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A GEOMETRIC IMPROVEMENT OF THE VELOCITY-PRESSURE LOCAL REGULARITY CRITERION FOR A SUITABLE WEAK SOLUTION TO THE NAVIER-STOKES EQUATIONS

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Abstract. We deal with a suitable weak solution $(\mathbf{v}, p)$ to the Navier-Stokes equations in a domain $\Omega \subset \mathbb{R}^3$. We refine the criterion for the local regularity of this solution at the point $(\mathbf{f}x_0, t_0)$, which uses the L^3-norm of $\mathbf{v}$ and the $L^{3/2}$-norm of p in a shrinking backward parabolic neighbourhood of $(\mathbf{x}_0, t_0)$. The refinement consists in the fact that only the values of $\mathbf{v}$, respectively p, in the exterior of a space-time paraboloid with vertex at $(\mathbf{x}_0, t_0)$, respectively in a "small" subset of this exterior, are considered. The consequence is that a singularity cannot appear at the point $(\mathbf{x}_0, t_0)$ if $\mathbf{v}$ and p are "smooth" outside the paraboloid.

Keywords: Navier-Stokes equation; suitable weak solution; regularity

MSC 2010: 35Q30, 76D03, 76D05

1. Introduction

Let Ω be a domain in $\mathbb{R}^3$ and $T > 0$. We deal with the Navier-Stokes system

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = \nabla p + \nu \Delta \mathbf{v} \tag{1.1}$$

$$\operatorname{div} \mathbf{v} = 0 \tag{1.2}$$

in $\Omega \times (0, T)$. The unknowns are $\mathbf{v} = (v_1, v_2, v_3)$ (the velocity) and p (the pressure). The coefficient of viscosity ν is supposed to be a positive constant.

The notion of a suitable weak solution to the system (1.1), (1.2) has been introduced in [1] and [11], the definitions and basic related results can also be found in

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papers [3]–[6] and others. Recall that the pair $(\mathbf{v}, p)$ is said to be a *suitable weak solution* of the system (1.1), (1.2) in $\Omega\times(0,T)$ if $\mathbf{v}$ is a weak solution, $p \in L^{5/4}(\Omega\times(0,T))$ is an associated pressure and the so called *generalized energy inequality*

$$2\nu \int_0^T\!\!\int_\Omega |\nabla\mathbf{v}|^2\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \leqslant \int_0^T\!\!\int_\Omega \big[|\mathbf{v}|^2(\partial_t\phi + \nu\Delta\phi) + (|\mathbf{v}|^2 + 2p)\mathbf{v}\cdot\nabla\phi\big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$

holds for every non-negative function ϕ from $C_0^\infty(\Omega\times(0,T))$. The existential theory for suitable weak solutions is developed in smooth domains in the case that the system (1.1), (1.2) is considered with the no-slip boundary condition $\mathbf{v} = \mathbf{0}$ on $\partial\Omega\times(0,T)$, see [3]. Regarding other boundary conditions, the theory of suitable weak solutions is so far less elaborated. If Ω is a general domain in $\mathbb{R}^3$ then, even with the no-slip boundary condition, the pressure associated with a weak solution $\mathbf{v}$ may exist only as a distribution (and not a function, see [14]). Thus, the suitable weak solution may not exist.

Following the definition from [1], we call the point $(\mathbf{x}_0, t_0) \in \Omega\times(0,T)$ a *regular point* of the suitable weak solution $(\mathbf{v}, p)$ if there exists a neighborhood U of $(\mathbf{x}_0, t_0)$ such that $\mathbf{v} \in \mathbf{L}^\infty(U)$.

There exist a series of criteria for regularity of the suitable weak solution $(\mathbf{v}, p)$ at the point $(\mathbf{x}_0, t_0)$, see, e.g., [1], [2], [6]–[8], [12], [13], [16] and others. Many of the criteria state that if some quantity is equal to zero or less than or equal to a certain sufficiently small constant $\varepsilon > 0$ (which is generally different in different criteria) then $(\mathbf{x}_0, t_0)$ is a regular point of solution $(\mathbf{v}, p)$. In this paper, we do not deal with the question of existence of a suitable weak solution—we assume from the beginning that a suitable weak solution $(\mathbf{v}, p)$ exists and we modify the criterion from [6], which uses the quantity $\delta^{-2}\int_{t_0-\delta^2}^{t_0}\int_{|\mathbf{x}-\mathbf{x}_0|<\delta}(|\mathbf{v}|^3 + |p|^{3/2})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t$. The modification consists in the reduction of the domains of the integral of $|\mathbf{v}|^3$ and the integral of $|p|^{3/2}$. The domains are subsets of the exterior of the space-time paraboloid P_a: $\sqrt{a(t_0-t)} = |\mathbf{x}-\mathbf{x}_0|$ with vertex at the point $(\mathbf{x}_0, t_0)$, where a is a certain positive parameter. We use no special assumptions on the behaviour of $\mathbf{v}$ or p in the interior of the paraboloid. This is in accordance with the approach introduced in [9] and [10], where $\mathbf{v}$ and p (paper [10]), respectively only $\mathbf{v}$ (paper [9]), have been supposed to satisfy the Serrin-type integrability conditions in some backward parabolic neighbourhood of $(\mathbf{x}_0, t_0)$, intersected with the exterior of paraboloid P_a.

Let $0 \leqslant \gamma_1 < \gamma_2$. We use the notation:

$$\begin{aligned}
\theta(t) &:= \sqrt{a(t_0-t)},\\
U_{a,\delta} &:= \{(\mathbf{x},t)\in\mathbb{R}^4;\ t_0-\delta^2 < t < t_0,\ \theta(t) < |\mathbf{x}-\mathbf{x}_0| < \sqrt{a}\delta\},\\
V_{a,\delta,\gamma_1,\gamma_2} &:= \{(\mathbf{x},t)\in\mathbb{R}^4;\ t_0-\delta^2 < t < t_0,\ \max\{\theta(t);\ \gamma_1\sqrt{a}\delta\} < |\mathbf{x}-\mathbf{x}_0| < \gamma_2\sqrt{a}\delta\}.
\end{aligned}$$

The shapes of the sets $U_{a,\delta}$ and $V_{a,\delta,\gamma_1,\gamma_2}$ are sketched in Figure 1. (The situation in Figure 1 corresponds to the case $\gamma_2 < 1$.) The main result of this paper says:

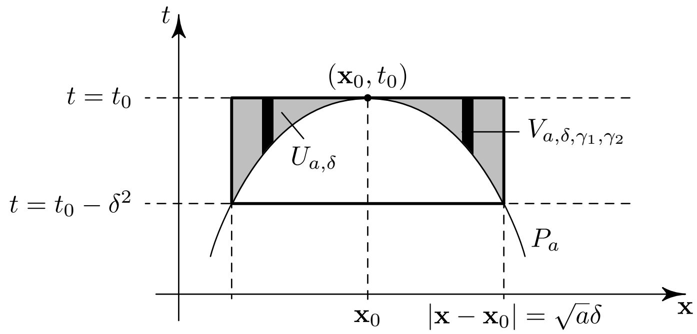

Figure 1.

