-1}, r_f)$ and $D(h^{-1}, r_g)$, respectively. Then*

(i) $\mathcal{L}_h\{(f * g)\}(s) = \mathcal{L}_h\{f\}(s) \cdot \mathcal{L}_h\{g\}(s)$ *on* $D(h^{-1}, r_*)$, *where* $r_* = \min\{r_f, r_g\}$.

(ii) *If* $\mu \in \mathbb{R}$, *then* $\mathcal{L}_h\{M_h^{\mu}\}(s) = s^{-\mu-1}$ *on* $D(h^{-1}, h^{-1})$.

(iii) *If* $0 < \alpha < 1$, *then* $\mathcal{L}_h\{{}_{RL}\nabla_h^{\alpha} f\}(s) = s^{\alpha}\mathcal{L}_h\{f\}(s) - \nabla_h^{-(1-\alpha)} f(0)$ *on* $D(h^{-1}, h^{-1}) \cap D(h^{-1}, r_f)$ *provided* $\nabla_h^{-(1-\alpha)} f(0)$ *is defined (see the comments in the following section).*

(iv) *If* $0 < \alpha < 1$ *and* f *is defined on* $h\mathbb{N}_0$, *then* $\mathcal{L}_h\{{}_{C}\nabla_h^{\alpha} f\}(s) = s^{\alpha}\mathcal{L}_h\{f\}(s) - s^{\alpha-1} f(0)$ *on* $D(h^{-1}, h^{-1}) \cap D(h^{-1}, r_f)$.

3. Two-term linear fractional difference equation

First, let us consider a linear fractional difference equation with the Riemann-Liouville h-difference operator in the form

$$ {}_{RL}\nabla_h^{\alpha} y(t_n) = \lambda y(t_n), \quad n = 1, 2, \ldots, \ 0 < \alpha < 1, \ \lambda \in \mathbb{C}. \tag{3.1} $$

Note that if $n = 1$, the definition of ${}_{RL}\nabla_h^{\alpha} f(t_n)$ requires the value $\nabla_h^{-(1-\alpha)} f(0)$ which is not introduced by (2.1). For this case, it seems to be convenient to involve an initial condition. Expanding ${}_{RL}\nabla_h^{\alpha} y(t_1)$ in (3.1), we (formally) get $\nabla_h^{-(1-\alpha)} y(t_1) - \nabla_h^{-(1-\alpha)} y(0) = h\lambda y(t_1)$. Prescribing a value to the symbol $\nabla_h^{-(1-\alpha)} y(0)$, i.e.,

$$ \nabla_h^{-(1-\alpha)} y(0) = \xi, \quad \xi \in \mathbb{R}, \tag{3.2} $$

we arrive at a one-to-one mapping (under the assumption $h^{\alpha}\lambda \neq 1$) between the values ξ and $y(t_1)$

$$ y(t_1) = \xi h^{\alpha-1}(1 - h^{\alpha}\lambda)^{-1} \tag{3.3} $$

(for details, see [9], Section 2). The condition (3.2) can be viewed as a discrete analogue of $D^{-(1-\alpha)} y(0+) = \xi$ from the continuous (differential) case, where any non-zero choice of ξ yields an unbounded solution in a right neighbourhood of 0. Overall, instead of (3.1) starting with $n = 2$ and equipped with the initial condition $y(t_1) = \xi_1$, we shall write our IVP in the form (3.1)–(3.2) taking into account (3.3).

The solvability is based on the fact that (3.1)–(3.2) can be rewritten into the equivalent form $y(t_{n+1}) = (1 - h^\alpha\lambda)^{-1} \sum\limits_{k=1}^{n} (-1)^{n-k} \binom{\alpha}{n-k+1} y(t_k)$, where the starting value $y(t_1)$ is given by (3.3) and $\binom{r}{s} = \Gamma(r+1)/(\Gamma(s+1)\Gamma(r-s+1))$, for details, see [9], Proof of Proposition 4. This equation is referred as the Volterra difference equation of convolution type and it is well known that it possesses a unique solution, see [13], Section 6.3.

