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ON THE GLOBAL REGULARITY OF N-DIMENSIONAL GENERALIZED BOUSSINESQ SYSTEM

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Abstract. We study the N-dimensional Boussinesq system with dissipation and diffusion generalized in terms of fractional Laplacians. In particular, we show that given the critical dissipation, a solution pair remains smooth for all time even with zero diffusivity. In the supercritical case, we obtain component reduction results of regularity criteria and smallness conditions for the global regularity in dimensions two and three.

Keywords: Boussinesq system; global regularity; regularity criteria; Besov space

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1. Introduction and statement of results

We study the following Boussinesq system $(B_{\alpha,\beta})$:

$$
\begin{cases}
\dfrac{\partial u}{\partial t} + (u\cdot\nabla)u + \nabla\pi + \nu\Lambda^{2\alpha}u = \theta e_N,\\[2mm]
\dfrac{\partial \theta}{\partial t} + (u\cdot\nabla)\theta + \eta\Lambda^{2\beta}\theta = 0,\\[2mm]
(u,\theta)(x,0) = (u_0,\theta_0)(x), \quad \nabla\cdot u = 0, \quad x\in\mathbb{R}^N,\ N\geqslant 2,
\end{cases}
\tag{1}
$$

where $u\colon \mathbb{R}^N\times\mathbb{R}^+\mapsto\mathbb{R}^N$ represents the velocity vector field, $\theta\colon \mathbb{R}^N\times\mathbb{R}^+\mapsto\mathbb{R}$ the temperature scalar field and $\pi\colon \mathbb{R}^N\times\mathbb{R}^+\mapsto\mathbb{R}$ the pressure scalar field in the context of thermal convection and density in the models of geophysical fluids (cf. [22]). The parameters $\nu,\eta\geqslant 0$ represent the kinematic viscosity and molecular diffusion coefficients, respectively, $e_N=(0,\dots,1)$ and $\Lambda=(-\Delta)^{1/2}$ with parameters $\alpha,\beta\geqslant 0$. Hereafter we shall denote the derivative with respect to time by ∂_t while the partial derivative with respect to the x_i-direction by ∂_i.

System (1) has caught much attention recently. The case $N = 2$ and $\nu = \eta = 0$ was investigated in [9], where the authors obtained local existence and blow-up criteria. Subsequently, in the case $N = 2$, $\nu > 0$, $\alpha = 1$ and $\eta = 0$, the authors in [19] proved the global regularity of the unique solution pair; simultaneously the author in [8] obtained the same result in this case and additionally in the case $\nu = 0$, $\eta > 0$ and $\beta = 1$. The author in [33] also showed that $\nu, \eta > 0$, $\alpha \in [1/2, 1)$ and $\beta \in (0, 1/2]$ such that $\alpha + \beta = 1$ suffice for the unique solution pair to remain smooth for all time. Finally, in [11] the authors obtained a global regularity result in the case $\nu, \eta > 0$ and $\beta > (2(1+\alpha))^{-1}$. For more recent interesting works on $B_{\alpha,\beta}$, we refer readers to [1], [2], [7], [13], [15], [16], [17], [18], [24] and references found therein.

We note that the work of [8] in particular solved the problem three in [25]. However, the global regularity issue in the case $N = 2$ is significantly different due to the fact that upon taking a curl of the first equation in (1), the vorticity formulation produces only one nonlinear term

$$\partial_t w + (u \cdot \nabla) w + \nu \Lambda^{2\alpha} w = -\partial_1 \theta, \quad w = \operatorname{curl} u,$$

whereas in dimension three, it produces two terms which makes the global regularity issue of the Navier-Stokes equation (NSE) extremely difficult. In [8], the author took an L^p-estimate, $p > 2$; however, this approach is not favourable without taking a curl as we must estimate the pressure term and that is not possible in the case $N > 3$.

For the generalized magnetohydrodynamics (MHD) system, we know from the work of [29], if $\alpha \geqslant 1/2 + N/4$, then the diffusion from the magnetic vector field must also have the power of the fractional Laplacian $\beta \geqslant 1/2 + N/4$ for the solution pair to remain smooth for all time.

In this paper we show that if $\alpha \geqslant 1/2 + N/4$, then diffusion is not necessary at all, extending the result of [8] and [19] to higher dimensions. Because the dissipation has the same power as what is sufficient for the hyper-dissipative NSE to remain smooth for all time, one may initiate the proof following the work on the hyper-dissipative NSE as we do in Proposition 3.1. The main difficulty arises thereafter in using the regularity of u to prove the regularity of θ. Simply applying a commutator estimate (see Lemma 2.3) on the θ-equation upon the H^{s_1}-estimate, one must face

$$\frac{1}{2}\partial_t \|\Lambda^{s_1}\theta\| \leqslant c(\|\nabla u\|_{L^\infty} \|\Lambda^{s_1-1}\nabla\theta\|_{L^2} + \|\Lambda^{s_1} u\|_{L^p} \|\nabla\theta\|_{L^q}) \|\Lambda^{s_1}\theta\|_{L^2},$$

where $1/p + 1/q = 1/2$, $p \in (2, \infty)$, due to the complete lack of diffusion. First, one must carefully apply Brezis-Wainger type argument on $\|\nabla u\|_{L^\infty}$ as $H^{N/2}$ is not embedded in L^∞. This implies that one must also take the H^{s_2}-estimate on u for $s_2 > 0$ sufficiently large. Second, to handle

$$\|\Lambda^{s_1} u\|_{L^p} \|\nabla\theta\|_{L^q},$$

it seems ideal to take $p = 2$, $q = \infty$, $s_1 = 1 + N/2$ to make best use of the Proposition 3.1. However, again, the $\|\nabla\theta\|_{L^\infty}$ term becomes problematic, because the Sobolev embedding $H^{N/2} \hookrightarrow L^\infty$ does not hold in general.

The difficulty here is that there is too much gap between the two powers of $\alpha = 1/2 + N/4$ and $\beta = 0$. Similar difficulty was seen in the work of [30] in which the author obtained the global regularity result of the N-dimensional logarithmically supercritical MHD system in the case

$$\alpha \geqslant \frac{1}{2} + \frac{N}{4}, \quad \beta > 0, \quad \alpha + \beta \geqslant 1 + \frac{N}{2}.$$

Indeed, the endpoint case $\alpha = 1 + N/2$, $\beta = 0$ was omitted for a technical reason. A favourable remedy to this situation is to take the H^{s_1}-estimate of θ and the H^{s_2}-estimate of u simultaneously with each parameter in an appropriate range as we do in Proposition 3.2 (see inequality (9)). Let us present our results.

Theorem 1.1. *Let $N \geqslant 3$, $\nu > 0$, $\eta = 0$, and $\alpha \geqslant 1/2 + N/4$. Suppose that $(u_0, \theta_0) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$, $s \geqslant 2 + N/2$. Then there exists a unique solution pair (u, θ) to (1); in particular,*

$$u \in L^\infty([0,\infty); H^s) \cap L^2([0,\infty); H^{s+\alpha}), \quad \theta \in L^\infty([0,\infty); H^s).$$

In the supercritical case, we obtain component reduction results for regularity criteria and smallness conditions that extend some previous results.

Theorem 1.2. *Let $N = 2$, $\nu, \eta > 0$, and $\alpha, \beta \in (0, 1/2)$. Suppose that $(u_0, \theta_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$, $s > 2$. If (u, θ) solves (1) in $[0, T]$ and*

$$\int_0^T \|\partial_1\theta\|_{L^p}\, \mathrm{d}\tau < \infty, \quad \frac{2(1-\alpha)(1-\beta)}{\alpha\beta} < p < \infty, \tag{2}$$

then there is no singularity up to time T. Moreover, if

$$\sup_{t\in[0,T]} \|\nabla u(t)\|_{L^{1/\beta}} \tag{3}$$

is sufficiently small, then there is no singularity up to time T.