Theorem 1.1. *Let* $(\mathbf{v}, p)$ *be a suitable weak solution of the system* (1.1), (1.2) *in* $\Omega \times (0,T)$, $(\mathbf{x}_0, t_0) \in \Omega \times (0,T)$, $0 < a < 3\nu(3\pi/2)^{2/3}$ *and* $0 \leqslant \gamma_1 < \gamma_2 \leqslant 1$. *There exists* $\varepsilon > 0$ *such that if*

$$\frac{1}{\delta^2} \iint_{U_{a,\delta}} |\mathbf{v}(\mathbf{x},t)|^3 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \leqslant \varepsilon \tag{1.3}$$

and

$$\frac{1}{\delta^2} \iint_{V_{a,\delta,\gamma_1,\gamma_2}} |p(\mathbf{x},t)|^{3/2} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \tag{1.4}$$

is bounded for all δ *in some interval* $(0,\delta_0)$ *(where* $\delta_0 > 0$*) then* $(\mathbf{x}_0, t_0)$ *is a regular point of the solution* $(\mathbf{v}, p)$.

2. The proof of Theorem 1.1

2.1. Introduction. We denote

$$G^I(\delta) := \frac{1}{\delta^2} \iint_{U_{a,\delta}} |\mathbf{v}|^3 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \frac{1}{\delta^2} \int_{t_0-\delta^2}^{t_0} \int_{\theta(t)<|\mathbf{x}-\mathbf{x}_0|<\sqrt{a}\delta} |\mathbf{v}|^3 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t,$$

$$G^{II}(\delta) := \frac{1}{\delta^2} \int_{t_0-\delta^2}^{t_0} \int_{|\mathbf{x}-\mathbf{x}_0|<\theta(t)} |\mathbf{v}|^3 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t,$$

and $G(\delta) := G^I(\delta) + G^{II}(\delta)$. The condition (1.3) implies that $\limsup_{\delta\to 0+} G^I(\delta) \leqslant \varepsilon$. We are going to prove that, provided that ε is sufficiently small and the conditions

(1.3), (1.4) hold, there exists a positive function f such that $f(\varepsilon) \to 0$ for $\varepsilon \to 0+$ and

$$\liminf_{\delta \to 0+} G^{II}(\delta) \leqslant f(\varepsilon). \tag{2.1}$$

Then $\liminf_{\delta\to 0+} G(\delta) \leqslant \varepsilon + f(\varepsilon)$, which implies that $(\mathbf{x}_0, t_0)$ is a regular point of the solution $(\mathbf{v}, p)$ by Wolf's regularity criterion, see [16].

Note that Wolf's criterion [16] assumes that $G(\delta)$ is "sufficiently small" for at least one $\delta > 0$. Also note that the criterion from [16] states that $\mathbf{v}$ is bounded and smooth only in a backward parabolic neighbourhood of point $(\mathbf{x}_0, t_0)$. However, using a standard localization procedure, one can show that $\mathbf{v}$ can be locally, in some neighbourhood of the point $\mathbf{x}_0$, extended as a smooth weak solution to some time interval $(t_0, t_0 + \Delta t)$. Applying the generalized energy inequality, one can also show that the extended solution coincides with the original solution $\mathbf{v}$ in the neighbourhood of $\mathbf{x}_0$. Thus, $\mathbf{v}$ is bounded in some neighbourhood (both backward and forward) of $(\mathbf{x}_0, t_0)$, see [8], pages 1395–1397, for more detailed explanation. Consequently, $(\mathbf{x}_0, t_0)$ is a regular point in the sense of the definition from [1].

Due to technical reasons, we use the additional assumption $\gamma_1 \leqslant 2$ in this section (see Subsection 2.6). However, the proof can also be carried out, with a small modification, for any $\gamma_1 \geqslant 0$.

2.2. Transformation to new coordinates. In order to estimate $G^{II}(\delta)$, we introduce new coordinates $\mathbf{x}'$ and t': we choose $\varrho \in (0, \sqrt{t_0})$ and put

$$\mathbf{x}' = \frac{\mathbf{x} - \mathbf{x}_0}{\theta(t)}, \quad t' = \int_{t_0 - \varrho^2}^{t} \frac{\mathrm{d}s}{\theta^2(s)} = \frac{1}{a} \ln \frac{\varrho^2}{t_0 - t}. \tag{2.2}$$

Then $t = t_0 - \varrho^2 \mathrm{e}^{-at'}$ and $\theta(t) = \sqrt{a}\varrho \mathrm{e}^{-at'/2}$. The interval $(t_0 - \varrho^2, t_0)$ on the t-axis now corresponds to the interval $(0, \infty)$ on the t'-axis and the interval $(t_0 - \delta^2, t_0)$ on the t-axis now corresponds to the interval (t'_δ, ∞) on the t'-axis, where

$$t'_\delta := \frac{2}{a} \ln \frac{\varrho}{\delta}. \tag{2.3}$$

Inverting this formula, we get

$$\delta = \varrho \mathrm{e}^{-at'_\delta/2}.$$

Obviously, $\delta \to 0+$ is equivalent to $t'_\delta \to \infty$. The equations (2.2) represent a one-to-one transformation of the region $\{(\mathbf{x}, t) \in \mathbb{R}^4;\ t_0 - \varrho^2 < t < t_0,\ |\mathbf{x} - \mathbf{x}_0| < \theta(t)\}$ in the interior of paraboloid P_a in the $\mathbf{x}$, t-space onto the infinite stripe $\{(\mathbf{x}', t') \in \mathbb{R}^4;$ $t' > 0$ and $|\mathbf{x}'| < 1\}$ in the $\mathbf{x}'$, t'-space. Similarly, (2.2) is a one-to-one transformation of the set $U_{a,\varrho}$ in the $\mathbf{x}$, t-space onto the set $\{(\mathbf{x}',t')\in\mathbb{R}^4;\ t'>0 \text{ and } 1<|\mathbf{x}'|<\mathrm{e}^{at'/2}\}$ in the $\mathbf{x}'$, t'-space. If we put

$$\mathbf{v}(\mathbf{x},t)=\frac{1}{\theta(t)}\mathbf{v}'\Big(\frac{\mathbf{x}-\mathbf{x}_0}{\theta(t)},\frac{1}{a}\ln\frac{\varrho^2}{t_0-t}\Big),$$
$$p(\mathbf{x},t)=\frac{1}{\theta^2(t)}p'\Big(\frac{\mathbf{x}-\mathbf{x}_0}{\theta(t)},\frac{1}{a}\ln\frac{\varrho^2}{t_0-t}\Big),$$

then the functions $\mathbf{v}'$, p' represent a suitable weak solution of the system of equations