Now, let us turn our attention to the Caputo case. In accordance with (3.1) we are going to study an equation in the form

$$ {}_C\nabla_h^\alpha y(t_n) = \lambda y(t_n), \quad n = 1, 2, \ldots, \ 0 < \alpha < 1, \ \lambda \in \mathbb{C} \tag{3.4} $$

equipped with the initial condition

$$ y(0) = \eta, \quad \eta \in \mathbb{R}. \tag{3.5} $$

Similarly to the Riemann-Liouville case, (3.4)–(3.5) is equivalent to the following non-homogeneous Volterra difference equation of the convolution type $y(t_{n+1}) = (1-h^\alpha\lambda)^{-1}\Big[\sum\limits_{k=1}^{n} (-1)^{n-k}\binom{\alpha}{n-k+1} y(t_k) + (-1)^n \binom{\alpha-1}{n}\eta\Big]$, where the starting value $y(t_1)$ is given by $y(t_1) = \eta(1-h^\alpha\lambda)^{-1}$ and again, a unique solution follows from the theory of Volterra difference equations.

To summarise the above considerations we can state

Proposition 3.1. *Let $h^\alpha\lambda \neq 1$. Then* (3.1)–(3.2) *and* (3.4)–(3.5) *have a unique solution.*

4. Discrete Mittag-Leffler function and its asymptotics

In view of (1.2), we propose the discrete Mittag-Leffler function $E_{\alpha,\beta}^{\lambda,h}\colon h\mathbb{N} \to \mathbb{C}$ as

$$ E_{\alpha,\beta}^{\lambda,h}(t_n) := \sum_{k=0}^{\infty} \lambda^k \frac{(t_n)_h^{(\alpha k+\beta-1)}}{\Gamma(\alpha k+\beta)}, \quad \lambda \in D(0, h^{-\alpha}), \ \alpha, \beta > 0. \tag{4.1} $$

The restriction $\lambda \in D(0, h^{-\alpha})$ ensures the convergence of the series. The main reason why we wish to discuss a discrete analogue to (1.2) and not to the original form (1.1), consists in the fact that the power law for the discrete power function does not hold, i.e., $(t_n)_h^{(\mu)} \cdot (t_n)_h^{(\nu)} \neq (t_n)_h^{(\mu+\nu)}$, hence, the form analogous to (1.1) does not seem to be convenient.

Let us show that $E_{\alpha,\alpha}^{\lambda,h}$ solves (3.1) and $E_{\alpha,1}^{\lambda,h}$ solves (3.4). Applying $\mathcal{L}_h$ on both sides of (3.1) and using Lemma 2.1 (iii), (3.2) and the linearity of $\mathcal{L}_h$, we obtain $s^\alpha \mathcal{L}_h\{y\}(s) - \xi = \lambda \mathcal{L}_h\{y\}(s)$ and thus $\mathcal{L}_h\{y\}(s) = \xi(s^\alpha - \lambda)^{-1}$. On the other hand,

$$\begin{aligned}
\mathcal{L}_h\{E_{\alpha,\beta}^{\lambda,h}\}(s) &= h\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\lambda^j M_h^{(\alpha j+\beta-1)}(t_k)(1-hs)^{k-1} \\
&= \sum_{j=0}^{\infty}\lambda^j h\sum_{k=1}^{\infty} M_h^{(\alpha j+\beta-1)}(t_k)(1-hs)^{k-1} = \sum_{j=0}^{\infty}\lambda^j \mathcal{L}_h\{M_h^{(\alpha j+\beta-1)}\}(s) \\
&= \sum_{j=0}^{\infty}\lambda^j s^{-\alpha j-\beta} = s^{-\beta}\sum_{j=0}^{\infty}(\lambda s^{-\alpha})^j = s^{\alpha-\beta}(s^\alpha-\lambda)^{-1},
\end{aligned}$$

where the last equality is true if λ lies in $D(0, |s|^\alpha)$. This restriction on $s \in \mathbb{C}$ together with the convergence region stated in Lemma 2.1 (ii) and in the definition of $E_{\alpha,\beta}^{\lambda,h}$ guarantee that $\mathcal{L}_h\{E_{\alpha,\beta}^{\lambda,h}\}(s)$ has a positive radius of convergence.