Theorem 1.3. *Let $N = 2$, $\nu, \eta > 0$, and $\alpha, \beta \in (0, 1/2)$. Suppose that $(u_0, \theta_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$, $s > 2$. If (u, θ) solves* (1) *in $[0, T]$ and*

$$\int_0^T \Big\|\operatorname{div}\Big(\frac{\partial_1 u}{\partial_1 \theta}\Big)\Big\|_{L^p}^r \,\mathrm{d}\tau < \infty \tag{4}$$

for some p, r such that

$$\frac{2}{p} + \frac{2\beta}{r} \leqslant 2\beta, \quad \max\Big\{\frac{2(1-\alpha)(1-\beta)^2}{\alpha\beta^2}, \frac{2}{\beta}\Big\} < p < \infty,$$

then there is no singularity up to time T. Moreover, if

$$\sup_{t\in[0,T]} \Big\|\operatorname{div}\Big(\frac{\partial_1 u}{\partial_1 \theta}\Big)\Big\|_{L^{1/\beta}} < \infty, \tag{5}$$

then there exists a constant $c = c(\beta)$ such that $\|\theta_0\|_{L^\infty} < c$ implies that there is no singularity up to time T.

Theorem 1.4. *Let $N = 3$, $\nu, \eta > 0$, and $\alpha \in [1, 5/4)$, $\beta \in (1/4, 1/2]$ satisfy $\alpha + \beta \geqslant 3/2$. Suppose that $(u_0, \theta_0) \in H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3)$, $s \geqslant 1 + 2\alpha$. Suppose that (u, θ) solves* (1) *in $[0, T]$ and for some p, r such that $2 < p < \infty$,*

$$\int_0^T \|\partial_3 u_3\|_{L^p}^r \,\mathrm{d}\tau < \infty, \quad \frac{3}{p} + \frac{(2\alpha)}{r} \leqslant \frac{3}{p} + \Big(\frac{p-2}{p}\Big)\frac{\alpha(5+4\alpha)}{4(5-2\alpha)} \tag{6}$$

or

$$\int_0^T \|u_3\|_{L^\infty}^{8(5-2\alpha)/(5+4\alpha)} \,\mathrm{d}\tau < \infty. \tag{7}$$

Then there is no singularity up to time T.

Remark 1.1. 1) Because $B_{\alpha,\beta}$ at $\theta \equiv 0$ is the NSE, any improvement on the power of the fractional Laplacian beyond our claim in Theorem 1.1 seems very challenging.

2) In the case $N = 3$, the regularity criteria results have been obtained in [26], [27], [31], and [32] in the whole space and [14] in a bounded domain. Recently, extensions of regularity criteria results by reducing the number of vector components or directions of derivatives have been obtained for various fluid dynamics partial differential equations: [4], [5], and [21] in the case of the NSE, [6], [35] in the case of the MHD system and [34] in the case of active scalars.

3) Theorem 1.3 was inspired by the work of [28], in which the author obtained a regularity criterion of a solution to the NSE in terms of the direction of the velocity vector field. The proof of Theorem 1.4 follows the approach of [5]. In contrast to Theorem 1.4, for the MHD system it remains unknown if the regularity criteria may be reduced to one entry of the Jacobian matrix of the velocity vector field (cf. [35]).

A local existence result was obtained in [9] for the case $N = 2$ using the mollifier method (cf. [23]) and it is standard to modify it for the N-dimensional case with fractional Laplacians. In Section 2, we state the key lemmas and thereafter we prove our theorems.

2. Preliminaries

Apart from situations when the dependence of a constant becomes of significance, we denote by $A \lesssim B$ the fact that there exists a non-negative constant c of no significance such that $A \leqslant cB$.

Lemma 2.1 (cf. [10]). *Let f be a divergence-free vector field such that $\nabla f \in L^p$, $p \in (1, \infty)$. Then there exists a constant $c > 0$ such that*

$$\|\nabla f\|_{L^p} \leqslant c\frac{p^2}{p-1}\|\mathrm{curl}\, f\|_{L^p}.$$

Lemma 2.2 (cf. [12]). *Let f, $\Lambda^{2\alpha} f \in L^p$, $p \geqslant 2$, $\alpha \in [0, 1]$. Then*

$$2\int |\Lambda^\alpha(f^{p/2})|^2 \leqslant p\int |f|^{p-2} f \Lambda^{2\alpha} f.$$

Lemma 2.3 (cf. [20]). *Let f, g be smooth such that $\nabla f \in L^{p_1}$, $\Lambda^{s-1} g \in L^{p_2}$, $\Lambda^s f \in L^{p_3}$, $g \in L^{p_4}$, $p \in (1, \infty)$, $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$, $p_2, p_3 \in (1, \infty)$, $s > 0$. Then there exists a constant $c > 0$ such that*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leqslant c(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}).$$

Lemma 2.4 (cf. [21]). *Let $f \in H^2(\mathbb{R}^3)$ be smooth and divergence-free. Then*

$$\begin{aligned}\sum_{i,j=1}^{2}\int_{\mathbb{R}^3} f_i \partial_i f_j \Delta_h f_j = & \sum_{i,j=1}^{2}\frac{1}{2}\int_{\mathbb{R}^3}\partial_i f_j \partial_i f_j \partial_3 f_3 \\ & - \int_{\mathbb{R}^3}\partial_1 f_1 \partial_2 f_2 \partial_3 f_3 + \int_{\mathbb{R}^3}\partial_1 f_2 \partial_2 f_1 \partial_3 f_3.\end{aligned}$$

We will use the following inequality from [5], simplified for our presentation.

Lemma 2.5 (cf. [5]). *Let $f, g, h \in C_c^\infty(\mathbb{R}^3)$. Then there exists a constant $c > 0$ such that for $2 < \gamma < 3$ and $\nabla_h = (\partial_1, \partial_2, 0)$, a horizontal gradient vector,*

$$\left|\int_{\mathbb{R}^3} fgh\right| \leqslant c\|f\|_{L^2}^{(\gamma-1)/\gamma}\|\partial_3 f\|_{L^{2/(3-\gamma)}}^{1/\gamma}\|g\|_{L^2}^{(\gamma-2)/\gamma}\|\nabla_h g\|_{L^2}^{2/\gamma}\|h\|_{L^2}.$$

Lemma 2.6. *For $0 \leqslant p < \infty$ and $a, b \geqslant 0$,*

$$(a+b)^p \leqslant 2^p(a^p + b^p).$$

Lemma 2.7 (cf. [3]). *Let $f \in L^2(\mathbb{R}^N) \cap W^{s,p}(\mathbb{R}^N)$, where $s \in \mathbb{R}$ such that $p \in [2, \infty)$, $N/p < s$. Then there exists a constant $c = c(s, N, p) > 0$ such that*

$$\|f\|_{L^\infty} \leqslant c(\|f\|_{L^2} + \|f\|_{H^{N/2}} \log_2(2 + \|f\|_{W^{s,p}}) + 1).$$

Remark 2.1. For convenience of readers, let us give a simple proof of Lemma 2.7 in Appendix. In fact, our proof gives a slightly sharper estimate in Besov space norms. We stated Lemma 2.7 so for simplicity in direct application to our proofs.

Finally, we recall a well-known fact that for all $\eta \geqslant 0$, $\beta \in [0, 1]$ if (u, θ) solves (1), then (cf. [12])

$$\|\theta(\cdot, t)\|_{L^p} \leqslant \|\theta_0(\cdot)\|_{L^p}, \quad p \in [1, \infty].$$

3. Proofs

When ν or η is positive, we shall always assume it to be one for simplicity.