$$\partial_{t'}\mathbf{v}'+\mathbf{v}'\cdot\nabla'\mathbf{v}'=-\nabla'p'+\nu\Delta'\mathbf{v}'-\frac{1}{2}a\mathbf{v}'-\frac{1}{2}a\mathbf{x}'\cdot\nabla'\mathbf{v}', \tag{2.4}$$

$$\operatorname{div}'\mathbf{v}'=0, \tag{2.5}$$

in any bounded sub-domain of $Q'_a:=\{(\mathbf{x}',t')\in\mathbb{R}^4;\ t'>0 \text{ and } |\mathbf{x}'|<\mathrm{e}^{at'/2}\}$. (The symbols ∇' and Δ' denote the nabla operator and the Laplace operator, acting in the spatial variable $\mathbf{x}'$.) As a suitable weak solution to the system (2.4), (2.5), $(\mathbf{v}',p')$ satisfies the generalized energy inequality

$$2\nu\int_{Q'_a}|\nabla'\mathbf{v}'|^2\phi\,\mathrm{d}\mathbf{x}'\,\mathrm{d}t'\leqslant\int_{Q'_a}\Big[|\mathbf{v}'|^2(\partial_{t'}\phi+\nu\Delta'\phi)+(|\mathbf{v}'|^2+2p')\mathbf{v}'\cdot\nabla'\phi+\frac{1}{2}a|\mathbf{v}'|^2\phi+\frac{1}{2}a(\mathbf{x}'\cdot\nabla'\phi)|\mathbf{v}'|^2\Big]\,\mathrm{d}\mathbf{x}'\,\mathrm{d}t' \tag{2.6}$$

for every non-negative function ϕ from $C_0^\infty(Q'_a)$. The inequality (2.6) can be modified by means of a special choice of the function ϕ: let, firstly, h be an infinitely differentiable non-increasing function in $[0,\infty)$ such that $h=1$ in $[0,1/4]$ and $h=0$ in $[1,\infty)$. We denote by $\dot{h}$ the derivative of h. Secondly, we choose $\mu\in(0,1/2)$ and put

$$\varphi(\mathbf{x}',t'):=\begin{cases}1 & \text{for } |\mathbf{x}'|\leqslant 1+\mu,\\ h\Big(\dfrac{|\mathbf{x}'|-1-\mu}{\mu}\mathrm{e}^{-a(t'-t'_\delta)/3}\Big) & \text{for } |\mathbf{x}'|>1+\mu.\end{cases}$$

Note that for each t', $\varphi(.,t')$ is supported in the closure of the set $M'(t'-t'_\delta)$, where $M'(\tau)$ denotes the ball with center at point $\mathbf{0}$ and radius $1+\mu+\mu\mathrm{e}^{a\tau/3}$. Finally, choosing $\phi(\mathbf{x}',t'):=\varphi^2(\mathbf{x}',t')\mathrm{e}^{-2a(t'-t'_\delta)/3}\mathcal{R}_{1/m}\chi(t')$, where χ is the characteristic function of the interval (t'_δ,t') and $\mathcal{R}_{1/m}$ is a one-dimensional mollifier with the kernel supported in $(-1/m, 1/m)$, and letting $m \to \infty$, we obtain

$$
\begin{aligned}
(2.7)\qquad & \|(\varphi\mathbf{v}')|_{t'}\|^2_{2;M'(t'-t'_\delta)}\mathrm{e}^{-2a(t'-t'_\delta)/3} \\
&+ \frac{a}{6}\int_{t'_\delta}^{t'} \|\varphi\mathbf{v}'\|^2_{2;M'(\tau-t'_\delta)}\mathrm{e}^{-2a(\tau-t'_\delta)/3}\,\mathrm{d}\tau \\
&+ 2\nu\int_{t'_\delta}^{t'} \|\nabla'(\varphi\mathbf{v}')\|^2_{2;M'(\tau-t'_\delta)}\mathrm{e}^{-2a(\tau-t'_\delta)/3}\,\mathrm{d}\tau \\
&\leqslant \|(\varphi\mathbf{v}')|_{t'_\delta}\|^2_{2;M'(0)} + \int_{t'_\delta}^{t'}\int_{M'(\tau-t'_\delta)} \big[2\nu|\nabla'\varphi|^2|\mathbf{v}'|^2 + 2\varphi(\partial_{t'}\varphi)|\mathbf{v}'|^2 \\
&+ (|\mathbf{v}'|^2+2p')(\mathbf{v}'\cdot\nabla'\varphi^2) + (a\mathbf{x}'\cdot\nabla'\varphi^2/2)|\mathbf{v}'|^2\big]\,\mathrm{d}\mathbf{x}'\mathrm{e}^{-2a(\tau-t'_\delta)/3}\,\mathrm{d}\tau.
\end{aligned}
$$

Note that $\|.\|_{2;M'(t'-t'_\delta)}$ denotes the norm in the space $\mathbf{L}^2(M'(t'-t'_\delta))$. Other norms are denoted by analogy. In order to derive (2.7), we have also used the identity $\varphi^2|\nabla'\mathbf{v}'|^2 = |\nabla'(\varphi\mathbf{v}')|^2 - |\nabla'\varphi^2||\mathbf{v}'|^2 - \nabla'\varphi^2\cdot\nabla'|\mathbf{v}'|^2/2$.

2.3. The first estimate of $G^{II}(\delta)$. Transforming $G^{II}(\delta)$ to the variables $\mathbf{x}'$, t', we get

$$
\begin{aligned}
(2.8)\qquad G^{II}(\delta) &= \frac{a\varrho^2}{\delta^2}\int_{t'_\delta}^{\infty} \|\mathbf{v}'\|^3_{3;B_1(\mathbf{0})}\mathrm{e}^{-at'}\,\mathrm{d}t' \\
&\leqslant \frac{a\varrho^2}{\delta^2}\int_{t'_\delta}^{\infty} \|\varphi\mathbf{v}'\|^3_{3;M'(t'-t'_\delta)}\mathrm{e}^{-at'}\,\mathrm{d}t' \\
&\leqslant \frac{a\varrho^2}{\delta^2}\int_{t'_\delta}^{\infty} \|\varphi\mathbf{v}'\|^{3/2}_{6;M'(t'-t'_\delta)}\|\varphi\mathbf{v}'\|^{3/2}_{2;M'(t'-t'_\delta)}\mathrm{e}^{-at'}\,\mathrm{d}t' \\
&\leqslant \frac{1}{3^{3/4}}\frac{2}{\pi}\frac{a\varrho^2}{\delta^2}\int_{t'_\delta}^{\infty} \|\nabla'(\varphi\mathbf{v}')\|^{3/2}_{2;M'(t'-t'_\delta)}\|\varphi\mathbf{v}'\|^{3/2}_{2;M'(t'-t'_\delta)}\mathrm{e}^{-at'}\,\mathrm{d}t' \\
&= \frac{1}{3^{3/4}}\frac{2}{\pi}a\int_{t'_\delta}^{\infty} \|\nabla'(\varphi\mathbf{v}')\|^{3/2}_{2;M'(t'-t'_\delta)}\|\varphi\mathbf{v}'\|^{3/2}_{2;M'(t'-t'_\delta)}\mathrm{e}^{-a(t'-t'_\delta)}\,\mathrm{d}t' \\
&\leqslant \frac{1}{3^{3/4}}\frac{2}{\pi}a\bigg(\int_{t'_\delta}^{\infty} \|\nabla'(\varphi\mathbf{v}')\|^2_{2;M'(t'-t'_\delta)}\mathrm{e}^{-2a(t'-t'_\delta)/3}\,\mathrm{d}t'\bigg)^{3/4} \\
&\quad\times\bigg(\int_{t'_\delta}^{\infty} \|\varphi\mathbf{v}'\|^6_{2;M'(t'-t'_\delta)}\mathrm{e}^{-2a(t'-t'_\delta)}\,\mathrm{d}t'\bigg)^{1/4}.
\end{aligned}
$$