Applying $\mathcal{L}_h$ on (3.4) with help of Lemma 2.1 (iv) and (3.5), we obtain $s^\alpha \mathcal{L}_h\{y\}(s) - s^{\alpha-1}\eta = \lambda \mathcal{L}_h\{y\}(s)$, hence, $\mathcal{L}_h\{y\}(s) = \eta s^{\alpha-1}(s^\alpha - \lambda)^{-1}$. Setting $\beta = \alpha$ and $\beta = 1$ in $\mathcal{L}_h\{E_{\alpha,\beta}^{\lambda,h}\}(s)$ with respect to the uniqueness of backward h-Laplace transform and Proposition 3.1, we immediately see

Proposition 4.1. *Let $\lambda \in D(0, h^{-\alpha})$. Then the function $\xi E_{\alpha,\alpha}^{\lambda,h}(t_n)$ uniquely solves* (3.1)–(3.2) *and the function $\eta E_{\alpha,1}^{\lambda,h}(t_n)$ uniquely solves* (3.4)–(3.5).

Using Lemma 2.1 (i) and (ii) we also immediately get the relation $E_{\alpha,1}^{\lambda,h}(t_n) = (M_h^{(-\alpha)} * E_{\alpha,\alpha}^{\lambda,h})(t_n)$. Note that Proposition 3.1 claims the existence and uniqueness of the solutions for any $\lambda \in \mathbb{C}$ such that $h^\alpha\lambda \neq 1$, while in Proposition 4.1 we have a restriction on λ. To the author's knowledge, explicit forms of the solutions for $\lambda \notin D(0, h^{-\alpha})$ are not known yet.

Recently, in [9] we have derived the following stability and asymptotic result for (3.1). Denoting $\mathcal{S}_{\alpha,h} := \{z \in \mathbb{C}\colon\ |\arg(z)| > \alpha\pi/2 \text{ or } |z| > h^{-\alpha}(2\cos(\arg(z)/\alpha))^\alpha\}$, we have

Theorem 4.1. *Let $\lambda \in \mathcal{S}_{\alpha,h}$ and let $y(t_n)$ be the solution of* (3.1)–(3.2). *Then $|y(t_n)| \to 0$ as $n \to \infty$. Moreover, if $|1 - h^\alpha\lambda| > 1$, then $y(t_n) = O(n^{-1-\alpha})$.*

In other words, (3.1) is asymptotically stable on $\mathcal{S}_{\alpha,h}$. It even can be shown that $y(t_n) \in l^1(\mathbb{N})$. If $\lambda \notin \operatorname{cl}\mathcal{S}_{\alpha,h}$ ("cl" stands for the closure of a set), then (3.1) is not stable. The condition $|1 - h^\alpha\lambda| > 1$ from the second part means that $\lambda \notin \operatorname{cl} D(h^{-\alpha}, h^{-\alpha})$ and expresses a subset of $\mathcal{S}_{\alpha,h}$. In this case we even have $|y(t_n)| \sim Cn^{-1-\alpha}$ ("$\sim$" means the asymptotic equivalency and the constant C depends on ξ, α, λ).

Now, let us consider $E^{\lambda,h}_{\alpha,1}$ with $\lambda \in D(0, h^{-\alpha}) \setminus \operatorname{cl} D(h^{-\alpha}, h^{-\alpha})$. Then, using $M^{(\mu)}_h(t_n) \sim Cn^\mu$ (see [9] for details), we have

$$\begin{aligned}
|E^{\lambda,h}_{\alpha,1}(t_n)| &= |(M^{(-\alpha)}_h * E^{\lambda,h}_{\alpha,\alpha})(t_n)| \\
&\leqslant h\sum_{k=1}^{n} M^{(-\alpha)}_h(t_k)|E^{\lambda,h}_{\alpha,\alpha}(t_{n-k+1})| \leqslant C_1 \sum_{k=1}^{n} \frac{1}{k^\alpha}|E^{\lambda,h}_{\alpha,\alpha}(t_{n-k+1})| \\
&= C_1\Biggl(\sum_{k=1}^{\lfloor n/2\rfloor} \frac{1}{k^\alpha}|E^{\lambda,h}_{\alpha,\alpha}(t_{n-k+1})| + \sum_{k=\lfloor n/2\rfloor+1}^{n} \frac{1}{k^\alpha}|E^{\lambda,h}_{\alpha,\alpha}(t_{n-k+1})|\Biggr),
\end{aligned}$$

