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Two notions which affected nonlinear analysis (Bernard Bolzano lecture)

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TWO NOTIONS WHICH AFFECTED NONLINEAR ANALYSIS (BERNARD BOLZANO LECTURE)

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Abstract. General mathematical theories usually originate from the investigation of particular problems and notions which could not be handled by available tools and methods. The Fučík spectrum and the p-Laplacian are typical examples in the field of nonlinear analysis. The systematic study of these notions during the last four decades led to several interesting and surprising results and revealed deep relationship between the linear and the nonlinear structures. This paper does not provide a complete survey. We focus on some pioneering works and present some contributions of the author. From this point of view the list of references is by no means exhaustive.

Keywords: Fučík spectrum; p-Laplacian

MSC 2010: 35J92, 34B15, 34B99, 35P30

1. Introduction

One of the most important and gifted Czech mathematicians of the second half of the twentieth century, Svatopluk Fučík, investigated, at the beginning of the 1970s, semilinear equations of the form

$$-u''(x) = f(x, u(x)), \quad x \in (a, b) \subset \mathbb{R} \tag{1.1}$$

subject to various kinds of boundary conditions. He applied topological methods based on the degree theory to get existence results both for (1.1) and also for its higher dimensional analogue

$$-\Delta u(x) = f(x, u(x)), \quad x \in \Omega \subset \mathbb{R}^N.$$

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Asymptotic properties of the nonlinear function f with respect to the second variable play the crucial role. In particular, if

$$f(\cdot, s) \sim \lambda s \quad \text{as } s \to \pm\infty, \tag{1.2}$$

the knowledge of the set of all eigenvalues of

$$-\Delta u(x) = \lambda u(x), \quad x \in \Omega$$

is an important tool for proving the existence of a solution. Problems of this type are studied systematically in two classical monographs by Fučík, Nečas, Souček and Souček [27] and Fučík [25].

It is an interesting fact that in 1976 two papers were published independently by Fučík [26] and Dancer [6], where the asymptotic condition (1.2) was generalized to

$$f(\cdot, s) \sim \mu s \quad \text{as } s \to \infty, \quad f(\cdot, s) \sim \nu s \quad \text{as } s \to -\infty \tag{1.3}$$

possibly with $\mu \neq \nu$. More precisely, the answer to the following question was the key to formulate existence and nonexistence results for the boundary value problem

$$-u''(x) = f(x, u(x)), \quad x \in (0,1), \quad u(0) = u(1) = 0: \tag{1.4}$$

Question: Find all couples $(\mu, \nu) \in \mathbb{R}^2$ for which there exists a nontrivial solution of

$$\begin{cases} -u''(x) - \mu u^+(x) + \nu u^-(x) = 0, & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{1.5}$$

Using the uniqueness of the initial value problem associated with the equation in (1.5), Fučík and Dancer provided an affirmative answer to this question. The set of all $(\mu, \nu) \in \mathbb{R}^2$ for which (1.5) has a nonzero solution is a collection of "hyperbolas" which can be expressed analytically and look like curves in Figure 1.

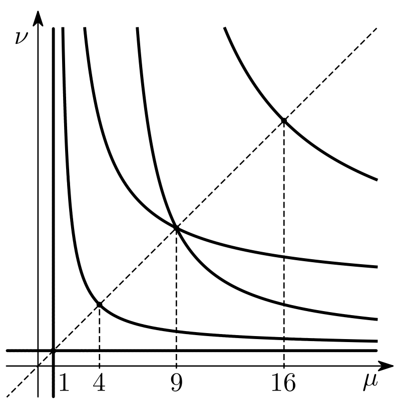

Figure 1. The Fučík spectrum.

Much later the collection of these curves was called the Fučík spectrum of (1.5). Under assumption (1.3) and depending on the position of (μ,ν) with respect to the Fučík spectrum, Fučík and Dancer derived several existence and nonexistence results for (1.4). Notice that Fučík's manuscript was received by the editors of Čas. Pěst. Mat. on August 30, 1974, while Dancer's manuscript was received by the editors of Bull. Austr. Math. Soc. on July 1, 1976. Fučík and Dancer never met before and their results provide an evidence that the Fučík spectrum arose naturally at a certain stage of development of nonlinear analysis.