The factor $3^{-3/4}2/\pi$ comes from Sobolev's inequality, see [15]. In order to estimate the integrals on the right hand side of (2.8), we use the inequality (2.7).

2.4. Notation. Let $\tau > 0$ and $\kappa := 2(\gamma_2/\gamma_1 - 1)$. We recall the definition of the set $M'(\tau)$ and define several other sets:

$$\begin{aligned} M'(\tau) &:= \left\{\mathbf{x}' \in \mathbb{R}^3;\ |\mathbf{x}'| < 1 + \mu + \mu\mathrm{e}^{a\tau/3}\right\},\\ B'_r &:= \left\{\mathbf{x}' \in \mathbb{R}^3;\ |\mathbf{x}'| < r\right\},\\ A'_0(\tau) &:= \left\{\mathbf{x}' \in \mathbb{R}^3;\ 1 + \mu < |\mathbf{x}'| < 1 + \mu + \mu\mathrm{e}^{a\tau/3}\right\},\\ A'_1(\tau) &:= \left\{\mathbf{x}' \in \mathbb{R}^3;\ 1 < |\mathbf{x}'| < (2+\kappa)\mathrm{e}^{a\tau/2}\right\},\\ A'_2(\tau) &:= \left\{\mathbf{x}' \in \mathbb{R}^3;\ 2\mathrm{e}^{a\tau/2} < |\mathbf{x}'| < (2+\kappa)\mathrm{e}^{a\tau/2}\right\}. \end{aligned}$$

Obviously, $M'(0) = B'_{1+2\mu}$. Except for τ, the sets $M'(\tau)$ and $A'_0(\tau)$ also depend on the parameter $\mu \in (0, 1/2)$. (This parameter will be later supposed to be "small enough", see (2.20).) Similarly, the sets $A'_1(\tau)$ and $A'_2(\tau)$ also depend on the parameter κ. The reason why κ is defined by the formula $\kappa := 2(\gamma_2/\gamma_1 - 1)$ is explained in Subsection 2.6.

We denote by C a generic constant, which may change its value from line to line. On the other hand, constants with indices preserve the same values throughout the whole paper.

2.5. First estimates of the integral on the right hand side of (2.7)**.** We denote

$$K(t'_\delta) := \int_{t'_\delta}^{\infty}\int_{A'_1(\tau - t'_\delta)} |\mathbf{v}'|^3 \mathrm{e}^{-a(\tau - t'_\delta)}\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\tau.$$

Transforming the integral to the $\mathbf{x}$, t-space and applying the condition (1.3), we obtain

$$\begin{aligned} K(t'_\delta) &= \frac{a}{\delta^2}\int_{t_0-\delta^2}^{t_0}\int_{\theta(t)<|\mathbf{x}-\mathbf{x}_0|<(2+\kappa)\sqrt{a}\delta} |\mathbf{v}|^3\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\leqslant a(2+\kappa)^2 G^I((2+\kappa)\delta) \leqslant a(2+\kappa)^2\varepsilon \end{aligned} \tag{2.9}$$

for $0 < \delta < \delta_0/(2+\kappa)$. The terms in the integral on the right hand side of (2.7) can now be successively estimated independently of t':

$$\begin{aligned} F_1(t'_\delta) &:= \int_{t'_\delta}^{t'}\int_{M'(\tau - t'_\delta)} 2\nu|\nabla'\varphi|^2|\mathbf{v}'|^2\,\mathrm{d}\mathbf{x}'\mathrm{e}^{-2a(\tau-t'_\delta)/3}\,\mathrm{d}\tau \\ &= \frac{C}{\mu^2}\int_{t'_\delta}^{t'}\int_{A'_0(\tau - t'_\delta)} 2\nu|\dot{h}|^2|\mathbf{v}'|^2\,\mathrm{d}\mathbf{x}'\mathrm{e}^{-4a(\tau-t'_\delta)/3}\,\mathrm{d}\tau \\ &\leqslant \frac{C}{\mu^2}\int_{t'_\delta}^{\infty}\left(\int_{A'_0(t'-t'_\delta)} |\mathbf{v}'|^3\,\mathrm{d}\mathbf{x}'\mathrm{e}^{-a(t'-t'_\delta)}\right)^{2/3}\mathrm{e}^{-a(t'-t'_\delta)/3}\,\mathrm{d}t' \end{aligned} \tag{2.10}$$

$$\leqslant C(\mu)K^{2/3}(t'_\delta)\Big(\int_{t'_\delta}^{\infty} \mathrm{e}^{-a(t'-t'_\delta)}\,\mathrm{d}t'\Big)^{1/3} \leqslant c_1(\mu,\kappa)\varepsilon^{2/3},$$

$$\begin{aligned}(2.11)\qquad &\int_{t'_\delta}^{t'}\int_{M'(\tau-t'_\delta)} \Big[2\varphi(\partial_{t'}\varphi)|\mathbf{v}'|^2 + \Big(\frac{a\mathbf{x}'\cdot\nabla'\varphi^2}{2}\Big)|\mathbf{v}'|^2\Big]\,\mathrm{d}\mathbf{x}'\mathrm{e}^{-2a(\tau-t'_\delta)/3}\,\mathrm{d}\tau \\ &= \frac{1}{\mu}\int_{t'_\delta}^{t'}\int_{A'_0(\tau-t'_\delta)} h\dot{h}\Big[\frac{2a(1+\mu)}{3}+\frac{a|\mathbf{x}'|}{3}\Big]|\mathbf{v}'|^2\,\mathrm{d}\mathbf{x}'\mathrm{e}^{-a(\tau-t'_\delta)}\,\mathrm{d}\tau \leqslant 0\end{aligned}$$

