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QUASITRIANGULAR HOPF GROUP ALGEBRAS
AND BRAIDED MONOIDAL CATEGORIES

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Abstract. Let $\pi$ be a group, and $H$ be a semi-Hopf $\pi$-algebra. We first show that the category $\mathcal{H}\mathcal{M}$ of left $\pi$-modules over $H$ is a monoidal category with a suitably defined tensor product and each element $\alpha$ in $\pi$ induces a strict monoidal functor $F_\alpha$ from $\mathcal{H}\mathcal{M}$ to itself. Then we introduce the concept of quasitriangular semi-Hopf $\pi$-algebra, and show that a semi-Hopf $\pi$-algebra $H$ is quasitriangular if and only if the category $\mathcal{H}\mathcal{M}$ is a braided monoidal category and $F_\alpha$ is a strict braided monoidal functor for any $\alpha \in \pi$. Finally, we give two examples of Hopf $\pi$-algebras and describe the categories of modules over them.

Keywords: Hopf $\pi$-algebra; $H$-$\pi$-modules; braided monoidal category; braided monoidal functor

MSC 2010: 16T05, 08C05

1. Introduction

The notion of a quasitriangular Hopf algebra was introduced by Drinfel’d [4], when he studied the Yang-Baxter equation. The category of modules over a quasitriangular Hopf algebra is a braided monoidal category. Moreover, the braiding structure of a braided monoidal category can supply solutions to the quantum Yang-Baxter equation. Recently, Turaev [9] introduced Hopf $\pi$-coalgebra, which generalizes the notion of Hopf algebra. Virelizier also studied algebraic properties of Hopf group-coalgebras and generalized the main properties of quasitriangular Hopf algebras to the setting of quasitriangular Hopf $\pi$-coalgebras in [10]. Wang introduced the concept of semi-Hopf group algebra and investigated properties of coquasitriangular Hopf group algebras.
in [11]. Zhu, Chen and Li studied the categories of modules and comodules over a Hopf group coalgebra in [13] and [14], respectively.

In this paper, we first investigate the category $H \mathcal{M}$ of left modules over a semi-Hopf $\pi$-algebra $H$, where $\pi$ is a group. We define a tensor product module of two modules over $H$, and show that $H \mathcal{M}$ is a monoidal category with respect to such a tensor product, and each element $\alpha$ in $\pi$ induces a strict monoidal functor $F_\alpha$ from $H \mathcal{M}$ to itself. Then we introduce the concept of quasitriangular semi-Hopf $\pi$-algebra, and show that a semi-Hopf $\pi$-algebra $H$ is quasitriangular if and only if the category $H \mathcal{M}$ is a braided monoidal category and $F_\alpha$ is a strict braided monoidal functor for any $\alpha \in \pi$. Finally, we give two examples of Hopf $\pi$-algebras and discuss the categories of modules over them.

2. Preliminaries

Throughout the paper, let $\pi$ be a discrete group (with neutral element 1) and $k$ be a fixed field. All algebras and coalgebras, $\pi$-algebras and Hopf $\pi$-algebras are defined over $k$. The definitions and properties of an algebra, coalgebra, Hopf algebra, category and monoidal category can be found in [5]–[7], [12]. We use the standard Sweedler notation for comultiplication. The tensor product $\otimes = \otimes_k$ is always assumed to be over $k$. If $U$ and $V$ are $k$-spaces, $\tau_{U,V}: U \otimes V \rightarrow V \otimes U$ will denote the twist map defined by $\tau_{U,V}(u \otimes v) = v \otimes u$. The following definitions and notations can be found in [1], [8]–[11].

**Definition 2.1.** A $\pi$-algebra (over $k$) is a family $A = \{A_\alpha\}_{\alpha \in \pi}$ of $k$-spaces endowed with a family $m = \{m_{\alpha,\beta}: A_\alpha \otimes A_\beta \rightarrow A_{\alpha\beta}\}_{\alpha,\beta \in \pi}$ of $k$-linear maps (the multiplication) and a $k$-linear map $u: k \rightarrow A_1$ (the unit) such that $m$ is associative in the sense that for any $\alpha, \beta, \gamma \in \pi$,

$$m_{\alpha\beta,\gamma}(m_{\alpha,\beta} \otimes \text{id}_{A_\gamma}) = m_{\alpha,\beta\gamma}(\text{id}_{A_\alpha} \otimes m_{\beta,\gamma}),$$

$$m_{\alpha,1}(\text{id}_{A_\alpha} \otimes u) = \text{id}_{A_\alpha} = m_{1,\alpha}(u \otimes \text{id}_{A_\alpha}).$$

Note that $(A_1, m_{1,1}, u)$ is an algebra in the usual sense.

**Definition 2.2.** Let $A = \{A_\alpha\}_{\alpha \in \pi}, m, u$ be a $\pi$-algebra. A left $\pi$-module over $A$ is a family $M = \{M_\alpha\}_{\alpha \in \pi}$ of $k$-spaces endowed with a family $\eta = \{\eta^M_{\alpha,\beta}: A_\alpha \otimes M_\beta \rightarrow M_{\alpha\beta}\}_{\alpha,\beta \in \pi}$ of $k$-linear maps such that for any $\alpha, \beta, \gamma \in \pi$,

1. $\eta^M_{\alpha,\beta\gamma}(\text{id}_{A_\alpha} \otimes \eta^M_{\beta,\gamma}) = \eta^M_{\alpha,\beta\gamma}(m_{\alpha,\beta} \otimes \text{id}_{M_\gamma});$
2. $\eta^M_{1,\alpha}(u \otimes \text{id}_{M_\alpha}) = \text{id}_{M_\alpha}.$
Definition 2.3. Assume that $A = \{A_\alpha\}_{\alpha \in \pi}$ is a $\pi$-algebra. Let $M = \{M_\alpha\}_{\alpha \in \pi}$ and $N = \{N_\alpha\}_{\alpha \in \pi}$ be two left $\pi$-modules over $A$. A left $A$-$\pi$-module map from $M$ to $N$ is a family $f = \{f_\alpha : M_\alpha \to N_\alpha\}_{\alpha \in \pi}$ of $k$-linear maps such that

$$u_\alpha^N(id_{A_\alpha} \otimes f_\beta) = f_\alpha u_\alpha^M, \quad \alpha, \beta \in \pi.$$ 

Definition 2.4. A semi-Hopf $\pi$-algebra is a $\pi$-algebra $H = \{H_\alpha\}_{\alpha \in \pi}, m, u\}$ such that:

1. Each $H_\alpha$ is a $k$-coalgebra with comultiplication $\Delta_\alpha$ and counit $\varepsilon_\alpha$, $\alpha \in \pi$.
2. $u : k \to H_1$ and $m_{\alpha, \beta} : H_\alpha \otimes H_\beta \to H_{\alpha\beta}$ are coalgebra maps, $\alpha, \beta \in \pi$.

Furthermore, if there is a family $S = \{S_\alpha : H_\alpha \to H_{\alpha^{-1}}\}_{\alpha \in \pi}$ of $k$-linear maps (the antipode) such that the following condition (3) is satisfied, then $H = \{H_\alpha\}_{\alpha \in \pi}, m, u\}$ is called a Hopf $\pi$-algebra.

3. $m_{\alpha^{-1}, \alpha}(S_\alpha \otimes id_{H_\alpha})\Delta_\alpha = u\varepsilon_\alpha = m_{\alpha, \alpha^{-1}}(id_{H_\alpha} \otimes S_\alpha)\Delta_\alpha$, $\alpha \in \pi$.

3. Category of modules over a semi-Hopf $\pi$-algebra

Throughout this section, assume that $H = \{H_\alpha\}_{\alpha \in \pi}, m, u\}$ is a semi-Hopf $\pi$-algebra. Denote by $H\mathcal{M}$ the category of all left $\pi$-modules over $H$, whose morphisms are left $H$-$\pi$-module maps.