3.1. Proof of Theorem 1.1. We consider the case $\alpha = 1/2 + N/4$, as the case $\alpha > 1/2 + N/4$ may be done by a simple modification. We first prove a proposition:

Proposition 3.1. *Let $N \geqslant 3$, $\nu > 0$, $\eta = 0$, $\alpha = 1/2 + N/4$. If (u, θ) solves (1) in $[0, T]$, then*

$$\sup_{t\in[0,T]} \|\Lambda^{1/2+N/4} u(t)\|_{L^2}^2 + \int_0^T \|\Lambda^{1+N/2} u\|_{L^2}^2 \,\mathrm{d}\tau < \infty.$$

P r o o f. We take an L^2-inner product of the first equation of (1) with u and the second with θ to obtain

$$\frac{1}{2}\partial_t(\|u\|_{L^2}^2+\|\theta\|_{L^2}^2)+\|\Lambda^{1/2+N/4}u\|_{L^2}^2=\int\theta e_N\cdot u\leqslant\|\theta\|_{L^2}\|u\|_{L^2}$$
$$\leqslant\frac{1}{2}(\|\theta\|_{L^2}^2+\|u\|_{L^2}^2)$$

by Hölder's and Young's inequalities. Thus,

$$\sup_{t\in[0,T]}\|u(t)\|_{L^2}^2+\|\theta(t)\|_{L^2}^2+\int_0^T\|\Lambda^{1/2+N/4}u\|_{L^2}^2\,\mathrm{d}\tau\lesssim 1. \tag{8}$$

Now we apply $\Lambda^{1/2+N/4}$ on the first equation, take an L^2-inner product with $\Lambda^{1/2+N/4}u$ to estimate, using Lemma 2.3,

$$\begin{aligned}
&\frac{1}{2}\partial_t\|\Lambda^{1/2+N/4}u\|_{L^2}^2+\|\Lambda^{1+N/2}u\|_{L^2}^2\\
&\quad=-\int\Lambda^{1/2+N/4}((u\cdot\nabla)u)\cdot\Lambda^{1/2+N/4}u\\
&\quad\quad-u\cdot\nabla\Lambda^{1/2+N/4}u\cdot\Lambda^{1/2+N/4}u+\int\Lambda^{1/2+N/4}\theta e_N\cdot\Lambda^{1/2+N/4}u\\
&\quad\lesssim(\|\nabla u\|_{L^{4N/(N+2)}}\|\Lambda^{N/4-1/2}\nabla u\|_{L^2}\\
&\quad\quad+\|\Lambda^{1/2+N/4}u\|_{L^2}\|\nabla u\|_{L^{4N/(N+2)}})\|\Lambda^{1/2+N/4}u\|_{L^{4N/(N-2)}}\\
&\quad\quad+\|\theta\|_{L^2}\|\Lambda^{1+N/2}u\|_{L^2}.
\end{aligned}$$

With the Sobolev embeddings of $\dot H^{N/4-1/2}\hookrightarrow L^{4N/(N+2)}$ and $\dot H^{1/2+N/4}\hookrightarrow L^{4N/(N-2)}$ and Young's inequality, we further bound the last line by

$$\begin{aligned}
&c\|\Lambda^{1/2+N/4}u\|_{L^2}^2\|\Lambda^{1+N/2}u\|_{L^2}+\|\theta_0\|_{L^2}\|\Lambda^{1+N/2}u\|_{L^2}\\
&\quad\leqslant\frac{1}{2}\|\Lambda^{1+N/2}u\|_{L^2}^2+c(1+\|\Lambda^{1/2+N/4}\|_{L^2}^4).
\end{aligned}$$

Thus, absorbing the dissipative term, (8) and Gronwall's inequality complete the proof of Proposition 3.1. □

Now we consider the case $s:=2+N/2$. We fix $k\in(1+N/2,3/2+3N/4)$ so that

$$k+\frac{1}{2}+\frac{N}{4}>\frac{3}{2}+\frac{3N}{4}>s. \tag{9}$$

Let us denote

$$X(t):=\|\Lambda^k u(t)\|_{L^2}^2+\|\Lambda^{k-1/2-N/4}\theta(t)\|_{L^2}^2,\quad Y(t):=\|\Lambda^s u(t)\|_{L^2}^2+\|\Lambda^s\theta(t)\|_{L^2}^2.$$

The next proposition is the core of the proof of Theorem 1.1:

Proposition 3.2. *Let $N \geqslant 3$, $\nu > 0$, $\eta = 0$, $\alpha = 1/2 + N/4$. If (u, θ) solves* (1) *in* $[0, T]$, *then*

$$\sup_{t \in [0,T]} (\|\Lambda^k u(t)\|_{L^2}^2 + \|\Lambda^{k-1/2-N/4}\theta(t)\|_{L^2}^2) + \int_0^T \|\Lambda^{k+1/2+N/4} u\|_{L^2}^2 \, d\tau < \infty.$$

P r o o f. We apply Λ^k on the first equation of (1), take an L^2-inner product with $\Lambda^k u$ to estimate, by Lemma 2.3,

$$\begin{aligned}
&\frac{1}{2}\partial_t \|\Lambda^k u\|_{L^2}^2 + \|\Lambda^{k+1/2+N/4} u\|_{L^2}^2 \\
&= -\int \Lambda^k((u\cdot\nabla)u)\cdot\Lambda^k u - \int \nabla\Lambda^k\pi\cdot\Lambda^k u + \int \Lambda^k\theta e_N \cdot \Lambda^k u \\
&\lesssim (\|\nabla u\|_{L^{4N/(N+2)}}\|\Lambda^{k-1}\nabla u\|_{L^2} + \|\Lambda^k u\|_{L^2}\|\nabla u\|_{L^{4N/(N+2)}})\|\Lambda^k u\|_{L^{4N/(N-2)}} \\
&\quad + \|\Lambda^{k-1/2-N/4}\theta\|_{L^2}\|\Lambda^{k+1/2+N/4}u\|_{L^2} \\
&\lesssim \|\Lambda^{N/4+1/2}u\|_{L^2}\|\Lambda^k u\|_{L^2}\|\Lambda^{k+1/2+N/4}u\|_{L^2} + \|\Lambda^{k-1/2-N/4}\theta\|_{L^2}\|\Lambda^{k+1/2+N/4}u\|_{L^2} \\
&\leqslant \frac{1}{4}\|\Lambda^{k+1/2+N/4}u\|_{L^2}^2 + c(\|\Lambda^{N/4+1/2}u\|_{L^2}^2\|\Lambda^k u\|_{L^2}^2 + \|\Lambda^{k-1/2-N/4}\theta\|_{L^2}^2) \\
&\leqslant \frac{1}{4}\|\Lambda^{k+1/2+N/4}u\|_{L^2}^2 + cX(t),
\end{aligned}$$

due to $\dot{H}^{N/4-1/2} \hookrightarrow L^{4N/(N+2)}$ and $\dot{H}^{1/2+N/4} \hookrightarrow L^{4N/(N-2)}$, Young's inequality and Proposition 3.1. Next, we apply $\Lambda^{k-1/2-N/4}$ on the second equation of (1), take an L^2-inner product with $\Lambda^{k-1/2-N/4}\theta$ to estimate, by Lemma 2.3,

$$\begin{aligned}
\frac{1}{2}\partial_t\|\Lambda^{k-1/2-N/4}\theta\|_{L^2}^2 &\lesssim (\|\nabla u\|_{L^\infty}\|\Lambda^{k-3/2-N/4}\nabla\theta\|_{L^2} \\
&\quad + \|\Lambda^{k-1/2-N/4}u\|_{L^{4N/(4k-6-N)}}\|\nabla\theta\|_{L^{4N/(3N-4k+6)}})\|\Lambda^{k-1/2-N/4}\theta\|_{L^2},
\end{aligned}$$