$$\begin{aligned}(2.12)\qquad F_2(t'_\delta) &:= \int_{t'_\delta}^{t'}\int_{M'(\tau-t'_\delta)} |\mathbf{v}'|^2(\mathbf{v}'\cdot\nabla'\varphi^2)\,\mathrm{d}\mathbf{x}'\mathrm{e}^{-2a(\tau-t'_\delta)/3}\,\mathrm{d}\tau \\ &\leqslant C(\mu)K(t'_\delta) \leqslant c_2(\mu,\kappa)\varepsilon,\end{aligned}$$

$$\begin{aligned}(2.13)\qquad F_3(t'_\delta) &:= \int_{t'_\delta}^{t'}\int_{M'(\tau-t'_\delta)} 2p'(\mathbf{v}'\cdot\nabla'\varphi^2)\,\mathrm{d}\mathbf{x}'\mathrm{e}^{-2a(\tau-t'_\delta)/3}\,\mathrm{d}\tau \\ &\leqslant C(\mu)K^{1/3}(t'_\delta)P^{2/3}(t'_\delta) \leqslant c_3(\mu,\kappa)\varepsilon^{1/3}P^{2/3}(t'_\delta),\end{aligned}$$

where

$$P(t'_\delta) := \int_{t'_\delta}^{\infty}\int_{A'_0(\tau-t'_\delta)} |p'|^{3/2}\mathrm{e}^{-a(\tau-t'_\delta)}\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\tau.$$

Note that the inequality in (2.11) holds because $h \geqslant 0$ and $\dot{h} \leqslant 0$. In order to estimate $P(t'_\delta)$, we use the next lemma.

Lemma 2.1. *Let $t'_\delta > 0$ and $t' > t'_\delta$. There exist constants $c_4 = c_4(\mu)$, $c_5 = c_5(\mu,\kappa)$ and $c_6 = c_6(\mu,\kappa)$ so that*

$$\begin{aligned}(2.14)\qquad &\int_{A'_0(t'-t'_\delta)} |p'(\mathbf{x}',t')|^{3/2}\,\mathrm{d}\mathbf{x}' \\ &\leqslant c_4\Big(\int_{B'_1}|\mathbf{v}'(\mathbf{x}',t')|^2\,\mathrm{d}\mathbf{x}'\Big)^{3/2} + c_5\int_{A_1(t'-t'_\delta)}|\mathbf{v}'(\mathbf{x}',t')|^3\,\mathrm{d}\mathbf{x}' \\ &\quad + c_6\mathrm{e}^{-a(t'-t'_\delta)/2}\int_{A'_2(t'-t'_\delta)}|p'(\mathbf{x}',t')|^{3/2}\,\mathrm{d}\mathbf{x}'.\end{aligned}$$

P r o o f. Let η be an infinitely differentiable cut-off function in $\mathbb{R}^3$ such that

$$\eta(\mathbf{x}',t')\begin{cases} = 1 & \text{for } |\mathbf{x}'| \leqslant 2\mathrm{e}^{a(t'-t'_\delta)/2},\\ \in [0,1] & \text{for } 2\mathrm{e}^{a(t'-t'_\delta)/2} \leqslant |\mathbf{x}'| \leqslant (2+\kappa)\mathrm{e}^{a(t'-t'_\delta)/2},\\ = 0 & \text{for } (2+\kappa)\mathrm{e}^{a(t'-t'_\delta)/2} \leqslant |\mathbf{x}'|,\end{cases}$$

and $|\nabla'\eta| \leqslant 2\kappa^{-1}\mathrm{e}^{-a(t'-t'_\delta)/2}$, $|\nabla'^2\eta| \leqslant 4\kappa^{-2}\mathrm{e}^{-a(t'-t'_\delta)}$. The function η can be split into the sum $\eta_1+\eta_2$, where both the functions η_1 and η_2 are from $C_0^\infty(\mathbb{R}^3)$, with values in $[0,1]$, and such that $\eta_1 = 1$ on B'_1 and $\eta_1 = 0$ on $\mathbb{R}^3 \setminus B'_{1+\mu/2}$. Thus, the function η_1 is supported in the closure of $B'_{1+\mu/2}$ and η_2 is supported in the closure of $A'_1(t'-t'_\delta)$. The function $\eta p'$ satisfies the identity

$$\eta(\mathbf{x}',t')p'(\mathbf{x}',t') = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|\mathbf{x}'-\mathbf{y}'|}[\Delta'(\eta p')](\mathbf{y}',t')\,\mathrm{d}\mathbf{y}'$$

for $\mathbf{x}' \in \mathbb{R}^3$. Using the equation $\Delta' p' = -\partial'_i\partial'_j(v'_iv'_j)$ and integrating by parts, we derive the formula

$$\eta(\mathbf{x}',t')p'(\mathbf{x}',t') = p'_1(\mathbf{x}',t') + p'_2(\mathbf{x}',t') + p'_3(\mathbf{x}',t'),$$

where

$$\begin{aligned}
p'_1(\mathbf{x}',t') &= \frac{1}{4\pi}\int_{B'_{1+\mu/2}} \frac{\partial^2}{\partial y'_i\partial y'_j}\Big(\frac{1}{|\mathbf{x}'-\mathbf{y}'|}\Big)[\eta_1 v'_iv'_j](\mathbf{y}',t')\,\mathrm{d}\mathbf{y}',\\
p'_2(\mathbf{x}',t') &= \frac{1}{4\pi}\int_{A'_1(t'-t'_\delta)} \frac{\partial^2}{\partial y'_i\partial y'_j}\Big(\frac{1}{|\mathbf{x}'-\mathbf{y}'|}\Big)[\eta_2 v'_iv'_j](\mathbf{y}',t')\,\mathrm{d}\mathbf{y}',\\
p'_3(\mathbf{x}',t') &= \frac{1}{2\pi}\int_{A'_2(t'-t'_\delta)} \frac{x'_i-y'_i}{|\mathbf{x}'-\mathbf{y}'|^3}\Big(\frac{\partial\eta}{\partial y'_j}v'_iv'_j\Big)(\mathbf{y}',t')\,\mathrm{d}\mathbf{y}'\\
&\quad + \frac{1}{4\pi}\int_{A'_2(t'-t'_\delta)} \frac{1}{|\mathbf{x}'-\mathbf{y}'|}\Big(\frac{\partial^2\eta}{\partial y'_i\partial y'_j}v'_iv'_j\Big)(\mathbf{y}',t')\,\mathrm{d}\mathbf{y}'\\
&\quad + \frac{1}{2\pi}\int_{A'_2(t'-t'_\delta)} \frac{x'_i-y'_i}{|\mathbf{x}'-\mathbf{y}'|^3}\Big(\frac{\partial\eta}{\partial y'_i}p'\Big)(\mathbf{y}',t')\,\mathrm{d}\mathbf{y}'\\
&\quad + \frac{1}{4\pi}\int_{A'_2(t'-t'_\delta)} \frac{1}{|\mathbf{x}'-\mathbf{y}'|}[\Delta'\eta p'](\mathbf{y}',t')\,\mathrm{d}\mathbf{y}'.
\end{aligned}$$