Lemma 3.1. Suppose that $(M, \eta^M)$ and $(N, \eta^N)$ are left $\pi$-modules over $H$. Then the tensor product $M \otimes N = \{(M \otimes N)_\alpha\}_{\alpha \in \pi}$ is also a left $\pi$-module over $H$, where $(M \otimes N)_\alpha = M_\alpha \otimes N_\alpha$, the structure maps $\eta^{M \otimes N} = \{\eta^{M \otimes N}_{\alpha, \beta} : H_\alpha \otimes M_\beta \otimes N_\beta \to M_{\alpha\beta} \otimes N_{\alpha\beta}\}_{\alpha, \beta \in \pi}$ are given by

$$\eta^{M \otimes N}_{\alpha, \beta} := (\eta^M_{\alpha, \beta} \otimes \eta^N_{\alpha, \beta})(id_{H_\alpha} \otimes H_{\alpha, M_\beta} \otimes id_{N_\beta})(\Delta_\alpha \otimes id_{M_\beta} \otimes id_{N_\beta}).$$

Proof. On the one hand, for any $h \in H_\alpha$, $l \in H_\beta$, $m \in M_\gamma$ and $n \in N_\gamma$, we have

$$\eta^{M \otimes N}_{\alpha, \beta, \gamma}(id_{H_\alpha} \otimes \eta^{M \otimes N}_{\beta, \gamma})(h \otimes l \otimes m \otimes n)$$

$$= \eta^{M \otimes N}_{\alpha, \beta, \gamma} \left(\sum h \otimes l_1 \cdot m \otimes l_2 \cdot n\right)$$

$$= \sum h_1 \cdot (l_1 \cdot m) \otimes h_2 \cdot (l_2 \cdot n)$$

$$= \sum (h_1 l_1) \cdot m \otimes (h_2 l_2) \cdot n$$

$$= \eta^{M \otimes N}_{\alpha, \beta, \gamma}(h l \otimes m \otimes n)$$

$$= \eta^{M \otimes N}_{\alpha, \beta, \gamma}(m_{\alpha, \beta} \otimes id_{(M \otimes N)_\gamma})(h \otimes l \otimes m \otimes n).$$
Hence $\eta^{M \otimes N}_{\alpha, \beta}(\text{id}_{H \alpha} \otimes \eta_{\beta, \gamma}^{M \otimes N}) = \eta^{M \otimes N}_{\alpha, \beta, \gamma}(m_{\alpha, \beta} \otimes \text{id}(M \otimes N)_{\gamma})$. On the other hand, for any $\lambda \in k$, $m \in M_{\alpha}$ and $n \in N_{\alpha}$, we have

$$\eta^{M \otimes N}_{1, \alpha}(u \otimes \text{id}(M \otimes N)_{\alpha})(\lambda \otimes m \otimes n) = \eta^{M \otimes N}_{1, \alpha}(\lambda 1_H \otimes m \otimes n) = \lambda(m \otimes n).$$

Hence $\eta^{M \otimes N}_{1, \alpha}(u \otimes \text{id}(M \otimes N)_{\alpha}) = \text{id}(M \otimes N)_{\alpha}$. Thus, $M \otimes N = \{(M \otimes N)_\alpha\}_{\alpha \in \pi}$ is a left $\pi$-module over $H$. $\square$

Let $M, N, P \in H.M$. Define $a_{M, N, P} = \{a_\alpha\}_{\alpha \in \pi} : (M \otimes N) \otimes P \to M \otimes (N \otimes P)$ by $a_\alpha : (M_\alpha \otimes N_\alpha) \otimes P_\alpha \to M_\alpha \otimes (N_\alpha \otimes P_\alpha)$, $(m \otimes n) \otimes p \mapsto m \otimes (n \otimes p)$, where $m \in M_\alpha$, $n \in N_\alpha$, $p \in P_\alpha$. Then we have the following lemma.

**Lemma 3.2.** The family $a_{M, N, P}$ is a family of left $H$-$\pi$-module natural isomorphisms, where $M, N, P \in H.M$.

**Proof.** For any $\alpha, \beta \in \pi$, $h \in H_\alpha$, $m \in M_\beta$, $n \in N_\beta$ and $p \in P_\beta$, we have

$$\eta^{M \otimes (N \otimes P)}_{\alpha, \beta}(\text{id}_{H_\alpha} \otimes a_\beta)(h \otimes ((m \otimes n) \otimes p))$$

$$= \eta^{M \otimes (N \otimes P)}_{\alpha, \beta}(h \otimes (m \otimes (n \otimes p)))$$

$$= \sum h_1 \cdot m \otimes h_2 \cdot (n \otimes p) = \sum h_1 \cdot m \otimes (h_2 \cdot n \otimes h_3 \cdot p)$$

$$= a_{\alpha, \beta}\left(\sum (h_1 \cdot m \otimes h_2 \cdot n) \otimes h_3 \cdot p\right)$$

$$= a_{\alpha, \beta}\left(\sum h_1 \cdot (m \otimes n) \otimes h_2 \cdot p\right)$$

$$= a_{\alpha, \beta}\eta^{(M \otimes N) \otimes P}_{\alpha, \beta}(h \otimes ((m \otimes n) \otimes p)).$$

This shows that $\eta^{M \otimes (N \otimes P)}_{\alpha, \beta}(\text{id}_{H_\alpha} \otimes a_\beta) = a_{\alpha, \beta}\eta^{(M \otimes N) \otimes P}_{\alpha, \beta}$, and so $a_{M, N, P}$ is a left $H$-$\pi$-module morphism. Consequently, $a_{M, N, P}$ is a left $H$-$\pi$-module isomorphism. Obviously, it is a family of natural isomorphisms of $H$-$\pi$-modules. $\square$

**Lemma 3.3.** Let $K = \{K_\alpha\}_{\alpha \in \pi}$ with $K_\alpha = k$. Define $\eta^K_{\alpha, \beta} : H_\alpha \otimes K_\beta \to K_{\alpha \beta}$ by $\eta^K_{\alpha, \beta}(h \otimes \lambda) = h \cdot \lambda := \varepsilon_\alpha(h)\lambda$. Then $K$ is a left $\pi$-module over $H$.

**Proof.** For any $h \in H_\alpha$, $l \in H_\beta$, $m \in K_\gamma = k$, $\lambda \in k$, $n \in K_\alpha = k$, we have

$$\eta^K_{\alpha, \beta, \gamma}(\text{id}_{H_\alpha} \otimes \eta^K_{\beta, \gamma})(h \otimes l \otimes m) = \eta^K_{\alpha, \beta, \gamma}(h \otimes \varepsilon_\beta(l)m)$$

$$= \varepsilon_\alpha(h)(\varepsilon_\beta(l)m) = \varepsilon_{\alpha, \beta}(hl)m = \eta^K_{\alpha, \beta, \gamma}(hl \otimes m)$$

and

$$\eta^K_{\alpha, \beta}(u \otimes \text{id}_{K_\alpha})(\lambda \otimes n) = \eta^K_{1, \alpha}(\lambda 1_H \otimes n) = \varepsilon_1(\lambda 1_H)n = \lambda n.$$ 

This shows that $\eta^K_{\alpha, \beta}(\text{id}_{H_\alpha} \otimes \eta^K_{\beta, \gamma}) = \eta^K_{\alpha, \beta, \gamma}(m_{\alpha, \beta} \otimes \text{id}_{K_\gamma})$ and $\eta^K_{1, \alpha}(u \otimes \text{id}_{K_\alpha}) = \text{id}_{K_\alpha}$. Thus, $K$ is a left $\pi$-module over $H$. $\square$
For any left $\pi$-module $M$ over $H$, we have $(K \otimes M)_\alpha = K_\alpha \otimes M_\alpha = k \otimes M_\alpha$ and $(M \otimes K)_\alpha = M_\alpha \otimes K_\alpha = M_\alpha \otimes k$, $\alpha \in \pi$. Define $l_M: K \otimes M \to M$ and $r_M: M \otimes K \to M$ by 
\[(l_M)_\alpha: k \otimes M_\alpha \to M_\alpha, \quad \lambda \otimes m \mapsto \lambda m,\]
\[(r_M)_\alpha: M_\alpha \otimes k \to M_\alpha, \quad m \otimes \lambda \mapsto \lambda m.\]

Then it is easy to see that $l = \{l_M\}$ and $r = \{r_M\}$ are two families of natural isomorphisms of left $H$-$\pi$-modules.

Summarizing the above discussion, one gets the following theorem.

**Theorem 3.4.** $(H^M, \otimes, K, a, l, r)$ is a monoidal category, where $K$ is the unit object.

For any $\alpha \in \pi$, define a functor $F_\alpha: H^M \to H^M$ by 
\[F_\alpha(M)_\beta = M_{\beta\alpha}, \quad \eta^{F_\alpha(M)}_{\beta,\gamma} = \eta^M_{\beta,\gamma\alpha}, \quad F_\alpha(f)_\beta = f_{\beta\alpha},\]

where $M$ is a left $\pi$-module over $H$ and $f$ is an $H$-$\pi$-module map. Obviously, $F_\alpha(K) = K$ and $(F_\alpha(M) \otimes F_\alpha(N))_\beta = F_\alpha(M)_\beta \otimes F_\alpha(N)_\beta = M_{\beta\alpha} \otimes N_{\beta\alpha} = (M \otimes N)_{\beta\alpha} = F_\alpha(M \otimes N)_{\beta\alpha}$, where $M$ and $N$ are left $\pi$-modules over $H$. Then by a straightforward verification, one can check the following theorem.