justified due to the careful selection of the range of k. Now by the Sobolev embeddings of $\dot{H}^{3/2+3N/4-k} \hookrightarrow L^{4N/(4k-N-6)}$ and $\dot{H}^{k-3/2-N/4} \hookrightarrow L^{4N/(3N-4k+6)}$, Lemma 2.7 and Gagliardo-Nirenberg inequality we estimate

$$\begin{aligned}
&\frac{1}{2}\partial_t\|\Lambda^{k-1/2-N/4}\theta\|_{L^2}^2 \\
&\lesssim (\|\nabla u\|_{L^2} + \|\nabla u\|_{H^{N/2}}\log_2(2+\|\nabla u\|_{H^{k-1}}) + 1)\|\Lambda^{k-1/2-N/4}\theta\|_{L^2}^2 \\
&\quad + \|\Lambda^{1+N/2}u\|_{L^2}\|\Lambda^{k-1/2-N/4}\theta\|_{L^2}^2 \\
&\lesssim (\|\nabla u\|_{L^2} + \|\Lambda^{1+N/2}u\|_{L^2}\log_2(2+X(t)) + 1)X(t) \\
&\lesssim (\|u\|_{L^2}^{(N-2)/(2+N)}\|\Lambda^{1/2+N/4}u\|_{L^2}^{1-(N-2)/(2+N)} \\
&\quad + \|\Lambda^{1+N/2}u\|_{L^2}\log_2(2+X(t)) + 1)X(t).
\end{aligned}$$

Therefore, by Proposition 3.1,

$$\frac{1}{2}\partial_t\|\Lambda^{k-1/2-N/4}\theta\|_{L^2}^2 \lesssim (1+\|\Lambda^{1+N/2}u\|_{L^2}\log_2(2+X(t)))X(t).$$

In sum, absorbing the dissipative term,

$$\partial_t X(t)+\|\Lambda^{k+1/2+N/4}u\|_{L^2}^2 \lesssim (1+\|\Lambda^{1+N/2}u\|_{L^2}\log_2(2+X(t)))X(t).$$

Integrating in time, Proposition 3.1 completes the proof of Proposition 3.2. □

We now complete the proof of Theorem 1.1. We apply Λ^s on both equations, take L^2-inner products with $\Lambda^s u$ and $\Lambda^s\theta$, respectively, to estimate by Lemma 2.7

$$\begin{aligned}
&\frac{1}{2}\partial_t Y(t)+\|\Lambda^{s+1/2+N/4}u\|_{L^2}^2\\
&\lesssim \|\nabla u\|_{L^{4N/(N+2)}}\|\Lambda^s u\|_{L^2}\|\Lambda^s u\|_{L^{4N/(N-2)}}+\|\Lambda^s\theta\|_{L^2}^2+\|\Lambda^s u\|_{L^2}^2\\
&\quad+(\|\nabla u\|_{L^\infty}\|\Lambda^s\theta\|_{L^2}+\|\Lambda^s u\|_{L^2}\|\nabla\theta\|_{L^\infty})\|\Lambda^s\theta\|_{L^2}\\
&\lesssim \|\Lambda^{N/4+1/2}u\|_{L^2}\|\Lambda^s u\|_{L^2}\|\Lambda^{s+1/2+N/4}u\|_{L^2}+Y(t)\\
&\quad+(\|u\|_{L^2}+\|\Lambda^k u\|_{L^2})\|\Lambda^s\theta\|_{L^2}^2+\|\Lambda^s u\|_{L^2}(\|\theta\|_{L^2}+\|\Lambda^s\theta\|_{L^2})\|\Lambda^s\theta\|_{L^2}\\
&\lesssim \|\Lambda^s u\|_{L^2}\|\Lambda^{s+1/2+N/4}u\|_{L^2}+Y(t)+\|\Lambda^s\theta\|_{L^2}^2+\|\Lambda^s u\|_{L^2}\|\Lambda^s\theta\|_{L^2}\|\Lambda^s\theta\|_{L^2}\\
&\leqslant \frac{1}{2}\|\Lambda^{s+1/2+N/4}u\|_{L^2}^2\\
&\quad+c(\|\Lambda^s u\|_{L^2}^2+Y(t)+\|\Lambda^s\theta\|_{L^2}^2+\|\Lambda^s u\|_{L^2}(\|\Lambda^s\theta\|_{L^2}^2+\|\Lambda^s\theta\|_{L^2}^2),
\end{aligned}$$

due to $\dot{H}^{N/4-1/2}\hookrightarrow L^{4N/(N+2)}$ and $\dot{H}^{1/2+N/4}\hookrightarrow L^{4N/(N-2)}$, Propositions 3.1 and 3.2, and Young's inequality. Absorbing the dissipative term, we have

$$\partial_t Y(t)+\|\Lambda^{s+1/2+N/4}u\|_{L^2}^2 \lesssim (1+\|\Lambda^s u\|_{L^2})Y(t).$$

By the Gagliardo-Nirenberg inequality and Proposition 3.2, we have

$$\begin{aligned}
\int_0^T \|\Lambda^s u\|_{L^2}\,\mathrm{d}\tau &\lesssim \sup_{t\in[0,T]}\|u\|_{L^2}^{((k+1/2+N/4)-s)/(k+1/2+N/4)}\\
&\quad\times\int_0^T \|\Lambda^{k+1/2+N/4}u\|_{L^2}^{1-((k+1/2+N/4)-s)/(k+1/2+N/4)}\,\mathrm{d}\tau\lesssim 1,
\end{aligned}$$

justified by the careful selection of the range of k and s in (9). Thus, Gronwall's inequality implies

$$\|\Lambda^s u(t)\|_{L^2}^2+\|\Lambda^s\theta(t)\|_{L^2}^2+\int_0^T\|\Lambda^{s+1/2+N/4}u\|_{L^2}^2\,\mathrm{d}\tau\lesssim 1.$$

Any higher regularity than $2 + N/2$ follows from the Sobolev embedding; this is because for any $r \in \mathbb{R}^+$, similarly as above, by Lemma 2.3

$$\begin{aligned}\partial_t(\|\Lambda^r u\|_{L^2}^2 + \|\Lambda^r \theta\|_{L^2}^2) + \|\Lambda^{r+1/2+N/4} u\|_{L^2}^2 \\ \lesssim (\|\Lambda^s u\|_{L^2} + \|\Lambda^s \theta\|_{L^2} + 1)(\|\Lambda^r u\|_{L^2}^2 + \|\Lambda^r \theta\|_{L^2}^2).\end{aligned}$$

This completes the proof of Theorem 1.1. □

3.2. Proof of Theorem 1.2. We first prove a proposition:

Proposition 3.3. *Let* $N = 2$, $\nu, \eta > 0$, *and* $\alpha, \beta \in (0, 1/2)$. *If* (u, θ) *solves* (1) *in* $[0, T]$ *and satisfies* (2), *then*

$$\sup_{t \in [0,T]} \|w(t)\|_{L^p}^p + \int_0^T \|w\|_{L^{p/(1-\alpha)}}^p \, d\tau < \infty.$$

P r o o f. We fix p that satisfies (2) and take a curl of the first equation in (1) to obtain the vorticity equation

$$\partial_t w + (u \cdot \nabla) w + \Lambda^{2\alpha} w = -\partial_1 \theta.$$