Svatopluk Fučík was the advisor of my master thesis which I submitted in May 1977. My task was to prove the existence and nonexistence results for the Dirichlet problem

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' = f(x,u(x)), \quad x\in(0,1),\\ u(0)=u(1)=0 \end{cases} \tag{1.6}$$

where $p\geqslant 2$ and

$$f(\cdot,s)\sim\mu|s|^{p-2}s \quad \text{as} \quad s\to\infty \quad \text{and} \quad f(\cdot,s)\sim\nu|s|^{p-2}s \quad \text{as} \quad s\to-\infty.$$

The results of my thesis relied on the characterization of all couples $(\mu,\nu)\in\mathbb{R}^2$ for which the problem

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' - \mu|u^+(x)|^{p-2}u^+(x) + \nu|u^-(x)|^{p-2}u^-(x) = 0,\\ u(0)=u(1)=0 \end{cases} \tag{1.7}$$

has a nontrivial solution in $(0,1)$. Actually I proved that the Fučík spectrum for (1.7) has a structure similar to that for (1.5) and the existence and nonexistence results for (1.6) along the lines of [6] and [26] hold true. However, different technique had to be employed in the quasilinear case $p>2$. The main results were published in Drábek [13] and then generalized and complemented in Boccardo, Drábek, Giachetti and Kučera [5].

Let us note that the quasilinear operator

$$u\mapsto(|u'|^{p-2}u')'$$

has its counterpart in higher dimensions

$$u\mapsto\operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

It is defined for any $p>1$ and called the p-Laplacian. Thus the main goal of my master thesis, expressed in this terminology, was the investigation of the Fučík spectrum for the p-Laplacian.

2. Common features and pioneering results

Both notions introduced in the previous section, the Fučík spectrum and the p-Laplacian, became meanwhile quite frequent in nonlinear analysis. They share the following common feature. In the language of the second order ODEs the Fučík spectrum as well as the p-Laplacian are well understood and there exist many accurate results involving both the notions. On the other hand, even very basic questions concerning the structure of the Fučík spectrum and the properties of the p-Laplacian are still open in the PDE case. In order to acquaint the reader with some of them, we consider the problem

$$\begin{cases} -\Delta u(x) - \mu u^+(x) + \nu u^-(x) = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^N$. The structure of the Fučík spectrum of (2.1) is not known even if Ω is a square or a disc in $\mathbb{R}^2$! There are many partial results showing that certain curves in the (μ, ν)-plane must be contained in the Fučík spectrum but there is no result which would reveal its complete description.

Let us consider the eigenvalue problem for the p-Laplacian $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ for $p > 1$:

$$\begin{cases} -\Delta_p u(x) - \lambda |u(x)|^{p-2} u(x) = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{2.2}$$

An eigenvalue of (2.2) is such an $\lambda \in \mathbb{R}$ for which there exists a nontrivial solution (called an eigenfunction) of (2.2). The set of all eigenvalues of (2.2) is not known for $\Omega \subset \mathbb{R}^N$ being a bounded smooth domain, $N \geqslant 2$. And this is the case even if Ω is such special domain as a square or a disc in $\mathbb{R}^2$!

The set of all eigenvalues is well understood in one dimension (Ω = bounded interval) when it forms a sequence

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots \to \infty.$$

In higher dimension there are several variational tools, like the Ljusternik-Schnirelmann method, which allow to construct a special sequence of eigenvalues of (2.2), $\{\lambda_n\}_{n=1}^{\infty}$, $\lambda_n \to \infty$ with a "minmax" characterization (see e.g. [27]). However, it is not clear if such a sequence exhausts the set of all eigenvalues of (2.2). For example, it is not clear whether or not the set of all eigenvalues of (2.2) contains an interval for some "special" domains Ω.

Let us mention some pioneering works on both the notions. As for the Fučík spectrum we mention the result of Krejčí [29]. He was also a student of Svatopluk Fučík and his paper [29] contains the results from his master thesis. Complete description of the Fučík spectrum for the fourth order ODE is given and the existence results along the lines of [6] and [26] are presented there. An interesting paper by Švarc [33], another student of Fučík, is devoted to the study of the Fučík spectrum for finite-dimensional operators. The author demonstrates (on examples in $\mathbb{R}^4$) that there is no relation between the value of the Leray-Schauder degree in different parts of the complement of the Fučík spectrum and the number of solutions of the associated equation.