**Theorem 3.5.** $F_\alpha$ is a strict monoidal functor from $(H^M, \otimes, K, a, l, r)$ to itself, where $\alpha \in \pi$.

4. QUASITRIANGULAR SEMI-HOPF $\pi$-ALGEBRAS

Throughout this section, assume that $H = (\{H_\alpha\}_{\alpha \in \pi}, m, u)$ is a semi-Hopf $\pi$-algebra, and that $H^M$ is the category of left $\pi$-modules over $H$, which is a monoidal category as stated in the last section.

**Definition 4.1.** $H$ is called a quasitriangular semi-Hopf $\pi$-algebra, if there exists an invertible element $R \in H_1 \otimes H_1$ such that the following conditions are satisfied:

1. $\Delta^\text{cop}_\alpha(h)R = R\Delta_\alpha(h)$;
2. $(\Delta_1 \otimes \text{id})(R) = R_{13}R_{23}$;
3. $(\text{id} \otimes \Delta_1)(R) = R_{13}R_{12},$

where $\alpha \in \pi$, $h \in H_\alpha$, $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$, $R_{13} = (\tau_{H_1,H_1} \otimes \text{id})(1 \otimes R) \in H_1 \otimes H_1 \otimes H_1$ and $\Delta^\text{cop}_\alpha = \tau_{H_\alpha,H_\alpha} \circ \Delta_\alpha$. In this case, $R$ is called a quasitriangular structure of $H$. 897
Remark 4.2. We remark that $H_1$ is a usual quasitriangular bialgebra if $H$ is quasitriangular, and that $H$ is called an almost cocommutative semi-Hopf $\pi$-algebra if only (1) is satisfied.

Let $R = \sum_i s_i \otimes t_i$. Then the three conditions in Definition 4.1 can be formulated as follows:

1. $\sum h_2 s_i \otimes h_1 t_i = \sum s_i h_1 \otimes t_i h_2$;
2. $\sum (s_i)_1 \otimes (s_i)_2 \otimes t_i = \sum s_i \otimes s_j \otimes t_i t_j$;
3. $\sum s_i \otimes (t_i)_1 \otimes (t_i)_2 = \sum s_i s_j \otimes t_j \otimes t_i$,

where $\alpha \in \pi$, $h \in H_\alpha$ and $\Delta_\alpha(h) = h_1 \otimes h_2$ as usual.

Lemma 4.3. If $H$ is almost cocommutative, then there exists a left $H$-$\pi$-module isomorphism $M \otimes N \cong N \otimes M$ for any left $\pi$-modules $M$ and $N$ over $H$.

Proof. Assume that $R = \sum_i s_i \otimes t_i \in H_1 \otimes H_1$ is an invertible element satisfying condition (1) of Definition 4.1. Let $M$ and $N$ be two left $\pi$-modules over $H$. For any $\alpha \in \pi$, define $(c_{M,N})_\alpha: M_\alpha \otimes N_\alpha \rightarrow N_\alpha \otimes M_\alpha$ by

\[(c_{M,N})_\alpha(m \otimes n) := \tau_{M_\alpha,N_\alpha}(R \cdot (m \otimes n)) = \sum_i t_i \cdot n \otimes s_i \cdot m,\]

where $m \in M_\alpha$ and $n \in N_\alpha$. Since $R$ is invertible, $(c_{M,N})_\alpha$ is a $k$-linear isomorphism. Now for any $\alpha, \beta \in \pi$, $m \in M_\beta$, $n \in N_\beta$ and $h \in H_\alpha$, we have

\[\eta_{\alpha,\beta}^N \otimes^M (\text{id}_{H_\alpha} \otimes (c_{M,N})_\beta)(h \otimes m \otimes n)\]

\[= \eta_{\alpha,\beta}^N \otimes^M \left( \sum_i h \otimes t_i \cdot n \otimes s_i \cdot m \right)\]

\[= \sum_i h_1 \cdot (t_i \cdot n) \otimes h_2 \cdot (s_i \cdot m) = \sum_i (h_1 t_i) \cdot n \otimes (h_2 s_i) \cdot m\]

\[= \sum_i (t_i h_2) \cdot n \otimes (s_i h_1) \cdot m = \sum_i t_i \cdot (h_2 \cdot n) \otimes s_i \cdot (h_1 \cdot m)\]

\[= (c_{M,N})_\alpha \beta \left( \sum_i h_1 \cdot m \otimes h_2 \cdot n \right) = (c_{M,N})_\alpha \beta \eta_{\alpha,\beta}^M \otimes^N (h \otimes m \otimes n).\]

Hence $\eta_{\alpha,\beta}^N \otimes^M (\text{id}_{H_\alpha} \otimes (c_{M,N})_\beta) = (c_{M,N})_\alpha \beta \eta_{\alpha,\beta}^M \otimes^N$. This shows that $c_{M,N}$ is a left $H$-$\pi$-module map, and so

\[c_{M,N} = \{(c_{M,N})_\alpha\}_{\alpha \in \pi}: M \otimes N \rightarrow N \otimes M\]

is a left $H$-$\pi$-module isomorphism.
Theorem 4.4. Assume that $H$ is quasitriangular with a quasitriangular structure $R$. Then the category $\mathcal{H} \mathcal{M}$ is a braided monoidal category and $F_\alpha$ is a strict braided monoidal functor for any $\alpha \in \pi$.

Proof. By Theorems 3.4 and 3.5, it follows that $\mathcal{H} \mathcal{M}$ is a monoidal category and $F_\alpha$ is a strict monoidal functor for any $\alpha \in \pi$.

For any $M, N \in \mathcal{H} \mathcal{M}$, let

$$c_{M,N} = \{(c_{M,N})_\alpha\}_{\alpha \in \pi} : M \otimes N \rightarrow N \otimes M$$

be defined as in Lemma 4.3. Then $c_{M,N}$ is a left $H$-$\pi$-module isomorphism. Let $f = \{f_\alpha\}_{\alpha \in \pi} : M \rightarrow M'$ and $g = \{g_\alpha\}_{\alpha \in \pi} : N \rightarrow N'$ be two left $H$-$\pi$-module maps. Then for any $\alpha \in \pi$, $m \in M_\alpha$ and $n \in N_\alpha$, we have

$$(g_\alpha \otimes f_\alpha)(c_{M,N})_\alpha(m \otimes n) = (g_\alpha \otimes f_\alpha)(\sum_i t_i \cdot n \otimes s_i \cdot m)$$

$$= \sum_i g_\alpha(t_i \cdot n) \otimes f_\alpha(s_i \cdot m) = \sum_i t_i \cdot g_\alpha(n) \otimes s_i \cdot f_\alpha(m)$$

$$= (c_{M',N'})_\alpha(f_\alpha(m) \otimes g_\alpha(n)) = (c_{M',N'})_\alpha(f_\alpha \otimes g_\alpha)(m \otimes n).$$

Hence $(g \otimes f)c_{M,N} = c_{M',N'}(f \otimes g)$, which shows that $c_{M,N}$ is a family of natural isomorphisms of left $H$-$\pi$-modules.

Now let $M, N, P \in \mathcal{H} \mathcal{M}$ and $\alpha \in \pi$. Then for any $m \in M_\alpha$, $n \in N_\alpha$ and $p \in P_\alpha$, we have

$$(c_{M,N \otimes P})_\alpha(m \otimes n \otimes p) = \sum_i t_i \cdot (n \otimes p) \otimes s_i \cdot m = \sum_i (t_i)_1 \cdot n \otimes (t_i)_2 \cdot p \otimes s_i \cdot m$$

$$= \sum_{i,j} t_i \cdot n \otimes t_j \cdot p \otimes (s_j s_i) \cdot m = \sum_{i,j} t_i \cdot n \otimes t_j \cdot p \otimes s_j \cdot (s_i \cdot m)$$

$$= (\text{id}_{N_\alpha} \otimes (c_{M,P})_\alpha)((\sum_i t_i \cdot n \otimes s_i \cdot m) \otimes p)$$

$$= (\text{id}_{N_\alpha} \otimes (c_{M,P})_\alpha)((c_{M,N})_\alpha \otimes \text{id}_{P_\alpha})(m \otimes n \otimes p)$$

and

$$(c_{M \otimes N,P})_\alpha(m \otimes n \otimes p) = \sum_i t_i \cdot p \otimes (s_i)_1 \cdot (m \otimes n) = \sum_i t_i \cdot p \otimes (s_i)_1 \cdot m \otimes (s_i)_2 \cdot n$$

$$= \sum_{i,j} (t_j t_i) \cdot p \otimes s_j \cdot m \otimes s_i \cdot n = \sum_{i,j} t_j \cdot (t_i \cdot p) \otimes s_j \cdot m \otimes s_i \cdot n$$

$$= ((c_{M,P})_\alpha \otimes \text{id}_{N_\alpha})(\sum_i m \otimes t_i \cdot p \otimes s_i \cdot n)$$

$$= ((c_{M,P})_\alpha \otimes \text{id}_{N_\alpha})(\text{id}_{M_\alpha} \otimes (c_{N,P})_\alpha)(m \otimes n \otimes p).$$
This shows that \( c_{M,N} \otimes F = (\text{id}_N \otimes c_{M,P}))(c_{M,N} \otimes \text{id}_P) \) and \( c_{M,N} \otimes P = (c_{M,P} \otimes \text{id}_N)(\text{id}_M \otimes c_{N,P}) \). Therefore, \( H \mathcal{M} \) is a braided monoidal category with the braiding \( c \).