By Lemma 2.2 and the Sobolev embedding of $\dot{H}^\alpha \hookrightarrow L^{2/(1-\alpha)}$ we obtain

$$c(p, \alpha) \|w\|_{L^{p/(1-\alpha)}}^p \leqslant 2 \||w|^{p/2}\|_{\dot{H}^\alpha}^2 \leqslant p \int (\Lambda^{2\alpha} w) |w|^{p-2} w,$$

so that multiplying the vorticity equation by $p|w|^{p-2}w$, integrating in space,

$$\partial_t \|w\|_{L^p}^p + c(p, \alpha) \|w\|_{L^{p/(1-\alpha)}}^p \leqslant -p \int \partial_1 \theta |w|^{p-2} w \leqslant p \|\partial_1 \theta\|_{L^p} \|w\|_{L^p}^{p-1}$$

by Hölder's inequality. Thus, in particular we have

$$\partial_t \|w\|_{L^p} \leqslant \|\partial_1 \theta\|_{L^p},$$

so that by (2) the proof of Proposition 3.3 is complete. □

Proposition 3.4. *Let $N=2$, $\nu,\eta>0$, and $\alpha,\beta\in(0,1/2)$. If (u,θ) solves* (1) *in $[0,T]$ and satisfies* (2), *then*

$$\sup_{t\in[0,T]}\|\nabla\theta(t)\|_{L^{p\beta/(1-\alpha)}}^{p\beta/(1-\alpha)}+\int_0^T\|\nabla\theta\|_{L^{p\beta/(1-\alpha)(1-\beta)}}^{p\beta/(1-\alpha)}\,\mathrm{d}\tau<\infty.$$

Proof. We fix p that satisfies (2) and define $q:=p\beta/(1-\alpha)$. We apply ∇ on the second equation of (1), multiply by $q|\nabla\theta|^{q-2}\nabla\theta$, integrate in space and estimate using Lemma 2.2 and the Sobolev embedding $\dot{H}^\beta\hookrightarrow L^{2/(1-\beta)}$ as before

$$\begin{aligned}\partial_t\|\nabla\theta\|_{L^q}^q+c(q,\beta)\|\nabla\theta\|_{L^{q/(1-\beta)}}^q&\lesssim\|\nabla u\|_{L^{q/\beta}}\|\nabla\theta\|_{L^{q/(1-\beta)}}\|\nabla\theta\|_{L^q}^{q-1}\\&\leqslant\frac{c(q,\beta)}{2}\|\nabla\theta\|_{L^{q/(1-\beta)}}^q+c\|w\|_{L^{q/\beta}}^{q/(q-1)}\|\nabla\theta\|_{L^q}^q,\end{aligned}$$

by Hölder's and Young's inequalities and Lemma 2.1. Thus, absorbing the diffusive term, Gronwall's and Hölder's inequalities and Proposition 3.3 imply by definition of q

$$\begin{aligned}&\sup_{t\in[0,T]}\|\nabla\theta(t)\|_{L^{p\beta/(1-\alpha)}}^{p\beta/(1-\alpha)}+\int_0^T\|\nabla\theta\|_{L^{p\beta/(1-\alpha)(1-\beta)}}^{p\beta/(1-\alpha)}\,\mathrm{d}\tau\\&\qquad\lesssim\|\nabla\theta_0\|_{L^{p\beta/(1-\alpha)}}^{p\beta/(1-\alpha)}\mathrm{e}^{\int_0^T\|w\|_{L^{p/(1-\alpha)}}^{p\beta/(p\beta-1+\alpha)}\,\mathrm{d}\tau}\\&\qquad\lesssim\|\nabla\theta_0\|_{L^{p\beta/(1-\alpha)}}^{p\beta/(1-\alpha)}\mathrm{e}^{\int_0^T\|w\|_{L^{p/(1-\alpha)}}^{p}\,\mathrm{d}\tau}\lesssim 1.\end{aligned}$$

This completes the proof of Proposition 3.4. □

We are now ready to obtain a higher regularity. On the first equation of (1), we apply Λ^s and take an L^2-inner product with $\Lambda^s u$ to estimate, using Lemma 2.3,

$$\begin{aligned}&\frac12\partial_t\|\Lambda^s u\|_{L^2}^2+\|\Lambda^{s+\alpha}u\|_{L^2}^2\\&\quad\lesssim(\|\nabla u\|_{L^{p/(1-\alpha)}}\|\Lambda^{s-1}\nabla u\|_{L^2}+\|\Lambda^s u\|_{L^2}\|\nabla u\|_{L^{p/(1-\alpha)}})\|\Lambda^s u\|_{L^{2p/(p-2(1-\alpha))}}\\&\qquad+\|\Lambda^s\theta\|_{L^2}^2+\|\Lambda^s u\|_{L^2}^2.\end{aligned}$$

By the Gagliardo-Nirenberg and Young's inequalities and Lemma 2.1, we bound the last line by

$$\begin{aligned}&c\|\nabla u\|_{L^{p/(1-\alpha)}}\|\Lambda^s u\|_{L^2}^{2-2(1-\alpha)/(p\alpha)}\|\Lambda^{s+\alpha}u\|_{L^2}^{2(1-\alpha)/(p\alpha)}+\|\Lambda^s\theta\|_{L^2}^2+\|\Lambda^s u\|_{L^2}^2\\&\qquad\leqslant\frac14\|\Lambda^{s+\alpha}u\|_{L^2}^2+c\|w\|_{L^{p/(1-\alpha)}}^{p\alpha/(p\alpha+\alpha-1)}\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s\theta\|_{L^2}^2+\|\Lambda^s u\|_{L^2}^2.\end{aligned}$$

Next, we apply Λ^s on the second equation of (1), take an L^2-inner product with $\Lambda^s\theta$ to estimate, using Lemma 2.3,

$$\frac{1}{2}\partial_t\|\Lambda^s\theta\|_{L^2}^2 + \|\Lambda^{s+\beta}\theta\|_{L^2}^2 \lesssim (\|\nabla u\|_{L^{p/(1-\alpha)}}\|\Lambda^{s-1}\nabla\theta\|_{L^{2p/(p-2(1-\alpha))}} + \|\Lambda^s u\|_{L^{2p\beta/(p\beta-2(1-\alpha)(1-\beta))}}\|\nabla\theta\|_{L^{p\beta/(1-\alpha)(1-\beta)}})\|\Lambda^s\theta\|_{L^2}.$$

Now by the Gagliardo-Nirenberg inequality we bound the last line by

$$\begin{aligned}
&c\big(\|\nabla u\|_{L^{p/(1-\alpha)}}\|\Lambda^s\theta\|_{L^2}^{1-2(1-\alpha)/(\beta p)}\|\Lambda^{s+\beta}\theta\|_{L^2}^{2(1-\alpha)/(\beta p)}\\
&\quad + \|\Lambda^s u\|_{L^2}^{(p\beta\alpha-2(1-\alpha)(1-\beta))/(p\alpha\beta)}\|\Lambda^{s+\alpha}u\|_{L^2}^{1-(p\beta\alpha-2(1-\alpha)(1-\beta))/(p\alpha\beta)}\\
&\quad \times \|\nabla\theta\|_{L^{p\beta/(1-\alpha)(1-\beta)}}\big)\|\Lambda^s\theta\|_{L^2}\\
&\lesssim \big(\|\nabla u\|_{L^{p/(1-\alpha)}}(\|\Lambda^s\theta\|_{L^2} + \|\Lambda^{s+\beta}\theta\|_{L^2})\\
&\quad + (\|\Lambda^s u\|_{L^2} + \|\Lambda^{s+\alpha}u\|_{L^2})\|\nabla\theta\|_{L^{p\beta/(1-\alpha)(1-\beta)}}\big)\|\Lambda^s\theta\|_{L^2}\\
&\leqslant \frac{1}{2}\big(\|\Lambda^{s+\beta}\theta\|_{L^2}^2 + \|\Lambda^{s+\alpha}u\|_{L^2}^2\big) + c\big(\|\nabla u\|_{L^{p/(1-\alpha)}}^2\\
&\quad + \|\nabla\theta\|_{L^{p\beta/(1-\alpha)(1-\beta)}}^2 + 1\big)\big(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s\theta\|_{L^2}^2\big),
\end{aligned}$$

due to Young's inequality. Absorbing the dissipative and diffusive terms, in sum we have by Lemma 2.1

$$\begin{aligned}
&\partial_t(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s\theta\|_{L^2}^2) + \|\Lambda^{s+\alpha}u\|_{L^2}^2 + \|\Lambda^{s+\beta}\theta\|_{L^2}^2\\
&\quad \lesssim (\|w\|_{L^{p/(1-\alpha)}}^2 + \|\nabla\theta\|_{L^{p\beta/(1-\alpha)(1-\beta)}}^2 + 1)(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s\theta\|_{L^2}^2).
\end{aligned}$$