As for the p-Laplacian, we mention here the paper by Elbert [24], as one of the very first publications about the eigenvalue problem (2.2) in one-dimension. Elbert himself (and some other mathematicians working in oscillatory theory, too) called one-dimensional p-Laplacian a half-linear operator. The reason is that only homogeneity is preserved while additivity is lost for $p \neq 2$. It should be emphasized that Elbert's paper [24] had not been recognized for relatively long period of time. In particular, in 1980s several papers of other authors "rediscovered" what had been already published in [24]. In particular, the generalization of goniometric functions, nowadays called $\sin_p$ and $\cos_p$, were studied in detail already by Elbert.

3. Contributions of the author: Fučík spectrum

In this section we present some modest contributions of the author (and his co-authors) to the study of the Fučík spectrum. In 1980s many mathematicians generalized the famous result of Landesman and Lazer [30] in different directions. In order to get existence results for a rather general growth restriction on the nonlinear term, the knowledge of the Fučík spectrum appeared to be very useful. This was observed in author's paper [12], where the boundary value problem

$$\begin{cases} u''(x) + u(x) + g(x, u(x)) = h(x), & x \in (0, \pi), \\ u(0) = u(\pi) = 0 \end{cases} \tag{3.1}$$

was studied. Let

$$g^{-\infty}(x) := \limsup_{s \to -\infty} g(x, s) \quad \text{and} \quad g_{\infty}(x) := \liminf_{s \to \infty} g(x, s).$$

We assume that $g^{-\infty}(x)$ and $g_{\infty}(x)$ are bounded from above and from below, respectively. The existence result for (3.1) holds if

$$\int_0^{\pi} g^{-\infty}(x) \sin x \,\mathrm{d}x < \int_0^{\pi} h(x) \sin x \,\mathrm{d}x < \int_0^{\pi} g_{\infty}(x) \sin x \,\mathrm{d}x.$$

Here $\limsup\limits_{s\to\infty} g(x,s)/s$ might be arbitrarily large but this fact must be compensated by a suitable restriction given by the Fučík spectrum on $\limsup\limits_{s\to-\infty} g(x,s)/s$.

Some other results along these lines were published in author's monograph [10] which was aimed to be a "continuation" of Fučík's book [25].

The models of suspension bridges are another interesting motivation for the study of the Fučík spectrum. Let us mention here the seminal paper by Lazer and McKenna [31] which was (and still is) a source of many inspirations and open problems. The simplest model of a suspension bridge is given by the beam equation

$$\begin{cases} mu_{tt} + ku_{xxxx} + \delta u_t + bu^+ = W(x) + f(x,t), & (x,t) \in (0,L) \times \mathbb{R}, \\ u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0, \end{cases} \tag{3.2}$$

see Figure 2.

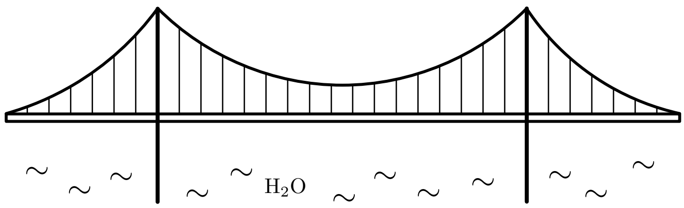

Figure 2. The model of suspension bridge.

The nonlinear term in the beam equation, bu^+, represents the influence of the cable stays which act as one-side springs. In our paper with Berkovits, Leinfelder, Mustonen and Tajčová (Holubová) [3] we proved the existence and uniqueness of periodic solutions of problem (3.2). Bifurcation of periodic solutions for a slightly different model were proved in our paper with Tajčová (Holubová) [23]. Discussion of the results for different and more complicated models (described by systems of PDEs) is contained in our joint paper with Holubová, Matas and Nečesal [16].

More recent results on the Fučík spectrum are contained in the joint papers of the author and Robinson [20], [21]. In [21] we study the problem

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' - \alpha(u^+(x))^{p-1} + \beta(u^-(x))^{p-1} = f(x), & x \in (0,T), \\ u(0) = u(T) = 0 \end{cases} \tag{3.3}$$

for $p > 1$, $T > 0$, $f \in L^1(0,T)$. Using the variational approach we prove existence results for (3.3) under the assumption that (α, β) belongs to the Fučík spectrum of the associated homogeneous problem. We also provide a new variational characterization for the points which belong to the third Fučík curve.