Let \( \alpha \in \pi \). Then for any \( M, N \in H \mathcal{M} \) and \( \beta \in \pi \), it is obvious that \( F_\alpha(c_{M,N})_\beta = (c_{M,N})_\beta \alpha = (c_{F_\alpha(M),F_\alpha(N)})_\beta \). Hence \( F_\alpha(c_{M,N}) = c_{F_\alpha(M),F_\alpha(N)} \), and consequently, \( F_\alpha \) is a strict braided monoidal functor for any \( \alpha \in \pi \).

**Theorem 4.5.** Suppose that \( H \mathcal{M} \) is a braided monoidal category, and \( F_\alpha \) is a strict braided monoidal functor for any \( \alpha \in \pi \). Then \( H \) is quasitriangular.

**Proof.** Suppose that \( H \mathcal{M} \) is a braided monoidal category with a braiding \( c \), and \( F_\alpha \) is a strict braided monoidal functor for any \( \alpha \in \pi \). Then \( c_{H,H} : H \otimes H \to H \otimes H \) is a left \( H\text{-}\pi\text{–}\text{module isomorphism} \), and hence \( (c_{H,H})_1 : H_1 \otimes H_1 \to H_1 \otimes H_1 \) is a \( k \)-linear isomorphism. Let \( R = \tau_{H_1,H_1}((c_{H,H})_1(1 \otimes 1)) \in H_1 \otimes H_1 \). Then Lemmas 4.8–4.10 below show that \( R \) is a quasitriangular structure of \( H \).

Throughout the following Lemma 4.6, Corollary 4.7 and Lemmas 4.8–4.10, assume that \( H \mathcal{M} \) is a braided monoidal category with a braiding \( c \), \( F_\alpha \) is a strict braided monoidal functor for any \( \alpha \in \pi \), and let \( R = \tau_{H_1,H_1}((c_{H,H})_1(1 \otimes 1)) = \sum_i s_i \otimes t_i \in H_1 \otimes H_1 \) be given as above. In this case, we have \( (c_{H,H})_1(1 \otimes 1) = \tau_{H_1,H_1}(R) = \sum_i t_i \otimes s_i \).

**Lemma 4.6.** Let \( M, N \in H \mathcal{M} \). Then we have

\[
(c_{M,N})_\alpha(m \otimes n) = \tau_{M_\alpha,N_\alpha}(R \cdot (m \otimes n)) = \sum_i t_i \cdot n \otimes s_i \cdot m,
\]

where \( \alpha \in \pi, m \in M_\alpha \) and \( n \in N_\alpha \).

**Proof.** Let \( \alpha \in \pi, m \in M_\alpha \) and \( n \in N_\alpha \). Then one can easily check that the two maps \( \overline{m} = \{\overline{m}_\beta\}_{\beta \in \pi} : H \to F_\alpha(M) \) and \( \overline{n} = \{\overline{n}_\beta\}_{\beta \in \pi} : H \to F_\alpha(N) \) defined by \( \overline{m}_\beta(h) = h \cdot m \) and \( \overline{n}_\beta(h) = h \cdot n, \beta \in \pi, h \in H_\beta \), are left \( H\text{-}\pi\text{–}\text{module maps} \). In this case, \( \overline{m}_1(1) = m \) and \( \overline{n}_1(1) = n \).

Since \( c_{M,N} \) is a family of natural isomorphisms of left \( H\text{-}\pi\text{–}\text{modules} \), we have \( c_{F_\alpha(M),F_\alpha(N)}(\overline{m} \otimes \overline{n}) = (\overline{m} \otimes \overline{n})c_{H,H} \). Since \( F_\alpha \) is a strict braided monoidal functor, \( F_\alpha(c_{M,N}) = c_{F_\alpha(M),F_\alpha(N)} \), and hence \( (c_{M,N})_\alpha = F_\alpha(c_{M,N})_1 = (c_{F_\alpha(M),F_\alpha(N)})_1 \).

Thus, we have

\[
(c_{M,N})_\alpha(m \otimes n) = (c_{M,N})_\alpha(\overline{m}_1 \otimes \overline{n}_1)(1 \otimes 1) = (c_{F_\alpha(M),F_\alpha(N)})(\overline{m}_1 \otimes \overline{n}_1)(1 \otimes 1) = (c_{F_\alpha(H),F_\alpha(H)})(\overline{m}_1 \otimes \overline{n}_1)(1 \otimes 1) = ((\overline{m} \otimes \overline{n})c_{H,H})(1 \otimes 1).
\]

\[
= (\overline{m}_1 \otimes \overline{n}_1)(c_{H,H})(1 \otimes 1) = (\overline{m}_1 \otimes \overline{n}_1)(\sum_i t_i \otimes s_i)
\]

\[
= \sum_i t_i \cdot n \otimes s_i \cdot m = \tau_{M_\alpha,N_\alpha}(R \cdot (m \otimes n)).
\]
Corollary 4.7. For any $\alpha \in \pi$ and $x, y \in H_\alpha$, we have
\[
(c_{H,H})_\alpha(x \otimes y) = \tau_{H_\alpha,H_\alpha}(R(x \otimes y)) = \sum_i t_i y \otimes s_i x.
\]

Proof. It follows by putting $M = N = H$ in Lemma 4.6. □

Lemma 4.8. $R$ is an invertible element in $H_1 \otimes H_1$.

Proof. Since $(c_{H,H})_1 : H_1 \otimes H_1 \to H_1 \otimes H_1$ is a $k$-linear isomorphism, there exists an element $a \in H_1 \otimes H_1$ such that $(c_{H,H})_1(a) = 1 \otimes 1$. From Corollary 4.7, it follows that $\tau_{H_1,H_1}(Ra) = 1 \otimes 1$, and so $Ra = 1 \otimes 1$. Then $(c_{H,H})_1(aR - 1 \otimes 1) = \tau_{H_1,H_1}(R(aR - 1 \otimes 1)) = \tau_{H_1,H_1}(RaR - R) = \tau_{H_1,H_1}(R - R) = 0$, which implies that $aR - 1 \otimes 1 = 0$, since $(c_{H,H})_1$ is a $k$-linear automorphism of $H_1 \otimes H_1$, and so $aR = 1 \otimes 1$. Thus, $R$ is an invertible element in $H_1 \otimes H_1$ with $R^{-1} = a$. □

Lemma 4.9. The following equations hold in $H_1 \otimes H_1 \otimes H_1$:

1. $(\text{id} \otimes \Delta_1)(R) = R_{13}R_{12}$;
2. $(\Delta_1 \otimes \text{id})(R) = R_{13}R_{23}$.