Thus, Gronwall's inequality and Lemma 2.1 complete the first claim of Theorem 1.2. Next, we take p that satisfies (2), apply ∂_1 to the second equation of (1), multiply by $p|\partial_1\theta|^{p-2}\partial_1\theta$, integrate in space, estimate using Lemma 2.2 and the Sobolev embedding $\dot{H}^\beta \hookrightarrow L^{2/(1-\beta)}$ as before,

$$\begin{aligned}
\partial_t\|\partial_1\theta\|_{L^p}^p + c(p,\beta)\|\partial_1\theta\|_{L^{p/(1-\beta)}}^p &\leqslant -p\int \partial_1((u\cdot\nabla)\theta)|\partial_1\theta|^{p-2}\partial_1\theta\\
&\leqslant p(\|\partial_1 u_1\|_{L^{1/\beta}}\|\partial_1\theta\|_{L^{p/(1-\beta)}}^p + \|\partial_1 u_2\|_{L^{1/\beta}}\|\partial_2\theta\|_{L^{p/(1-\beta)}}\|\partial_1\theta\|_{L^{p/(1-\beta)}}^{p-1})
\end{aligned}$$

by Hölder's inequality. Similarly

$$\begin{aligned}
&\partial_t\|\partial_2\theta\|_{L^p}^p + c(p,\beta)\|\partial_2\theta\|_{L^{p/(1-\beta)}}^p\\
&\quad \leqslant p(\|\partial_2 u_1\|_{L^{1/\beta}}\|\partial_1\theta\|_{L^{p/(1-\beta)}}\|\partial_2\theta\|_{L^{p/(1-\beta)}}^{p-1} + \|\partial_2 u_2\|_{L^{1/\beta}}\|\partial_2\theta\|_{L^{p/(1-\beta)}}^p).
\end{aligned}$$

With Young's inequality, in sum we have

$$\begin{aligned}
&\partial_t(\|\partial_1\theta\|_{L^p}^p + \|\partial_2\theta\|_{L^p}^p)\\
&\quad \leqslant (\|\partial_1\theta\|_{L^{p/(1-\beta)}}^p + \|\partial_2\theta\|_{L^{p/(1-\beta)}}^p)(c\sup_{t\in[0,T]}\|\nabla u(t)\|_{L^{1/\beta}} - c(p,\beta)).
\end{aligned}$$

This implies that for $\sup_{t\in[0,T]} \|\nabla u(t)\|_{L^{1/\beta}}$ sufficiently small,

$$\int_0^T \|\partial_1\theta\|_{L^p}\,\mathrm{d}\tau \leqslant \|\partial_1\theta_0\|_{L^p}T.$$

By the first claim of Theorem 1.2 already proven, because we chose p to satisfy (2), this completes the proof of the second claim of Theorem 1.2. □

3.3. Proof of Theorem 1.3. We fix (p, r) that satisfies (4) and define $q := p\beta$. We apply ∂_1 on the second equation of (1), multiply by $q|\partial_1\theta|^{q-2}\partial_1\theta$ and integrate in space to obtain by Lemma 2.2 and the Sobolev embedding $\dot{H}^\beta \hookrightarrow L^{2/(1-\beta)}$ as before

$$\begin{aligned}
\partial_t\|\partial_1\theta\|_{L^q}^q + c(q,\beta)\|\partial_1\theta\|_{L^{q/(1-\beta)}}^q &\leqslant -q\sum_{i=1}^2\int \partial_1 u_i\partial_i\theta|\partial_1\theta|^{q-2}\partial_1\theta \\
= q(q-1)\sum_{i=1}^2\int \theta\partial_1 u_i|\partial_1\theta|^{q-2}\partial_{1i}\theta &= -q(q-1)\int\theta|\partial_1\theta|^q\mathrm{div}\Big(\frac{\partial_1 u}{\partial_1\theta}\Big) \\
\leqslant \frac{c(q,\beta)}{2}\|\partial_1\theta\|_{L^{q/(1-\beta)}}^q + c\|\theta_0\|_{L^\infty}^{q/(q-1)}&\|\partial_1\theta\|_{L^q}^q\Big\|\mathrm{div}\Big(\frac{\partial_1 u}{\partial_1\theta}\Big)\Big\|_{L^{q/\beta}}^{q/(q-1)},
\end{aligned}$$

due to Hölder's and Young's inequalities. Hence, after absorbing the diffusive term, Gronwall's inequality implies by (4)

$$\int_0^T \|\partial_1\theta\|_{L^{q/(1-\beta)}}^q\,\mathrm{d}\tau \lesssim 1.$$

By the condition on p, because

$$\frac{q}{1-\beta} \geqslant \frac{2(1-\alpha)(1-\beta)}{\alpha\beta},$$

by Theorem 1.2 this completes the proof of the first claim of Theorem 1.3. □

Next, going back, with p satisfying (2), by the same estimate as above, we obtain

$$\partial_t\|\partial_1\theta\|_{L^p}^p + c(q,\beta)\|\partial_1\theta\|_{L^{p/(1-\beta)}}^p \leqslant p(p-1)\|\theta_0\|_{L^\infty}\|\partial_1\theta\|_{L^{p/(1-\beta)}}^p\Big\|\mathrm{div}\Big(\frac{\partial_1 u}{\partial_1\theta}\Big)\Big\|_{L^{1/\beta}},$$

due to Hölder's inequality. Hence,

$$\partial_t\|\partial_1\theta\|_{L^p}^p \leqslant \Big(p(p-1)\|\theta_0\|_{L^\infty}\sup_{t\in[0,T]}\Big\|\mathrm{div}\Big(\frac{\partial_1 u}{\partial_1\theta}\Big)\Big\|_{L^{1/\beta}} - c(q,\beta)\Big)\|\partial_1\theta\|_{L^{p/(1-\beta)}}^p.$$

Therefore, there exists a constant $c = c(\beta)$ such that $\|\theta_0\|_{L^\infty} < c$ implies

$$\int_0^T \|\partial_1\theta\|_{L^p}\,\mathrm{d}\tau \leqslant \|\partial_1\theta_0\|_{L^p}T \lesssim 1.$$

By Theorem 1.2, this completes the proof of Theorem 1.3. □

3.4. Proof of Theorem 1.4. First, let us consider the case $\alpha > 1$ and leave the case $\alpha = 1$ in the Appendix. To start, we have the following bound as before

$$\sup_{t\in[0,T]} \|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + \int_0^T \|\Lambda^\alpha u\|_{L^2}^2 + \|\Lambda^\beta \theta\|_{L^2}^2 \,\mathrm{d}\tau \lesssim 1.$$

We prove a proposition:

Proposition 3.5. *Let $N = 3$, $\nu > 0$, $\eta \geqslant 0$, and $\alpha \in [1, 5/4)$, $\beta \geqslant 0$. Suppose (u, θ) solves* (1) *in* $[0, T]$ *and satisfies* (6) *or* (7). *Then*

$$\sup_{t\in[0,T]} \|\nabla u(t)\|_{L^2}^2 + \int_0^T \|\Lambda^\alpha \nabla u\|_{L^2}^2 \,\mathrm{d}\tau < \infty.$$