Paper [20] deals with the boundary value problem

$$(3.4)\qquad \begin{cases} -\Delta u(x) = \alpha u^+(x) - \beta u^-(x) + g(u(x)) + h(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $g\colon \mathbb{R} \to \mathbb{R}$ is a bounded continuous function and $h \in L^2(\Omega)$. Even if the structure of the entire Fučík spectrum of the associated homogeneous equation is not known, we prove existence results for (3.4) depending on the position of (α, β) with respect to the Fučík spectrum. In the nonresonance case, where (α, β) is not an element of the Fučík spectrum, the existence result for (3.4) follows without further restriction on g and h. In the resonance case, where (α, β) is an element of the Fučík spectrum, we present a new and more general Landesman-Lazer-type condition (see [30] for the original results of Landesman and Lazer) to prove existence results for (3.4). The proofs are variational and rely strongly on the variational characterization of certain parts of the Fučík spectrum.

4. Contributions of the author: p-Laplacian

Let $\{\lambda_n\}_{n=1}^{\infty}$ be the sequence of eigenvalues of (2.2) obtained by the Ljusternik-Schnirelmann method. It is known, see e.g. Anane [1], that λ_1 is a simple eigenvalue (i.e., all eigenfunctions associated with λ_1 are mutually proportional), the corresponding eigenfunctions do not change sign in Ω, and there are no eigenvalues λ of (2.2) with $\lambda \in (-\infty, \lambda_1) \cup (\lambda_1, \lambda_2)$. It is also known that every eigenfunction associated with λ_2 has exactly two modal domains, see e.g. Anane and Tsouli [2]. As was already mentioned before, the structure of the set of all eigenvalues beyond λ_2 in the case $N \geqslant 2$ is not clear at all. However, the isolatedness and simplicity of λ_1 leads naturally to the question of bifurcation from λ_1.

Let us consider the problem

$$(4.1)\qquad \begin{cases} -\Delta_p u(x) = \lambda |u(x)|^{p-2} u(x) + g(x, u(x), \lambda), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

$\Omega \subset \mathbb{R}^N$ a smooth bounded domain, $\lambda \in \mathbb{R}$ a bifurcation parameter, $g(x, 0, \lambda) = 0$, $(x, \lambda) \in \Omega \times \mathbb{R}$,

$$\lim_{s \to 0} \frac{g(x, s, \lambda)}{|s|^{p-2}} = 0,$$

g being of subcritical growth. The existence of global bifurcation branches of solutions $(\lambda, u), (\lambda, v) \in \mathbb{R} \times W_0^{1,p}(\Omega)$ of (4.1), $u > 0$ and $v < 0$ in Ω, bifurcating from

$(\lambda_1, 0)$, was proved independently in Drábek [11] (received by the editors in May 2, 1988) and del Pino, Manásevich [8] (received by the editors on November 2, 1989). Since the contacts between Czech and Chilean mathematicians in 1980s were inconceivable, this is another evidence of the fact that, at a certain stage, the research follows the same track independently in different parts of the globe. After the change of the political climate both in Czech Republic and Chile we have all become close collaborators, see our joint paper [7] below.

Another natural question connected with the properties of the principal eigenvalue of the p-Laplacian, λ_1, is the possibility of a generalization of the Fredholm alternative. It is simply formulated as follows: "Find sufficient (and possibly also necessary) conditions on f such that the problem

$$\begin{cases} -\Delta_p u(x) - \lambda_1 |u(x)|^{p-2} u(x) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega \end{cases} \tag{4.2}$$

has at least one solution." In the case $p = 2$ there is a necessary and sufficient condition

$$\int_\Omega f(x)\varphi_1(x)\,\mathrm{d}x = 0 \tag{4.3}$$

where $\varphi_1 = \varphi_1(x) > 0$ is the normalized eigenfunction associated with λ_1. However, in the case $p \neq 2$, the analogue of (4.3) was not apparent at all. In the joint paper of the author, Binding and Huang [4] we gave an example which shows that for $p \neq 2$ there exist right-hand sides f such that