Proof. Since $c$ is a braiding and $H \in \mathcal{H}$, we have
\[
c_{H \otimes H} = (\text{id}_H \otimes c_{H,H})(c_{H,H} \otimes \text{id}_H), \quad c_{H,H,H} = (c_{H,H} \otimes \text{id}_H)(\text{id}_H \otimes c_{H,H}),
\]
and hence
\[
(c_{H,H,H})_1 = (\text{id}_{H_1} \otimes (c_{H,H})_1)((c_{H,H})_1 \otimes \text{id}_{H_1}),
\]
\[
(c_{H \otimes H,H})_1 = ((c_{H,H})_1 \otimes \text{id}_{H_1})(\text{id}_{H_1} \otimes (c_{H,H})_1).
\]
By Lemma 4.6 (and Corollary 4.7), we have
\[
(c_{H,H \otimes H})_1(1 \otimes 1 \otimes 1) = \sum_i t_i \cdot (1 \otimes 1) \otimes s_i = \sum_i \Delta(t_i) \otimes s_i
\]
and
\[
(\text{id}_{H_1} \otimes (c_{H,H})_1)((c_{H,H})_1 \otimes \text{id}_{H_1})(1 \otimes 1 \otimes 1) = (\text{id}_{H_1} \otimes (c_{H,H})_1)(\sum_i t_i \otimes s_i \otimes 1) = \sum_i t_i \otimes t_j \otimes s_js_i.
\]
Hence $\sum_i \Delta(t_i) \otimes s_i = \sum_{i,j} t_i \otimes t_j \otimes s_j s_i$, and so $\sum_i s_i \otimes \Delta(t_i) = \sum_{i,j} s_j s_i \otimes t_i \otimes t_j$. This shows equation (1). Equation (2) can be proved similarly. □
Lemma 4.10. Let \( \alpha \in \pi \) and \( h \in H_\alpha \). Then we have
\[
\Delta^{\text{cop}}(h)R = R\Delta_{\alpha}(h).
\]

Proof. Since \( c_{H,H} \) is a left \( H-\pi \)-module map, we have
\[
\eta^{H \otimes H}_{\alpha,1}((\text{id}_{H_{\alpha}} \otimes (c_{H,H})_1))(h \otimes 1 \otimes 1) = \eta^{H \otimes H}_{\alpha,1}(h \otimes \sum_i t_i \otimes s_i) = \sum_i h_1 t_i \otimes h_2 s_i
\]
and
\[
(c_{H,H})_{\alpha} \eta^{H \otimes H}_{\alpha,1}(h \otimes 1 \otimes 1) = (c_{H,H})_{\alpha}(\sum_i h_1 \otimes h_2) = \sum_i t_i h_2 \otimes s_i h_1.
\]
Hence \( \sum_i h_1 t_i \otimes h_2 s_i = \sum_i t_i h_2 \otimes s_i h_1 \), and so \( \sum_i h_2 s_i \otimes h_1 t_i = \sum_i s_i h_1 \otimes t_i h_2 \). That is, \( \Delta^{\text{cop}}(h)R = R\Delta_{\alpha}(h) \). \( \square \)

Combining Theorems 4.4 and 4.5, one gets the following theorem.

Theorem 4.11. Let \( H = (\{H_{\alpha}\}_{\alpha \in \pi}, m, u) \) be a semi-Hopf \( \pi \)-algebra. Then \( H \) is a quasitriangular semi-Hopf \( \pi \)-algebra if and only if the category \( H_{\pi} \mathcal{M} \) is a braided monoidal category and \( F_{\alpha} \) is a strict braided monoidal functor for any \( \alpha \in \pi \).

5. Examples

In this section, we will give two examples of Hopf \( \pi \)-algebras, and consider the category of modules over them.

Let \( H = (\{H_{\alpha}\}_{\alpha \in \pi}, m, u) \) be a semi-Hopf \( \pi \)-algebra. Then \( H_1 \) is a usual bialgebra, and hence the category \( H_1 \mathcal{M} \) of the left \( H_1 \)-modules is a monoidal category as usual. Let \( V \in H_1 \mathcal{M} \). For any \( \alpha, \beta \in \pi \), let \( M_{\alpha} = H_\alpha \otimes H_1 V \) and \( \eta^{M}_{\alpha,\beta} = m_{\alpha,\beta} \otimes \text{id}_V : H_\alpha \otimes H_\beta \otimes H_1 V \rightarrow H_{\alpha \beta} \otimes H_1 V \). Then it is easy to see that \( M = \{M_{\alpha}\}_{\alpha \in \pi} \) is a left \( \pi \)-module over \( H \) with the module structure map \( \eta = \{\eta^{M}_{\alpha,\beta}\}_{\alpha,\beta \in \pi} \). Denote \( M \) by \( H \otimes H_1 V \). Let \( f : U \rightarrow V \) be a left \( H_1 \)-module map. Then \( \text{id}_H \otimes H_1 f = \{\text{id}_{H_{\alpha}} \otimes H_1 f : H_\alpha \otimes H_1 U \rightarrow H_{\alpha} \otimes H_1 V\}_{\alpha \in \pi} \) is a left \( H-\pi \)-module map. Thus, we have a functor \( F \) from \( H_1 \mathcal{M} \) to \( H \mathcal{M} \) as follows:
\[
F : H_1 \mathcal{M} \rightarrow H \mathcal{M}, \quad F(V) = H \otimes H_1 V, \quad F(f) = \text{id}_H \otimes H_1 f.
\]
where $V$ is an object of $H_1 \mathcal{M}$ and $f$ is a morphism of $H_1 \mathcal{M}$. We have another functor $G$ from $H \mathcal{M}$ to $H_1 \mathcal{M}$ as follows:

$$G: H \mathcal{M} \rightarrow H_1 \mathcal{M}; \quad G(M) = M_1, \quad F(f) = f_1,$$

where $M = \{M_\alpha\}_{\alpha \in \pi}$ is an object of $H \mathcal{M}$ and $f = \{f_\alpha\}_{\alpha \in \pi}$ is a morphism of $H \mathcal{M}$. For the unit object $K$ of the monoidal category $H \mathcal{M}$ as stated in the last two sections, $G(K) = K_1 = k$ is exactly the unit object $k$ of the monoidal category $H_1 \mathcal{M}$. For any $M, N \in H \mathcal{M}$, $G(M \otimes N) = (M \otimes N)_1 = M_1 \otimes N_1 = G(M) \otimes G(N)$. Then one can easily check that $G$ is a strict monoidal functor from $H \mathcal{M}$ to $H_1 \mathcal{M}$.

For any $H_1$-module $V$, let $\theta(V): GF(V) \rightarrow V$ be the canonical $H_1$-module isomorphism $H_1 \otimes H_1V \rightarrow V$, $h \otimes v \mapsto h \cdot v$. Then one can easily check that $\theta$ is a natural isomorphism from $GF$ to $\text{id}_{H_1 \mathcal{M}}$.

**Example 5.1.** Let $\pi$ be a cyclic group of order 2 generated by $\alpha$. Then, $\pi = \{1, \alpha\}$ with $\alpha^2 = 1$. Let $H_1$ be a 2-dimensional $k$-space with a $k$-basis $\{h_0, h_2\}$, and $H_\alpha$ a 2-dimensional $k$-space with a $k$-basis $\{h_1, h_3\}$. Define $k$-linear maps $m_{1,1}: H_1 \otimes H_1 \rightarrow H_1$ by $m_{1,1}(h_0 \otimes h_0) = m_{1,1}(h_2 \otimes h_2) = h_0$ and $m_{1,1}(h_0 \otimes h_2) = m_{1,1}(h_2 \otimes h_0) = h_2$; $m_{\alpha,\alpha}: H_\alpha \otimes H_\alpha \rightarrow H_1$ by $m_{\alpha,\alpha}(h_1 \otimes h_3) = m_{\alpha,\alpha}(h_3 \otimes h_1) = h_0$ and $m_{\alpha,\alpha}(h_1 \otimes h_1) = m_{\alpha,\alpha}(h_3 \otimes h_3) = h_2$; $m_{1,\alpha}: H_1 \otimes H_\alpha \rightarrow H_\alpha$ by $m_{1,\alpha}(h_0 \otimes h_1) = m_{1,\alpha}(h_2 \otimes h_3) = h_1$ and $m_{1,\alpha}(h_0 \otimes h_3) = m_{1,\alpha}(h_2 \otimes h_1) = h_3$; and $m_{\alpha,1}: H_\alpha \otimes H_1 \rightarrow H_\alpha$ by $m_{\alpha,1} = m_{1,\alpha} \tau_{H_\alpha,H_1}$. Define a $k$-linear map $u \rightarrow H_1$ by $u(\lambda) = \lambda h_0$, $\lambda \in k$. Then one can check that $H = (\{H_1, H_\alpha\}, m, u)$ is a $\pi$-algebra with $h_0 = 1$.

Define $k$-linear maps $\Delta_1: H_1 \rightarrow H_1 \otimes H_1$ by $\Delta(h_i) = h_i \otimes h_i$, and $\varepsilon_1: H_1 \rightarrow k$ by $\varepsilon_1(h_i) = 1$, $i = 0, 2$. Then one can see that $H_1$ is a coalgebra. Similarly, $H_\alpha$ is also a coalgebra with comultiplication and counit given by $\Delta_\alpha: H_\alpha \rightarrow H_\alpha \otimes H_\alpha$, $\Delta(h_i) = h_i \otimes h_i$, and $\varepsilon_\alpha: H_\alpha \rightarrow k$, $\varepsilon_\alpha(h_i) = 1$, $i = 1, 3$.