If $\mathbf{x}' \in A'_0(t'-t'_\delta)$ then the distance between $\mathbf{x}'$ and any $\mathbf{y}' \in B'_{1+\mu/2}$ is greater than $|\mathbf{x}'| - (1+\mu/2)$, which is further greater than $\mu/2$. Hence

$$\begin{aligned}
|p'_1(\mathbf{x}',t')| &\leqslant \frac{C}{\big[|\mathbf{x}'|-(1+\mu/2)\big]^3}\int_{B'_{1+\mu/2}} |\mathbf{v}'|^2\,\mathrm{d}\mathbf{y}'\\
&\leqslant C(\mu)\bigg[\int_{B'_1} |\mathbf{v}'|^2\,\mathrm{d}\mathbf{y}' + \Big(\int_{1<|\mathbf{y}'|<1+\mu/2} |\mathbf{v}'|^3\,\mathrm{d}\mathbf{y}'\Big)^{2/3}\bigg].
\end{aligned}$$

Similarly, the distance between $\mathbf{x}'$ and any $\mathbf{y}' \in A'_2(t'-t'_\delta)$ is greater than expression $(1-2\mu)\mathrm{e}^{a(t'-t'_\delta)/2}$. Thus,

$$\begin{aligned}
|p'_3(\mathbf{x}',t')| &\leqslant C(\mu)\mathrm{e}^{-3a(t'-t'_\delta)/2}\int_{A'_2(t'-t'_\delta)} \big(|\mathbf{v}'|^2 + |p'|\big)\,\mathrm{d}\mathbf{x}'\\
&\leqslant C(\kappa,\mu)\mathrm{e}^{-a(t'-t'_\delta)}\Big(\int_{A'_2(t'-t'_\delta)} \big(|\mathbf{v}'|^3 + |p'|^{3/2}\big)\,\mathrm{d}\mathbf{x}'\Big)^{2/3}.
\end{aligned}$$

Finally, applying the Calderon-Zygmund theorem, we can estimate the integral of $|p_2'|^{3/2}$:

$$\int_{A_0'(t'-t_\delta')} \bigl|p_2'(\mathbf{x}',t')\bigr|^{3/2}\,\mathrm{d}\mathbf{x}' \leqslant C\int_{A_1'(t'-t_\delta')} \bigl|\mathbf{v}'(\mathbf{x}',t')\bigr|^{3}\,\mathrm{d}\mathbf{x}'.$$

These inequalities imply (2.10). □

2.6. Estimates of $P(t_\delta')$. Applying Lemma 2.1, we obtain

$$P(t_\delta') \leqslant H_1(t_\delta') + H_2(t_\delta') + H_3(t_\delta'), \tag{2.15}$$

where

$$\begin{aligned}
H_1(t_\delta') &:= c_4\int_{t_\delta'}^{\infty}\biggl(\int_{B_1'} |\mathbf{v}'|^2\,\mathrm{d}\mathbf{x}'\biggr)^{3/2} \mathrm{e}^{-a(t'-t_\delta')}\,\mathrm{d}t',\\
H_2(t_\delta') &:= c_5 K(t_\delta'),\\
H_3(t_\delta') &:= c_6\int_{t_\delta'}^{\infty}\int_{A_2'(t'-t_\delta')} |p'|^{3/2}\,\mathrm{d}\mathbf{x}'\mathrm{e}^{-3a(t'-t_\delta')/2}\,\mathrm{d}t'.
\end{aligned}$$

The first term $H_1(t_\delta')$ can be estimated by the means of inequality (2.7):

$$\begin{aligned}
H_1(t_\delta') &\leqslant \operatorname*{ess\,sup}_{t'>t_\delta'}\biggl(\int_{B_1'} |\mathbf{v}'|^2\,\mathrm{d}\mathbf{x}'\mathrm{e}^{-2a(t'-t_\delta')/3}\biggr)^{1/2}\biggl[c_4\int_{t_\delta'}^{\infty}\int_{B_1'} |\mathbf{v}'|^2\,\mathrm{d}\mathbf{x}'\mathrm{e}^{-2a(t'-t_\delta')/3}\,\mathrm{d}t'\biggr]\\
&\leqslant \frac{6c_4}{a}\Bigl[\|(\varphi\mathbf{v}')|_{t_\delta'}\|_{2;M'(0)}^2 + F_1(t_\delta') + F_2(t_\delta') + F_3(t_\delta')\Bigr]^{3/2}.
\end{aligned}$$

The second term $H_2(t_\delta')$ can be estimated by means of (2.9). In order to estimate the third term $H_3(t_\delta')$, we put $\delta_* := (2/\gamma_1)\delta$. The assumption $\gamma_1 \leqslant 2$ implies that $\delta_* \geqslant \delta$. Recall that $\kappa := 2(\gamma_2/\gamma_1 - 1)$. This special choice of κ guarantees that $(2+\kappa)\sqrt{a}\delta = \gamma_2\sqrt{a}\delta_*$, which is used in the forthcoming integrals. Now, transforming the integral in $H_3(t_\delta')$ to the original coordinates $\mathbf{x}$, t, we get

$$\begin{aligned}
H_3(t_\delta') &= \frac{c_6}{a^{3/2}\delta^2}\int_{t_0-\delta^2}^{t_0} \frac{\theta(t)}{\delta}\int_{2\sqrt{a}\delta<|\mathbf{x}-\mathbf{x}_0|<(2+\kappa)\sqrt{a}\delta} |p|^{3/2}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\leqslant \frac{c_6}{a^{3/2}\delta^2}\int_{t_0-\delta^2}^{t_0}\int_{2\sqrt{a}\delta<|\mathbf{x}-\mathbf{x}_0|<(2+\kappa)\sqrt{a}\delta} |p|^{3/2}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&= \frac{c_6}{a^{3/2}\delta^2}\int_{t_0-\delta^2}^{t_0}\int_{\max\{\theta(t);2\sqrt{a}\delta\}<|\mathbf{x}-\mathbf{x}_0|<(2+\kappa)\sqrt{a}\delta} |p|^{3/2}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&= \frac{c_6}{a^{3/2}}\frac{4}{\gamma_1^2}\frac{1}{\delta_*^2}\int_{t_0-\delta^2}^{t_0}\int_{\max\{\theta(t);\gamma_1\sqrt{a}\delta_*\}<|\mathbf{x}-\mathbf{x}_0|<\gamma_2\sqrt{a}\delta_*} |p|^{3/2}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\leqslant \frac{c_6}{a^{3/2}}\frac{4}{\gamma_1^2}\frac{1}{\delta_*^2}\int_{t_0-\delta_*^2}^{t_0}\int_{\max\{\theta(t);\gamma_1\sqrt{a}\delta_*\}<|\mathbf{x}-\mathbf{x}_0|<\gamma_2\sqrt{a}\delta_*} |p|^{3/2}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&= \frac{c_6}{a^{3/2}}\frac{4}{\gamma_1^2}\frac{1}{\delta_*^2}\iint_{V_{\delta_*,a,\gamma_1,\gamma_2}} |p|^{3/2}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.
\end{aligned}$$