$$\int_\Omega f(x)\varphi_1(x)\,\mathrm{d}x \neq 0$$

and problem (4.2) still has a solution. In other words, (4.3) is not a necessary condition. However, this was not surprising at all. On the other hand, it was very surprising that (4.3) is still sufficient even if $p \neq 2$! This fact was discovered in the paper del Pino, Drábek and Manásevich [7] in the case $N = 1$ ($\Omega =$ bounded interval). In fact, much more was proved in [7]. It became quite clear what is the characterization of all right-hand sides f for which (4.2) has a solution. Namely, split f as follows: $f = t\varphi_1 + f^\perp$, where $t \in \mathbb{R}$ and $\int_\Omega f^\perp(x)\varphi_1(x)\,\mathrm{d}x = 0$. Given $f^\perp$ fixed, there exist $t_1(f^\perp) < 0 < t_2(f^\perp)$ such that (4.2) has at least one solution if $f = f^\perp$, $f = t_i\varphi_1 + f^\perp$, $i = 1, 2$, and it has at least two distinct solutions provided $f = t\varphi_1 + f^\perp$ with $t \in (t_1, 0) \cup (0, t_2)$.

To recover this result for the PDE case was not obvious at all. We mention here the paper Drábek, Girg, Takáč and Ulm [14] where the bifurcation from infinity argument was used to prove the results along the lines of the one-dimensional case. Moreover, some other multiplicity results (existence of at least three solutions) were proved for the parameter λ close to λ_1 but not equal to λ_1.

It is worth mentioning also the variational approach to (4.2). The energy functional associated with this problem has the form

$$\mathrm{E}_p(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \frac{\lambda_1}{p}\int_\Omega |u|^p \,\mathrm{d}x - \int_\Omega f u \,\mathrm{d}x.$$

Its critical points correspond to solutions of (4.2). Let $\int_\Omega f\varphi_1 \,\mathrm{d}x = 0$. In the case $p = 2$ this functional is bounded below and its critical points, which are its global minimizers at the same time, form an affine set of one dimension "parallel" to $\{t\varphi_1\colon t \in \mathbb{R}\}$. The graph of E_2 can be thus visualized as an infinite trough and its bottom as solutions of (4.2). The geometry of E_p is different for $p \in (1,2)$ and $p \in (2,\infty)$. Indeed, in Drábek [9] we show that for $p \in (1,2)$ the functional E_p is unbounded from below and from above and has geometry of the saddle point. On the other hand, for $p \in (2,\infty)$, E_p is bounded below and has "global minimizer" geometry. In both the cases the set of all critical points is a priori bounded (in contrast with the case $p = 2$) but the Palais-Smale condition is not available. The proof of existence of critical points thus requires more involved arguments, different from the standard ones, see [9].

In our joint paper with Robinson [15] we introduced a new variational characterization of the sequence of eigenvalues of (2.2). For this purpose consider the functional

$$\mathrm{I}(u) := \frac{\int_\Omega |\nabla u|^p \,\mathrm{d}x}{\int_\Omega |u|^p \,\mathrm{d}x}$$

defined on $W_0^{1,p}(\Omega) \setminus \{0\}$ and the manifold

$$\mathscr{S} := \bigl\{u \in W_0^{1,p}(\Omega)\colon\ \|u\|_{L^p(\Omega)} = 1\bigr\}.$$

For any $k \in \mathbb{N}$ let $\mathcal{F}_k := \{\mathcal{A} \subset \mathscr{S}\colon \exists$ continuous odd surjection $h\colon \mathscr{S}^{k-1} \to \mathcal{A}\}$, where $\mathscr{S}^{k-1}$ represents the unit sphere in $\mathbb{R}^k$. Then

$$\lambda_k := \inf_{\mathcal{A}\in\mathcal{F}_k} \sup_{u\in\mathcal{A}} \mathrm{I}(u) \tag{4.4}$$

is a critical value of $\mathrm{I}|_{\mathscr{S}}$ and an eigenvalue of (2.2) at the same time. It is important to note that the given characterization of λ_k is not the same as the usual Ljusternik-Schnirelmann characterization involving a minimax over sets of genus greater than k. Let $\{\mu_k\}_{k=1}^\infty$ be the eigenvalues defined by the Ljusternik-Schnirelmann charakterization. Since $\mathcal{F}_k$ contains only the sets of genus k, it follows that $\lambda_k \geqslant \mu_k$. Thus

$\mu_k \to \infty$ implies $\lambda_k \to \infty$. It is clear that $\lambda_1 = \mu_1$ and it can be shown that $\lambda_2 = \mu_2$ (see [15]). However, the question whether $\lambda_k = \mu_k$, $k \geqslant 3$, remains open.