With the above structure, a straightforward verification shows that $H$ is a semi-Hopf $\pi$-algebra. Moreover, $H$ is a Hopf $\pi$-algebra with the antipode $S = \{S_1, S_\alpha\}$ given by

$$S_1: H_1 \rightarrow H_1, \quad h_0 \mapsto h_0, \quad h_2 \mapsto h_2;$$

$$S_\alpha: H_\alpha \rightarrow H_\alpha, \quad h_1 \mapsto h_3, \quad h_3 \mapsto h_1.$$ 

It is easy to see that $R = 1 \otimes 1$ is a (trivial) quasitriangular structure of $H$. If $\text{Char}(k) \neq 2$, then $H$ has a nontrivial quasitriangular structure as follows:

$$R = \frac{1}{2}(1 \otimes 1 + 1 \otimes h_2 + h_2 \otimes 1 - h_2 \otimes h_2).$$

Now we consider the functors $F: H_1 \mathcal{M} \rightarrow H \mathcal{M}$ and $G: H \mathcal{M} \rightarrow H_1 \mathcal{M}$ given as above. We have already shown that $G$ is a strict monoidal functor. Let $(\varphi_0)_1:$

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$K_1 = k \to F(k)_1 = H_1 \otimes_{H_1} k$, $\lambda \mapsto \lambda h_0 \otimes_{H_1} 1 = 1 \otimes_{H_1} \lambda$ be the canonical $k$-linear isomorphism, and let $(\varphi_0)_\alpha: K_\alpha = k \to F(k)_\alpha = H_\alpha \otimes_{H_1} k$ be the $k$-linear map defined by $(\varphi_0)_\alpha(\lambda) = \lambda h_1 \otimes_{H_1} 1 = h_1 \otimes_{H_1} \lambda$. Then one can easily check that $\varphi_0 = \{(\varphi_0)_1, (\varphi_0)_\alpha\}$ is a left $H$-$\pi$-module isomorphism from $K$ to $F(k)$.

Let $V, W \in H_1 \mathcal{M}$. Define $\varphi_2(V, W)_1: (F(V) \otimes F(W))_1 \to F(V \otimes W)_1$ by

$$
\varphi_2(V, W)_1((h \otimes_{H_1} v) \otimes (l \otimes_{H_1} w)) = 1 \otimes_{H_1} (h \cdot v \otimes l \cdot w),
$$

$$
h, l \in H_1, \ v \in V, \ w \in W;
$$

and $\varphi_2(V, W)_\alpha: (F(V) \otimes F(W))_\alpha \to F(V \otimes W)_\alpha$ by

$$
\varphi_2(V, W)_\alpha((h \otimes_{H_1} v) \otimes (l \otimes_{H_1} w)) = h_1 \otimes_{H_1} ((h_3 h) \cdot v \otimes (h_3 l) \cdot w),
$$

$$
h, l \in H_\alpha, \ v \in V, \ w \in W.
$$

Then a straightforward verification shows that $\varphi_2(V, W) = \{\varphi_2(V, W)_1, \varphi_2(V, W)_\alpha\}$ is a left $H$-$\pi$-module isomorphism from $F(V) \otimes F(W)$ to $F(V \otimes W)$. Moreover, one can easily check that $\varphi_2(V, W)$ is a family of natural isomorphisms of left $\pi$-modules over $H$ indexed by all couples $(V, W)$ of objects of $H_1 \mathcal{M}$. Now by a standard verification, one can check that $(F, \varphi_0, \varphi_2)$ is a monoidal functor from $H_1 \mathcal{M}$ to $H \mathcal{M}$.

We have already seen that there is a natural isomorphism $\theta: GF \to \text{id}_{H_1 \mathcal{M}}$ as given before. It is easy to check that $\theta$ is a natural monoidal isomorphism from $GF$ to $\text{id}_{H_1 \mathcal{M}}$.

Let $M = \{M_1, M_\alpha\} \in H \mathcal{M}$. Let $\sigma(M)_1: M_1 \to FG(M)_1 = H_1 \otimes_{H_1} M_1$ be the canonical left $H_1$-module isomorphism, and let $\sigma(M)_\alpha: M_\alpha \to FG(M)_\alpha = H_\alpha \otimes_{H_1} M_1$ be the $k$-linear map defined by $\sigma(M)_\alpha(m) = h_1 \otimes_{H_1} h_3 \cdot m, \ m \in M_\alpha$. Then one can check that $\sigma(M)_\alpha$ is a bijection with the inverse given by $\sigma(M)_\alpha^{-1}(h \otimes m) = h \cdot m$, where $h \in H_\alpha$ and $m \in M_1$. Now by a straightforward verification, one can check that $\sigma(M) = \{\sigma(M)_\alpha\}_{\alpha \in \pi}$ is a left $H$-$\pi$-module map, and so it is an $H$-$\pi$-module isomorphism. Moreover, $\sigma$ is a natural isomorphism from $\text{id}_{H \mathcal{M}}$ to $FG$. Then a standard verification shows that $\sigma$ is a natural monoidal isomorphism from $\text{id}_{H \mathcal{M}}$ to $FG$. This shows that $H \mathcal{M}$ and $H_1 \mathcal{M}$ are equivalent monoidal categories.

Finally, since $H_1$ is the group algebra of the cyclic group $\{1, h_2\}$ of order 2, the category $H_1 \mathcal{M}$ can be well described. When $\text{Char}(k) \neq 2$, $H_1$ is semisimple. There are only two simple $H_1$-modules $V_0$ and $V_1$ in this case. $V_0$ and $V_1$ are both one-dimensional with the actions given by $h_2 \cdot v = v$ for $v \in V_0$ and $h_2 \cdot v = -v$ for $v \in V_1$. When $\text{Char}(k) = 2$, there is a unique simple $H_1$-module $V_0$ as given above, and the regular module $H_1$ is the unique non-simple indecomposable $H_1$-module, which is projective and uniserial.

In order to give another example, we first give some properties of a semi-Hopf $\pi$-algebra.
Definition 5.1. Let $H = \{H_{\alpha}\}_{\alpha \in \pi}$ be a semi-Hopf $\pi$-algebra. A family $e = \{e_\alpha\}_{\alpha \in \pi}$ of nonzero elements with $e_\alpha \in H_\alpha$ is called a generalized idempotent if $e_\alpha e_\beta = e_{\alpha\beta}$ for all $\alpha, \beta \in \pi$. Furthermore,

1. if $e_1 = 1$, then $e$ is called a strong generalized idempotent;
2. if $\Delta_\alpha(e_\alpha) = e_\alpha \otimes e_\alpha$ for all $\alpha \in \pi$, then $e$ is called a group-like generalized idempotent;
3. if $\pi$ is abelian and $e_\alpha h = he_\alpha$ for all $\alpha, \beta \in \pi$ and $h \in H_\beta$, then $e$ is called a central generalized idempotent.

Remark 5.2. Assume that $H = \{H_{\alpha}\}_{\alpha \in \pi}$ is a semi-Hopf $\pi$-algebra and $e = \{e_\alpha\}_{\alpha \in \pi}$ is a generalized idempotent in $H$. Then the set $\{e_\alpha; \alpha \in \pi\}$ forms a group, which is isomorphic to $\pi$. If $e$ is strong, then $e_\alpha e_{\alpha^{-1}} = e_{\alpha^{-1}} e_\alpha = e_1 = 1$ for all $\alpha \in \pi$. If $e$ is group-like, then $e_\alpha(e_\alpha) = 1$ for all $\alpha \in \pi$.

Lemma 5.3. Assume that $H = \{H_{\alpha}\}_{\alpha \in \pi}$ is a semi-Hopf $\pi$-algebra and that $H$ has a strong generalized idempotent $e = \{e_\alpha\}_{\alpha \in \pi}$. Then $H\mathcal{M}$ and $H_1\mathcal{M}$ are equivalent categories.

Proof. We use the functors $F$ and $G$ given before. We have already seen that $\theta$ is a natural isomorphism from $GF$ to $\text{id}_{H_1\mathcal{M}}$.