The last inequality holds for $\delta_* \in (0, \delta_0)$, i.e., for $\delta \in (0, \gamma_1\delta_0/2)$. If we denote by c_7 the upper bound in the condition (1.4), we obtain

$$H_3(t'_\delta) \leqslant \frac{c_6}{a^{3/2}} \frac{4}{\gamma_1^2} c_7.$$

Substituting the estimates for $H_1(t'_\delta)$, $H_2(t'_\delta)$, $H_3(t'_\delta)$ into (2.15) and applying the inequalities (2.11)–(2.13), we obtain

$$\begin{aligned} P^{2/3}(t'_\delta) \leqslant \Big(\frac{6c_4}{a}\Big)^{2/3} &\Big[\|(\varphi\mathbf{v}')|_{t'_\delta}\|^2_{2;M'(0)} + c_1\varepsilon^{2/3} + c_2\varepsilon + c_3\varepsilon^{1/3}P^{2/3}(t'_\delta)\Big] \\ &+ [c_5 a(2+\kappa)^2\varepsilon]^{2/3} + \Big[\frac{c_6}{a^{3/2}}\frac{4}{\gamma_1^2}c_7\Big]^{2/3}. \end{aligned}$$

Assuming that ε is so small that $(6c_4/a)^{2/3}c_3\varepsilon^{1/3} \leqslant 1/2$, we obtain

$$\begin{aligned} P^{2/3}(t'_\delta) \leqslant 2\Big(\frac{6c_4}{a}\Big)^{2/3} &\big[\|(\varphi\mathbf{v}')|_{t'_\delta}\|^2_{2;M'(0)} + c_1\varepsilon^{2/3} + c_2\varepsilon\big] \\ &+ 2\big[c_5 a(2+\kappa)^2\varepsilon\big]^{2/3} + 2\Big[\frac{4}{\gamma_1^2}\frac{c_6}{a^{3/2}}c_7\Big]^{2/3}. \end{aligned} \tag{2.16}$$

2.7. Consequences of the inequality (2.7)**.** Using the inequalities (2.9)–(2.13) and (2.16), we observe that the right hand side of (2.7) is

$$\begin{aligned} &\leqslant \|(\varphi\mathbf{v}')|_{t'_\delta}\|^2_{2;M'(0)} + F_1(t'_\delta) + F_2(t'_\delta) + F_3(t'_\delta) \\ &\leqslant \|(\varphi\mathbf{v}')|_{t'_\delta}\|^2_{2;M'(0)} + c_1\varepsilon^{2/3} + c_2\varepsilon \\ &\quad + 2c_3\varepsilon^{1/3}\Big(\frac{6c_4}{a}\Big)^{2/3}\big[\|(\varphi\mathbf{v}')|_{t'_\delta}\|^2_{2;M'(0)} + c_1\varepsilon^{2/3} + c_2\varepsilon\big] \\ &\quad + 2c_3\varepsilon^{1/3}\big[c_5 a(2+\kappa)^2\varepsilon\big]^{2/3} + 2c_3\varepsilon^{1/3}\Big[\frac{4}{\gamma_1^2}\frac{c_6}{a^{3/2}}c_7\Big]^{2/3} \\ &\quad =: (1 + c_8\varepsilon^{1/3})\|(\varphi\mathbf{v}')|_{t'_\delta}\|^2_{2;M'(0)} + c_9\varepsilon + c_{10}\varepsilon^{2/3} + c_{11}\varepsilon^{1/3}. \end{aligned} \tag{2.17}$$

The term $\|\nabla'(\varphi\mathbf{v}')\|^2_{2;M'(\tau-t'_\delta)}$ on the left hand side of the inequality (2.7) can be estimated from below by means of Sobolev's and Hölder's inequalities:

$$\begin{aligned} \|\nabla'(\varphi\mathbf{v}')|_\tau\|^2_{2;M'(\tau-t'_\delta)} &\geqslant 3\Big(\frac{\pi}{2}\Big)^{4/3}\|(\varphi\mathbf{v}')|_\tau\|^2_{6;M'(\tau-t'_\delta)} \\ &\geqslant 3\Big(\frac{\pi}{2}\Big)^{4/3}\|(\varphi\mathbf{v}')|_\tau\|^2_{6;M'(0)} \\ &\geqslant 3\Big(\frac{\pi}{2}\Big)^{4/3}\frac{3^{2/3}}{(1+2\mu)^2(4\pi)^{2/3}}\|(\varphi\mathbf{v}')|_\tau\|^2_{2;M'(0)} \\ &= \frac{3}{(1+2\mu)^2}\Big(\frac{3\pi}{16}\Big)^{2/3}\|(\varphi\mathbf{v}')|_\tau\|^2_{2;M'(0)}. \end{aligned}$$

(See [15] for the optimal constant in Sobolev's inequality.) Thus, omitting the first term on the left hand side of (2.7) and letting $t' \to \infty$, the inequality (2.7) yields:

$$\begin{aligned}(2.18)\qquad &\left[\frac{a}{6}+\frac{6\nu}{(1+2\mu)^2}\Big(\frac{3\pi}{16}\Big)^{2/3}\right]\int_{t'_\delta}^{\infty}\|\varphi\mathbf{v}'\|^2_{2;M'(0)}\mathrm{e}^{-2a(\tau-t'_\delta)/3}\,\mathrm{d}\tau\\ &\qquad\leqslant(1+c_8\varepsilon^{1/3})\|(\varphi\mathbf{v}')|_{t'_\delta}\|^2_{2;M'(0)}+c_9\varepsilon+c_{10}\varepsilon^{2/3}+c_{11}\varepsilon^{1/3}.\end{aligned}$$

Denote $g(t'_\delta):=\int_{t'_\delta}^{\infty}\|\varphi\mathbf{v}'\|^2_{2;M'(0)}\mathrm{e}^{-2a(\tau-t'_\delta)/3}\,\mathrm{d}\tau$.