where $\lfloor x\rfloor$ denotes the floor function, i.e., the nearest integer smaller or equal to x. Since $E^{\lambda,h}_{\alpha,\alpha}$ belongs to $l^1(\mathbb{N})$, we have $E^{\lambda,h}_{\alpha,\alpha}(t_n) \leqslant Cn^{-1}$ and the first sum of the last line can be estimated as $\sum\limits_{k=1}^{\lfloor n/2\rfloor} k^{-\alpha}|E^{\lambda,h}_{\alpha,\alpha}(t_{n-k+1})| \leqslant C_2 n^{-1} \sum\limits_{k=1}^{\lfloor n/2\rfloor} k^{-\alpha} \leqslant C_3 n^{-\alpha}$, where we have used $\sum\limits_{k=2}^{n} 1/k^\alpha \leqslant \int_1^n 1/x^\alpha \,\mathrm{d}x$. The second sum is estimated as $\sum\limits_{k=\lfloor n/2\rfloor+1}^{n} k^{-\alpha}|E^{\lambda,h}_{\alpha,\alpha}(t_{n-k+1})| \leqslant C_4 n^{-\alpha} \sum\limits_{k=\lfloor n/2\rfloor+1}^{n} |E^{\lambda,h}_{\alpha,\alpha}(t_{n-k+1})| \leqslant C_5 n^{-\alpha}$. Consequently, by the above considerations we have

Corollary 4.1. *Let $0 < \alpha < 1$ and $\lambda \in D(0, h^{-\alpha}) \setminus \operatorname{cl} D(h^{-\alpha}, h^{-\alpha})$. Then $E^{\lambda,h}_{\alpha,\alpha}(t_n) = O(n^{-1-\alpha})$ and $E^{\lambda,h}_{\alpha,1}(t_n) = O(n^{-\alpha})$ as $n \to \infty$.*

It means that the answer to our second question from Introduction is positive. The decay rate of $E^{\lambda,h}_{\alpha,\beta}$ is the same as of $E^{\lambda}_{\alpha,\beta}$ (for the corresponding values of the parameters).

5. Final remarks

The form (1.2) represents a starting point to defining a discrete Mittag-Leffler function on any discrete time scale $\mathbb{T}$. If we are able to find an analogue $M^{(\mu)}_{\mathbb{T}}$ to the Taylor monomial $t^\mu/\Gamma(\mu+1)$ satisfying the key property $\nabla_{\mathbb{T}} M^{(\mu)}_{\mathbb{T}} = M^{(\mu-1)}_{\mathbb{T}}$ (the symbol $\nabla_{\mathbb{T}}$ denotes the backward nabla derivative on $\mathbb{T}$, see [8], Section 8.4), then also the backward fractional sum and difference of a function $f\colon \mathbb{T} \to \mathbb{C}$ can be defined analogously as in Section 2 and the discrete Mittag-Leffler will be introduced as $E^{\lambda,\mathbb{T}}_{\alpha,\beta}(t_n) := \sum\limits_{k=0}^{\infty} \lambda^k M^{(\alpha k+\beta-1)}_{\mathbb{T}}(t_n)$. However, finding an explicit form of the monomial

$M_{\mathbb{T}}^{(\mu)}$ is a difficult task and seems to be possible only in a few special cases (in [10] and [11] it has been done for a two-parametric time scale with a linear graininess).

The asymptotics of $E_{\alpha,\alpha}^{\lambda,h}$ (and consequently of $E_{\alpha,1}^{\lambda,h}$) has been derived by an indirect procedure using the qualitative properties of solutions to (3.1) stated by Theorem 4.1. This procedure does not seem to be applicable in a general case on any discrete time scale $\mathbb{T}$ (e.g. due to problems with an appropriate introduction of the backward Laplace transform on $\mathbb{T}$). Whether it is possible to use a direct approach similar to [18], Theorem 1.4, is an open question.

Note also that by Corollary 4.1 the asymptotics $E_{\alpha,\alpha}^{\lambda,h}(t_n) = O(n^{-1-\alpha})$ is valid on a smaller set than $D(0, h^{-\alpha}) \cap \mathcal{S}_{\alpha,h}$, where $E_{\alpha,\alpha}^{\lambda,h}$ also belongs to $l^1(\mathbb{N})$. One can guess that in this case the decay rate is still $O(n^{-1-\alpha})$, however, a verification of this hypothesis is an open problem.

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Author's address: *Luděk Nechvátal*, Faculty of Mechanical Engineering, Brno University of Technology, Technická 2, 616 69 Brno, Czech Republic, e-mail: `nechvatal@fme.vutbr.cz`.