For any $M = \{M_\alpha\}_{\alpha \in \pi} \in H\mathcal{M}$ and $\alpha \in \pi$, let $\sigma(M)_\alpha : M_\alpha \rightarrow FG(M)_\alpha = H_\alpha \otimes_{H_1} M_1$ be defined by $\sigma(M)_\alpha(m) = e_\alpha \otimes_{H_1} (e_{\alpha^{-1}} \cdot m)$, $m \in M_\alpha$. Then it is obvious that $\sigma(M)_\alpha$ is a $k$-linear map. Let $\tau(M)_\alpha : H_\alpha \otimes_{H_1} M_1 \rightarrow M_\alpha$ be the $k$-linear map defined by $\tau(M)_\alpha(h \otimes_{H_1} m) = h \cdot m$, where $h \in H_\alpha$ and $m \in M_1$. Then for any $\alpha \in \pi$, $m \in M_\alpha$, $h \in H_\alpha$ and $m' \in M_1$, we have $(\tau(M)_\alpha \sigma(M)_\alpha)(m) = \tau(M)_\alpha(e_\alpha \otimes_{H_1} (e_{\alpha^{-1}} \cdot m)) = e_\alpha \cdot (e_{\alpha^{-1}} \cdot m) = (e_\alpha e_{\alpha^{-1}}) \cdot m = 1 \cdot m = m$ and $(\sigma(M)_\alpha \tau(M)_\alpha)(h \otimes_{H_1} m') = e_\alpha \otimes_{H_1} (e_{\alpha^{-1}} \cdot (h \cdot m')) = e_\alpha \otimes_{H_1} ((e_{\alpha^{-1}} h) \cdot m') = e_\alpha e_{\alpha^{-1}} h \otimes_{H_1} m' = h \otimes_{H_1} m'$. This shows that $\sigma(M)_\alpha$ is a $k$-linear isomorphism with $\sigma(M)_\alpha^{-1} = \tau(M)_\alpha$ for all $\alpha \in \pi$. Now it is easy to see that $\tau(M) = \{\tau(M)_\alpha\}_{\alpha \in \pi}$ is a left $H$-$\pi$-module map, and so it is an isomorphism. It follows that $\sigma(M) = \{\sigma(M)_\alpha\}_{\alpha \in \pi}$ is a left $H$-$\pi$-module isomorphism from $M$ to $FG(M)$. Then it is easy to check that $\sigma(M)$ is a family of natural morphisms indexed by all objects $M$ of $H\mathcal{M}$. Therefore, $\sigma$ is a natural isomorphism from $\text{id}_{H\mathcal{M}}$ to $FG$.

Proposition 5.4. Assume that $\pi$ is abelian and that $H = \{H_{\alpha}\}_{\alpha \in \pi}$ is a semi-Hopf $\pi$-algebra with a generalized idempotent $e = \{e_\alpha\}_{\alpha \in \pi}$. If $e$ is a central, strong and group-like generalized idempotent, then $H\mathcal{M}$ and $H_1\mathcal{M}$ are equivalent monoidal categories.

Proof. Suppose that $e$ is a central, strong and group-like generalized idempotent. We use the notations introduced in the proof of Lemma 5.3.
Note that the unit object of the monoidal category $H_1\mathcal{M}$ is the trivial $H_1$-module $k$ with the action given by $h \cdot 1 = \epsilon_1(h)$, where $h \in H_1$. Hence $F(k) = H \otimes H_1$.

For any $\alpha, \beta \in \pi$, $H_\alpha = (e_\alpha e_{\alpha^{-1}})H_\alpha = e_\alpha(e_{\alpha^{-1}}H_\alpha) \subseteq e_\alpha H_1 \subseteq H_\alpha$, and hence $H_\alpha = e_\alpha H_1$. It follows that $H_\alpha$ is a free right $H_1$-module of rank one with an $H_1$-basis $e_\alpha$, since $e_{\alpha^{-1}}e_\alpha = 1$. Therefore, $H_\alpha \otimes H_1 k$ is a one-dimensional $k$-vector space with the $k$-basis $e_\alpha \otimes H_1 1$. Thus, there is a $k$-linear isomorphism $(\varphi_0)_\alpha: K_\alpha = k \rightarrow H_\alpha \otimes H_1 k$, $\lambda \mapsto \lambda e_\alpha \otimes H_1 1 = e_\alpha \otimes H_1 \lambda$ for any $\alpha \in \pi$. Now let $\alpha, \beta \in \pi$, $h \in H_\alpha$ and $\lambda \in K_\beta = k$. Then $h \cdot (\varphi_0)_\beta(\lambda) = h \cdot (e_\beta \otimes H_1 \lambda) = (e_\beta h) \otimes H_1 \lambda = (e_\alpha e_{\alpha^{-1}}h) \otimes H_1 \lambda = e_\alpha (e_{\alpha^{-1}}h) \cdot \lambda = e_\alpha \otimes H_1 \epsilon_1(e_{\alpha^{-1}}h)\lambda = e_\alpha \otimes H_1 \epsilon_{\alpha^{-1}}(e_{\alpha^{-1}}e_\alpha(h))\lambda = (\varphi_0)_\alpha\beta(e_\alpha(h))\lambda = (\varphi_0)_\alpha\beta(h \cdot \lambda)$. Thus, $\varphi_0$ is a left $H_\pi$-module isomorphism from $K$ to $F(k)$.

Let $U, V \in H_1\mathcal{M}$ and $\alpha \in \pi$. Define $\varphi_2(U, V)_\alpha: (F(U) \otimes F(V))_\alpha \rightarrow F(U \otimes V)_\alpha$ by

$$\varphi_2(U, V)_\alpha((h \otimes H_1 x) \otimes (l \otimes H_1 v)) = e_\alpha \otimes H_1 ((e_{\alpha^{-1}}h) \cdot x \otimes (e_{\alpha^{-1}}l) \cdot v),$$

where $h, l \in H_\alpha$, $x \in U$ and $v \in V$. Since $H_\alpha$ is a free right $H_1$-module of rank one with an $H_1$-basis $e_\alpha$ as stated before, it is easy to check that $\varphi_2(U, V)_\alpha$ is a $k$-linear isomorphism. Let $h, l \in H_\alpha$, $y \in H_\beta$ with $\alpha, \beta \in \pi$, $x \in U$ and $v \in V$. Then

$$y \cdot \varphi_2(U, V)_\alpha((h \otimes H_1 x) \otimes (l \otimes H_1 v))$$

$$= ye_\alpha \otimes H_1 ((e_{\alpha^{-1}}h) \cdot x \otimes (e_{\alpha^{-1}}l) \cdot v)$$

$$= e_\beta y e_{\alpha^{-1}}h \otimes H_1 ((e_{\alpha^{-1}}h) \cdot x \otimes (e_{\alpha^{-1}}l) \cdot v)$$

$$= e_\beta y \otimes H_1 ((e_{\alpha^{-1}}h) \cdot x \otimes (e_{\alpha^{-1}}l) \cdot v)$$

$$= \sum e_\beta e_\gamma \otimes H_1 (((e_{\alpha^{-1}}h) \cdot x) \otimes (e_{\alpha^{-1}}l) \cdot v)$$

$$= \sum e_\beta e_\gamma \otimes H_1 (((e_{\alpha^{-1}}h) \cdot x) \otimes (e_{\alpha^{-1}}l) \cdot v)$$

$$= \sum e_\beta e_\gamma \otimes H_1 (((e_{\alpha^{-1}}h) \cdot x) \otimes (e_{\alpha^{-1}}l) \cdot v)$$

$$= \varphi_2(U, V)_\beta(y \cdot ((h \otimes H_1 x) \otimes (l \otimes H_1 v))).$$

It follows that $\varphi_2(U, V)$ is a left $H_\pi$-module isomorphism. A straightforward verification shows that $\varphi_2(U, V)$ is a family of natural isomorphisms of left $H_\pi$-modules indexed by all couples $(U, V)$ of objects of $H_1\mathcal{M}$.

Let $U, V, W \in H_1\mathcal{M}$ and $\alpha \in \pi$. For any $h, l, s \in H_\alpha$, $x \in U$, $v \in V$ and $w \in W$, we have

$$((\varphi_2(U, V \otimes W)_\alpha (\id_{F(U)_\alpha} \otimes \varphi_2(V, W)_\alpha a_\alpha))((h \otimes H_1 x) \otimes (l \otimes H_1 v)) \otimes (s \otimes H_1 w))$$

$$= (\varphi_2(U, V \otimes W)_\alpha (\id_{F(U)_\alpha} \otimes \varphi_2(V, W)_\alpha a_\alpha))((h \otimes H_1 x) \otimes ((l \otimes H_1 v) \otimes (s \otimes H_1 w)))$$

$$= \varphi_2(U, V \otimes W)_\alpha ((h \otimes H_1 x) \otimes (e_\alpha \otimes H_1 ((e_{\alpha^{-1}}l) \cdot v) \otimes (e_{\alpha^{-1}}s) \cdot w))).$$
Therefore, for any objects $U, V, W$ and $H$ can show that $F$ maps $(F \alpha \beta)$

Assume that $G$ is a strict monoidal functor from $\alpha$ and $\beta$.