The proof is divided into two parts:

3.5. The $\|\nabla_h u\|_{L^2}^2$ estimate. We first take an L^2-inner product of the first equation with $-\Delta_h u$ to estimate by Hölder's inequality

$$\frac{1}{2}\partial_t \|\nabla_h u\|_{L^2}^2 + \|\Lambda^\alpha \nabla_h u\|_{L^2}^2 \lesssim \left| \int (u\cdot\nabla)u \cdot \Delta_h u \right| + \|\nabla\nabla_h u\|_{L^2}.$$

We apply the Gagliardo-Nirenberg and Young's inequalities to obtain

$$\|\nabla\nabla_h u\|_{L^2} \lesssim \|\nabla_h u\|_{L^2}^{1-1/\alpha} \|\Lambda^\alpha \nabla_h u\|_{L^2}^{1/\alpha} \leqslant \frac{1}{4}\|\Lambda^\alpha \nabla_h u\|_{L^2}^2 + c(1 + \|\nabla_h u\|_{L^2}^2).$$

By Lemma 2.4, we have

$$\left| \int (u\cdot\nabla)u \cdot \Delta_h u \right| \lesssim \int |u_3||\nabla u||\nabla\nabla_h u|.$$

Now we use Lemma 2.5 to estimate using the Gagliardo-Nirenberg inequality

$$\begin{aligned}\int |u_3||\nabla u||\nabla\nabla_h u| &\lesssim \|\partial_3 u_3\|_{L^{2/(3-\gamma)}}^{1/\gamma} \|\nabla u\|_{L^2}^{(\gamma-2)/\gamma} \|\nabla\nabla_h u\|_{L^2}^{1+2/\gamma} \\ &\lesssim \|\partial_3 u_3\|_{L^{2/(3-\gamma)}}^{1/\gamma} \|\nabla u\|_{L^2}^{(\gamma-2)/\gamma} \|\nabla_h u\|_{L^2}^{(1-1/\alpha)(1+2/\gamma)} \|\Lambda^\alpha \nabla_h u\|_{L^2}^{(1/\alpha)(1+2/\gamma)}.\end{aligned}$$

We then use Young's inequality to obtain

$$\int |u_3||\nabla u||\nabla\nabla_h u| \leqslant \frac{1}{4}\|\Lambda^\alpha \nabla_h u\|_{L^2}^2 + c(\|\partial_3 u_3\|_{L^{2/(3-\gamma)}}^{2/(\gamma-2)} \|\nabla u\|_{L^2}^2 + \|\nabla_h u\|_{L^2}^2). \tag{10}$$

Combined with the previous estimate, absorbing the dissipation term, integrating in time $[0,T]$, we obtain

$$\sup_{t\in[0,T]} \|\nabla_h u(t)\|_{L^2}^2 + \int_0^T \|\Lambda^\alpha \nabla_h u\|_{L^2}^2 \,\mathrm{d}\tau \lesssim 1 + \int_0^T \|\partial_3 u_3\|_{L^{2/(3-\gamma)}}^{2/(\gamma-2)} \|\nabla u\|_{L^2}^2 \,\mathrm{d}\tau. \tag{11}$$

Let us obtain another estimate: as before

$$\begin{aligned}\frac{1}{2}\partial_t \|\nabla_h u\|_{L^2}^2 + \|\Lambda^\alpha \nabla_h u\|_{L^2}^2 &\\ \leqslant c(\|u_3\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla_h u\|_{L^2}^{1-1/\alpha} \|\Lambda^\alpha \nabla_h u\|_{L^2}^{1/\alpha} + 1 + \|\nabla_h u\|_{L^2}^2) + \frac{1}{4}\|\Lambda^\alpha \nabla_h u\|_{L^2}^2 &\\ \leqslant c(\|u_3\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2 + \|\nabla_h u\|_{L^2}^2 + 1) + \frac{1}{2}\|\Lambda^\alpha \nabla_h u\|_{L^2}^2 &\end{aligned} \tag{12}$$

by Hölder's, Gagliardo-Nirenberg and Young's inequalities. Absorbing the dissipative term and integrating in time, we have

$$\sup_{t\in[0,T]} \|\nabla_h u(t)\|_{L^2}^2 \lesssim 1 + \int_0^T \|u_3\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2 \,\mathrm{d}\tau. \tag{13}$$

Going back to (12), this also implies

$$\int_0^T \|\Lambda^\alpha \nabla_h u\|_{L^2}^2 \,\mathrm{d}\tau \lesssim 1 + \int_0^T \|u_3\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2 \,\mathrm{d}\tau. \tag{14}$$

3.6. The $\|\nabla u\|_{L^2}^2$ estimate. Next, we estimate $\|\nabla u\|_{L^2}^2$; taking an L^2-inner product of the first equation of (1) with $-\Delta u$, we have

$$\begin{aligned}\frac{1}{2}\partial_t \|\nabla u\|_{L^2}^2 + \|\Lambda^\alpha \nabla u\|_{L^2}^2 &\lesssim \int (|u_3||\nabla u||\nabla\nabla_h u| + |\nabla_h u||\partial_3 u|^2 + |\theta||\Delta u|)\\ &= J_1 + J_2 + c\int |\theta||\Delta u|.\end{aligned}$$

First,

$$\begin{aligned}\int |\theta||\Delta u| &\lesssim \|\nabla u\|_{L^2}^{1-1/\alpha} \|\Lambda^\alpha \nabla u\|_{L^2}^{1/\alpha}\\ &\lesssim \|\nabla u\|_{L^2} + \|\Lambda^\alpha \nabla u\|_{L^2} \leqslant \frac{1}{4}\|\Lambda^\alpha \nabla u\|_{L^2}^2 + c\|\nabla u\|_{L^2}^2 + c,\end{aligned}$$

by Hölder's, Gagliardo-Nirenberg and Young's inequalities. Second,

$$\begin{aligned}J_2 = c\int |\nabla_h u||\partial_3 u|^2 &\lesssim \|\nabla_h u\|_{L^2} \|\nabla u\|_{L^4}^2\\ &\lesssim \|\nabla_h u\|_{L^2} \|\nabla u\|_{L^{6/(5-2\alpha)}}^{((4\alpha-3)/4)2} \|\Lambda^\alpha u\|_{L^6}^{((7-4\alpha)/4)2}\end{aligned}$$

by Hölder's and the Gagliardo-Nirenberg inequalities. Now the Sobolev embedding of $\dot H^{\alpha-1}\hookrightarrow L^{6/(5-2\alpha)}$ allows us to continue the estimate by

$$J_2 \lesssim \|\nabla_h u\|_{L^2}\|\Lambda^\alpha u\|_{L^2}^{((4\alpha-3)/4)2}\|\Lambda^\alpha\nabla_h u\|_{L^2}^{4/3((7-4\alpha)/4)}\|\Lambda^\alpha\nabla u\|_{L^2}^{2/3((7-4\alpha)/4)}, \tag{15}$$

where we used (cf. [6])

$$\|f\|_{L^6}\lesssim \|\nabla_h f\|_{L^2}^{2/3}\|\partial_3 f\|_{L^2}^{1/3}. \tag{16}$$