Then $\dot g(t'_\delta)=-\|\varphi\mathbf{v}'\|^2_{2;M'(0)}+2ag(t'_\delta)/3$. Substituting for $\|\varphi\mathbf{v}'\|^2_{2;M'(0)}$ from this formula into (2.18), we obtain

$$\begin{aligned}(2.19)\qquad &(1+c_8\varepsilon^{1/3})\dot g(t'_\delta)+\left[\frac{6\nu}{(1+2\mu)^2}\Big(\frac{3\pi}{16}\Big)^{2/3}-\frac{a}{2}-\frac{2a}{3}c_8\varepsilon^{1/3}\right]g(t'_\delta)\\ &\qquad\leqslant c_9\varepsilon+c_{10}\varepsilon^{2/3}+c_{11}\varepsilon^{1/3}.\end{aligned}$$

Recall that the parameter a is assumed to be less than $3\nu(3\pi/2)^{2/3}$. Thus $\zeta:=3\nu(3\pi/2)^{2/3}/2-a/2=6\nu(3\pi/16)^{2/3}-a/2>0$. Assume that $\mu\in(0,1/2)$ is so small that

$$(2.20)\qquad 6\nu\Big(\frac{3\pi}{16}\Big)^{2/3}\frac{1}{(1+2\mu)^2}-\frac{a}{2}\geqslant\frac{\zeta}{2}.$$

Then the inequality (2.19) yields

$$(1+c_8\varepsilon^{1/3})\dot g(t'_\delta)+\Big[\frac{\zeta}{2}-\frac{2a}{3}c_8\varepsilon^{1/3}\Big]g(t'_\delta)\leqslant c_9\varepsilon+c_{10}\varepsilon^{2/3}+c_{11}\varepsilon^{1/3}.$$

Dividing this inequality by $1+c_8\varepsilon^{1/3}$, we get

$$(2.21)\qquad \dot g(t'_\delta)+\Big[\frac{\zeta}{2}-f_1(\varepsilon)\Big]g(t'_\delta)\leqslant f_2(\varepsilon),$$

where f_1 and f_2 are appropriate positive functions, satisfying $f_1(\varepsilon)\to0$ and $f_2(\varepsilon)\to0$ for $\varepsilon\to0+$. Assuming that $\varepsilon>0$ is so small that $\zeta/2-f_1(\varepsilon)\geqslant\zeta/4$ and integrating the inequality (2.21) from an arbitrary fixed s to t'_δ, we obtain

$$\begin{aligned}g(t'_\delta)&\leqslant\mathrm{e}^{-\zeta(t'_\delta-s)/4}g(s)+\int_s^{t'_\delta}\mathrm{e}^{-\zeta(t'_\delta-\sigma)/4}f_2(\varepsilon)\,\mathrm{d}\sigma\\ &\leqslant\mathrm{e}^{-\zeta(t'_\delta-s)/4}g(s)+\frac{4f_2(\varepsilon)}{\zeta}.\end{aligned}$$

If t'_δ is sufficiently large then $\mathrm{e}^{-\zeta(t'_\delta-s)/4}g(s)<\varepsilon$, which yields $g(t'_\delta)<\varepsilon+4\zeta^{-1}f_2(\varepsilon)$. Recalling the definition of the function g, we deduce that there exists an increasing sequence of $t'_{\delta,n}$ such that $t'_{\delta,n}\to\infty$ for $n\to\infty$ and $\|(\varphi\mathbf{v}')|_{t'_{\delta,n}}\|^2_{2;M'(0)}\leqslant$

$2a[\varepsilon + 4\zeta^{-1} f_2(\varepsilon)]/3$. Applying this inequality to the right hand side of (2.7), which is estimated in (2.17), we finally obtain two inequalities:

$$\begin{aligned}(2.22)\qquad & \|(\varphi\mathbf{v}')|_{t'}\|^2_{2;M'(t'-t'_{\delta,n})}\mathrm{e}^{-2a(t'-t'_{\delta,n})/3} \\ &\quad\leqslant (1+c_8\varepsilon^{1/3})\frac{2a[\varepsilon + 4\zeta^{-1}f_2(\varepsilon)]}{3} + c_9\varepsilon + c_{10}\varepsilon^{2/3} + c_{11}\varepsilon^{1/3},\end{aligned}$$

$$\begin{aligned}(2.23)\qquad & \frac{a}{6}\int_{t'_{\delta,n}}^{\infty} \|\varphi\mathbf{v}'\|^2_{2;M'(\tau-t'_{\delta,n})}\mathrm{e}^{-2a(\tau-t'_{\delta,n})/3}\,\mathrm{d}\tau \\ &\quad + 2\nu\int_{t'_{\delta}}^{\infty} \|\nabla'(\varphi\mathbf{v}')\|^2_{2;M'(\tau-t'_{\delta,n})}\mathrm{e}^{-2a(\tau-t'_{\delta,n})/3}\,\mathrm{d}\tau \\ &\quad\leqslant (1+c_8\varepsilon^{1/3})\frac{2a[\varepsilon + 4\zeta^{-1}f_2(\varepsilon)]}{3} + c_9\varepsilon + c_{10}\varepsilon^{2/3} + c_{11}\varepsilon^{1/3}.\end{aligned}$$

2.8. Final estimates of $G^{II}(\delta_n)$. We define δ_n so that δ_n and $t'_{\delta,n}$ are connected through the formula (2.3): $\delta_n := \varrho\mathrm{e}^{-at'_{\delta,n}/2}$. The sequence $\{\delta_n\}$ satisfies $\delta_n \searrow 0$ for $n\to\infty$. Using the inequality (2.8) (with $\delta=\delta_n$), we estimate $G^{II}(\delta_n)$ as follows:

$$\begin{aligned}G^{II}(\delta_n) \leqslant{}& \frac{1}{3^{3/4}}\frac{2}{\pi}a\Big(\int_{t'_{\delta,n}}^{\infty} \|\nabla'(\varphi\mathbf{v}')\|^2_{2;M'(t'-t'_{\delta,n})}\mathrm{e}^{-2a(t'-t'_{\delta})/3}\,\mathrm{d}t'\Big)^{3/4} \\ &\times \operatorname*{ess\,sup}_{t'>t'_{\delta,n}}\Big(\|\varphi\mathbf{v}'\|^2_{2;M'(t'-t'_{\delta,n})}\mathrm{e}^{-2a(t'-t'_{\delta,n})/3}\Big)^{1/2} \\ &\times \Big(\int_{t'_{\delta_n}}^{\infty} \|\varphi\mathbf{v}'\|^2_{2;M'(t'-t'_{\delta_n})}\mathrm{e}^{-2a(t'-t'_{\delta,n})/3}\,\mathrm{d}t'\Big)^{1/4}.\end{aligned}$$

Applying (2.22) and (2.23) to the right hand side, we get

$$\begin{aligned}G^{II}(\delta_n) \leqslant{}& \frac{1}{3^{3/4}}\frac{2}{\pi}a\Big(\frac{1}{2\nu}\Big)^{3/4}\Big(\frac{6}{a}\Big)^{1/4}\Big[(1+c_8\varepsilon^{1/3})\frac{2a(\varepsilon + 4\zeta^{-1}f_2(\varepsilon))}{3} \\ &+ c_9\varepsilon + c_{10}\varepsilon^{2/3} + c_{11}\varepsilon^{1/3}\Big]^{3/2}.\end{aligned}$$

This inequality implies (2.1). The proof of Theorem 1.1 is completed. □

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