We have

$$(F(a) \varphi_2(U \otimes V, W)_\alpha(\varphi_2(U, V)_\alpha \otimes \text{id}_{F(W)}))((h \otimes H_1 \times) \otimes (l \otimes H_1 \cdot v)) \otimes (s \otimes H_1 \cdot w))

= (F(a) \varphi_2(U \otimes V, W)_\alpha((e_\alpha \otimes H_1 \cdot ((e_\alpha - l) \cdot x \otimes (e_\alpha - l) \cdot v)) \otimes (s \otimes H_1 \cdot w))

= F(a)(e_\alpha \otimes H_1 \cdot ((e_\alpha - l) \cdot x \otimes (e_\alpha - l) \cdot v) \otimes (e_\alpha - l) \cdot w))

= e_\alpha \otimes H_1 \cdot ((e_\alpha - l) \cdot x \otimes ((e_\alpha - l) \cdot v) \otimes (e_\alpha - l) \cdot w)).

Therefore, for any objects $U, V, W$ of $H, M$, we have

$$
\varphi_2(U, V \otimes W)(\text{id}_{F(U)} \otimes \varphi_2(V, W))_\alpha F(U, V, F(W)) = F(a, U, V, W)(\varphi_2(U, V) \otimes \text{id}_{F(W)}).
$$

For any $h \in H_\alpha, v \in V$ and $\lambda \in K_\alpha = k$ with $\alpha \in \pi$, we have

$$(F(l_V)_\alpha \varphi_2(k, V)_\alpha(\varphi_0)_\alpha \otimes \text{id}_{F(V)})(\lambda \otimes (h \otimes H_1 \cdot v))

= (F(l_V)_\alpha \varphi_2(k, V)_\alpha((e_\alpha \otimes H_1 \cdot \lambda) \otimes (h \otimes H_1 \cdot v))

= F(l_V)_\alpha(e_\alpha \otimes H_1 \cdot ((e_\alpha - l) \cdot \lambda) \otimes (e_\alpha - l) \cdot v))

= F(l_V)_\alpha(e_\alpha \otimes H_1 \cdot (\lambda(e_\alpha - l) \cdot v))

= e_\alpha \lambda e_\alpha - l \otimes H_1 \cdot v

= \lambda(h \otimes H_1 \cdot v)

= (l_{F(V)})_\alpha(\lambda \otimes (h \otimes H_1 \cdot v)).

Hence $F(l_V) \varphi_2(k, V)(\varphi_0 \otimes \text{id}_{F(V)}) = l_{F(V)}$ for any object $V$ of $H, M$. Similarly, one can show that $F(r_V) \varphi_2(V, k)(\text{id}_{F(V)} \otimes \varphi_0) = r_{F(V)}$ for any object $V$ of $H, M$. Thus, we have proved that $(F, \varphi_0, \varphi_2)$ is a monoidal functor.

Note that $G$ is a strict monoidal functor from $H, M$ to $H, M$ as stated before.

Finally, a straightforward verification shows that $\theta$ is a natural monoidal isomorphism from $GF$ to id$_{H, M}$, and $\sigma$ is a natural monoidal isomorphism from id$_{H, M}$ to FG. Hence $H, M$ and $H, M$ are equivalent monoidal categories.

**Example 5.2.** Assume that $\text{Char}(k) \neq 2$. Let $\pi$ be any group. For any $\alpha \in \pi$, let $H_\alpha$ be a 4-dimensional vector space with a $k$-basis $\{e_\alpha, g_\alpha, h_\alpha, x_\alpha\}$. Define $k$-linear maps $\Delta_\alpha: H_\alpha \rightarrow H_\alpha \otimes H_\alpha$ and $\varepsilon_\alpha: H_\alpha \rightarrow k$ by

\[
\begin{align*}
\Delta_\alpha(e_\alpha) &= e_\alpha \otimes e_\alpha, & \Delta_\alpha(h_\alpha) &= h_\alpha \otimes g_\alpha + e_\alpha \otimes h_\alpha, \\
\Delta_\alpha(g_\alpha) &= g_\alpha \otimes g_\alpha, & \Delta_\alpha(x_\alpha) &= x_\alpha \otimes e_\alpha + g_\alpha \otimes x_\alpha, \\
\varepsilon_\alpha(e_\alpha) &= \varepsilon_\alpha(g_\alpha) = 1, & \varepsilon_\alpha(h_\alpha) &= \varepsilon_\alpha(x_\alpha) = 0.
\end{align*}
\]
Then a straightforward verification shows that \((H_\alpha, \Delta_\alpha, \varepsilon_\alpha)\) is a coalgebra over \(k\) for any \(\alpha \in \pi\).

For any \(\alpha, \beta \in \pi\), define a \(k\)-linear map \(m_{\alpha, \beta}: H_\alpha \otimes H_\alpha \rightarrow H_{\alpha \beta}\) by

\[
\begin{align*}
e_{\alpha}e_{\beta} &= e_{\alpha \beta}, \\
e_{\alpha}g_{\beta} &= g_{\alpha \beta}, \\
e_{\alpha}h_{\beta} &= h_{\alpha \beta}, \\
e_{\alpha}x_{\beta} &= x_{\alpha \beta}, \\
g_{\alpha}e_{\beta} &= g_{\alpha \beta}, \\
g_{\alpha}g_{\beta} &= e_{\alpha \beta}, \\
g_{\alpha}h_{\beta} &= x_{\alpha \beta}, \\
g_{\alpha}x_{\beta} &= h_{\alpha \beta}, \\
h_{\alpha}e_{\beta} &= h_{\alpha \beta}, \\
h_{\alpha}g_{\beta} &= -x_{\alpha \beta}, \\
h_{\alpha}h_{\beta} &= 0, \\
h_{\alpha}x_{\beta} &= 0,
\end{align*}
\]

where we denote \(m_{\alpha, \beta}(y \otimes z)\) by \(yz\) for any \(y \in H_\alpha\) and \(z \in H_\beta\). Then define a \(k\)-linear map \(u: k \rightarrow H_1\) by \(u(1) = e_1\). A tedious but standard verification shows that \(H = \langle \{H_\alpha\}_{\alpha \in \pi}, m, u \rangle\) is a \(\pi\)-algebra with \(e_1 = 1\). Moreover, one can check that \(H\) is a semi-Hopf \(\pi\)-algebra.

For any \(\alpha \in \pi\), define a \(k\)-linear map \(S_\alpha: H_\alpha \rightarrow H_{\alpha^{-1}}\) by \(S_\alpha(e_{\alpha}) = e_{\alpha^{-1}}\), \(S_\alpha(g_{\alpha}) = g_{\alpha^{-1}}\), \(S_\alpha(h_{\alpha}) = x_{\alpha^{-1}}\) and \(S_\alpha(x_{\alpha}) = -h_{\alpha^{-1}}\). Then one can check that \(H = \langle \{H_\alpha\}_{\alpha \in \pi}, m, u, S \rangle\) is a Hopf \(\pi\)-algebra.

For any \(\lambda \in k\), let

\[
R_\lambda = \frac{1}{2}(1 \otimes 1 + 1 \otimes g_1 + g_1 \otimes 1 - g_1 \otimes g_1) + \frac{1}{2}\lambda(x_1 \otimes x_1 - x_1 \otimes h_1 + h_1 \otimes x_1 + h_1 \otimes h_1).
\]

Then one can check that \(R_\lambda\) is a quasitriangular structure of \(H\) for any \(\lambda \in k\).

Let \(e = \{e_\alpha\}_{\alpha \in \pi}\). Then \(e\) is a strong group-like generalized idempotent. Now assume that \(\pi\) is abelian. Then \(e\) is central. It follows from Proposition 5.4 that \(H_1\) is a \(\pi\)-algebra. Hence there are only 4 non-isomorphic finite-dimensional indecomposable modules \(V_0, V_1, U_0\) and \(U_1\). Modules \(V_0\) and \(V_1\) are \(h_1\)-dimensional with the actions given by \(g_1 \cdot v = (-1)^i v \cdot h_1 = 0\) for all \(v \in V_i\), where \(i = 0, 1\). Modules \(U_0\) and \(U_1\) are both \(2\)-dimensional. The matrix representation \(\varphi_i: H_1 \rightarrow M_2(k)\) corresponding to \(U_i\) is given by

\[
\varphi_i(g_1) = \begin{pmatrix} (-1)^i & 0 \\ 0 & (-1)^{i-1} \end{pmatrix}, \quad \varphi_i(h_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]

where \(i = 0, 1\). Moreover, \(U_0\) and \(U_1\) are both projective and uniserial. For details, one can see [2] and [3].

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References


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