Even if it is not clear whether or not $\{\lambda_k\}_{k=1}^{\infty}$ exhausts the set of all eigenvalues of (2.2), the variational characterization (4.4) appears to be very useful to recover the Landesman-Lazer-type results to the p-Laplacian. Indeed, let us consider

$$\begin{cases} -\Delta_p u(x) - \lambda |u(x)|^{p-2} u(x) + f(x, u(x)) = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega \end{cases} \tag{4.5}$$

where f is a bounded function and the limits

$$f^{\pm}(x) := \lim_{t \to \pm\infty} f(x, t)$$

exist. Let λ be an eigenvalue of (2.2) (not necessarily given by (4.4)). Then the problem (4.5) has a solution provided

$$\int_{v>0} f^{+}(x) v(x) \,\mathrm{d}x + \int_{v<0} f^{-}(x) v(x) \,\mathrm{d}x > 0 \quad (\text{or} < 0)$$

for any eigenfunction $v = v(x)$ associated with λ.

Another useful application of (4.4) concerns the generalization of the Courant nodal domain theorem for the eigenfunctions of (4.5), see Drábek and Robinson [22]. Let u_λ be an eigenfunction associated with an eigenvalue λ of (2.2) and let $\{\lambda_n\}_{n=1}^{\infty}$ be the sequence of eigenvalues given by (4.4). We proved in [22]:

(1) For $\lambda < \lambda_{n+1}$ the eigenfunction u_λ has at most n nodal domains (maximal connected sets in Ω where u_λ is of definite sign).

(2) For $\lambda = \lambda_n$ the eigenfunction u_{λ_n} has at most $2n - 2$ nodal domains.

Note that if $\lambda_n < \lambda_{n+1}$ then in view of (1) u_{λ_n} has at most n nodal domains.

We say that the unique continuation property (UCP for short) is satisfied for (2.2) if no eigenfunction of (2.2) can vanish in a set with nonempty interior in Ω. UCP holds if $p = 2$ but it is an open problem if $p \neq 2$. Under the assumption that UCP holds for (2.2) also with $p \neq 2$, we recovered in [22] the Courant nodal domain theorem as follows:

(3) For $\lambda \leqslant \lambda_n$ the eigenfunction u_λ has at most n nodal domains.

Systematic study of the degenerated and/or singular boundary value problems involving the p-Laplacian with weights

$$\begin{cases} -\operatorname{div}(a(x,u)|\nabla u|^{p-2}\nabla u) = f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is contained in the monograph Drábek, Kufner and Nicolosi [18]. In particular, we focus on the connection between the singularities and degenerations of the weight a on the one hand and the rate of growth of the nonlinearity f on the other.

Some recent results in the ODE case concern the Sturm-Liouville property for the quasilinear problem

$$\begin{cases} (\varrho(t)|u'(t)|^{p-2}u'(t))' + \lambda\sigma(t)|u(t)|^{p-2}u(t) = 0 & \text{on } (a,b), \\ \lim\limits_{t\to a_+} \varrho(t)|u'(t)|^{p-2}u'(t) = \lim\limits_{t\to b_-} u(t) = 0, \end{cases}$$

$-\infty \leqslant a < b \leqslant \infty$. The reader is referred to papers Drábek and Kufner [17] and Drábek and Kuliev [19] for the details.

The flow of papers about the p-Laplacian retains its intensity. Many other generalizations are considered in the literature. Let us mention, e.g., the p-Laplacian with variable $p = p(x)$, $x \in \Omega$,

$$\Delta_{p(x)} u := \operatorname{div}(|\nabla u(x)|^{p(x)-2}\nabla u(x)). \tag{4.6}$$

The systematic study of suitable function spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ was invented as an "artificial" problem by Alois Kufner and was treated systematically in the paper by Kováčik and Rákosník [28]. Several years later very interesting connections to rheological models involving the operator (4.6) were discovered (see e.g. Růžička [32]). This led to the vast literature on this topic and brought a lot of citations for [28]. This is to say that any generalization makes sense if a good mathematics is behind it. And very often also useful applications may appear later on.

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