Therefore, combined with the previous estimate on J_1 from (10), we absorb the dissipative term, integrate in time and obtain by the Gagliardo-Nirenberg inequality

$$\begin{aligned}
&\|\nabla u(t)\|_{L^2}^2+\int_0^T\|\Lambda^\alpha\nabla u\|_{L^2}^2\,\mathrm{d}\tau\\
&\quad\lesssim \int_0^T\|\partial_3u_3\|_{L^{2/3-\gamma}}^{2/(\gamma-2)}\|\nabla u\|_{L^2}^2\,\mathrm{d}\tau+1\\
&\qquad+\sup_{t\in[0,T]}\|\nabla_h u(t)\|_{L^2}\biggl(\int_0^T\|\Lambda^\alpha u\|_{L^2}^{((4\alpha-3)/4)2(4/(4\alpha-3))}\,\mathrm{d}\tau\biggr)^{(4\alpha-3)/4}\\
&\qquad\times\biggl(\int_0^T\|\Lambda^\alpha\nabla_h u\|_{L^2}^{4((7-4\alpha)/4)(6/(7-4\alpha))/3}\,\mathrm{d}\tau\biggr)^{(7-4\alpha)/6}\\
&\qquad\times\biggl(\int_0^T\|\Lambda^\alpha\nabla u\|_{L^2}^{2((7-4\alpha)/4)(12/(7-4\alpha))/3}\,\mathrm{d}\tau\biggr)^{(7-4\alpha)/12}
\end{aligned}$$

due to Hölder's inequality. By (11), we have the second term bounded by

$$\begin{aligned}
&\sup_{t\in[0,T]}\|\nabla_h u\|_{L^2}\biggl(\int_0^T\|\Lambda^\alpha u\|_{L^2}^2\,\mathrm{d}\tau\biggr)^{(4\alpha-3)/4}\\
&\qquad\times\biggl(\int_0^T\|\Lambda^\alpha\nabla_h u\|_{L^2}^2\,\mathrm{d}\tau\biggr)^{(7-4\alpha)/6}\biggl(\int_0^T\|\Lambda^\alpha\nabla u\|_{L^2}^2\,\mathrm{d}\tau\biggr)^{(7-4\alpha)/12}\\
&\quad\leqslant c\biggl(1+\biggl(\int_0^T\|\partial_3u_3\|_{L^{2/(3-\gamma)}}^{2/(\gamma-2)}\|\nabla u\|_{L^2}^2\,\mathrm{d}\tau\biggr)^{4(5-2\alpha)/(5+4\alpha)}\biggr)\\
&\qquad+\frac14\biggl(\int_0^T\|\Lambda^\alpha\nabla u\|_{L^2}^2\,\mathrm{d}\tau\biggr)
\end{aligned}$$

where we used Young's inequality. Absorbing the dissipative term, we have

$$\begin{aligned}
\|\nabla u(t)\|_{L^2}^2+\int_0^T\|\Lambda^\alpha\nabla u\|_{L^2}^2\,\mathrm{d}\tau &\lesssim \int_0^T\|\partial_3u_3\|_{L^{2/(3-\gamma)}}^{2/(\gamma-2)}\|\nabla u\|_{L^2}^2\,\mathrm{d}\tau\\
&\quad+1+\biggl(\int_0^T\|\partial_3u_3\|_{L^{2/(3-\gamma)}}^{2/(\gamma-2)}\|\nabla u\|_{L^2}^2\,\mathrm{d}\tau\biggr)^{4(5-2\alpha)/(5+4\alpha)}.
\end{aligned}$$

We now estimate the last term by

$$\left(\int_0^T \|\partial_3 u_3\|_{L^{2/(3-\gamma)}}^{2/(\gamma-2)} \|\nabla u\|_{L^2}^{2(5+4\alpha)/(4(5-2\alpha))} \|\nabla u\|_{L^2}^{2(15-12\alpha)/(4(5-2\alpha))} \,\mathrm{d}\tau\right)^{4(5-2\alpha)/(5+4\alpha)}$$
$$\leqslant \left(\int_0^T \|\partial_3 u_3\|_{L^{2/(3-\gamma)}}^{(8/(\gamma-2))(5-2\alpha)/(5+4\alpha)} \|\nabla u\|_{L^2}^2 \,\mathrm{d}\tau\right)\left(\int_0^T \|\nabla u\|_{L^2}^2 \,\mathrm{d}\tau\right)^{(15-12\alpha)/(5+4\alpha)}$$

due to Hölder's inequality and thus we now have

$$\|\nabla u(t)\|_{L^2}^2 + \int_0^T \|\Lambda^\alpha \nabla u\|_{L^2}^2 \,\mathrm{d}\tau \lesssim \int_0^T \|\partial_3 u_3\|_{L^{2/(3-\gamma)}}^{2/(\gamma-2)} \|\nabla u\|_{L^2}^2 \,\mathrm{d}\tau$$
$$+ 1 + \int_0^T \|\partial_3 u_3\|_{L^{2/(3-\gamma)}}^{(8/(\gamma-2))(5-2\alpha)/(5+4\alpha)} \|\nabla u\|_{L^2}^2 \,\mathrm{d}\tau.$$

Therefore, the proof is complete in case of (6). Next, in case of (7) we estimate J_1 as done in (12) and J_2 as in (15)

$$\frac{1}{2}\partial_t \|\nabla u\|_{L^2}^2 + \|\Lambda^\alpha \nabla u\|_{L^2}^2$$
$$\leqslant c(\|u_3\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2 + \|\nabla_h u\|_{L^2}^2) + \frac{1}{2}\|\Lambda^\alpha \nabla_h u\|_{L^2}^2 + c$$
$$+ c\|\nabla_h u\|_{L^2} \|\Lambda^\alpha u\|_{L^2}^{2(4\alpha-3)/4} \|\Lambda^\alpha \nabla_h u\|_{L^2}^{4/3((7-4\alpha)/4)} \|\Lambda^\alpha \nabla u\|_{L^2}^{2/3((7-4\alpha)/4)} + c\|\nabla u\|_{L^2}^2.$$

Absorbing the dissipative term, we integrate in $[0,t]$ to obtain

$$\|\nabla u(t)\|_{L^2}^2 + \int_0^T \|\Lambda^\alpha \nabla u\|_{L^2}^2 \,\mathrm{d}\tau \lesssim \int_0^T \|u_3\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2 \,\mathrm{d}\tau + 1$$
$$+ \left(\int_0^T \|u_3\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2 \,\mathrm{d}\tau\right)^{(5-2\alpha)/3} \left(\int_0^T \|\Lambda^\alpha \nabla u\|_{L^2}^2 \,\mathrm{d}\tau\right)^{(7-4\alpha)/12} + 1$$
$$\leqslant c\left(\int_0^T \|u_3\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2 \,\mathrm{d}\tau + 1 + \left(\int_0^T \|u_3\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2 \,\mathrm{d}\tau\right)^{4(5-2\alpha)/(5+4\alpha)}\right)$$
$$+ \frac{1}{4}\int_0^T \|\Lambda^\alpha \nabla u\|_{L^2}^2 \,\mathrm{d}\tau$$

by Hölder's inequality, (13), (14) and Young's inequality as before. Absorbing the dissipative term, we obtain by Hölder's inequality

$$\|\nabla u(t)\|_{L^2}^2 + \int_0^T \|\Lambda^\alpha \nabla u\|_{L^2}^2 \,\mathrm{d}\tau \lesssim \int_0^T \|u_3\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2 \,\mathrm{d}\tau$$
$$+ \left(\int_0^T \|u_3\|_{L^\infty}^{8(5-2\alpha)/(5+4\alpha)} \|\nabla u\|_{L^2}^2 \,\mathrm{d}\tau\right)\left(\int_0^T \|\nabla u\|_{L^2}^2 \,\mathrm{d}\tau\right)^{(15-12\alpha)/(5+4\alpha)}.$$

Gronwall's inequality completes the proof of Proposition 3.5. □

We are now ready to complete the proof of Theorem 1.4.

3.7. The H^1-estimate. We apply ∇ on the second equation of (1), multiply by $\nabla\theta$ and integrate in space to obtain

$$\frac{1}{2}\partial_t\|\nabla\theta\|_{L^2}^2+\|\Lambda^\beta\nabla\theta\|_{L^2}^2=-\int\nabla((u\cdot\nabla)\theta)\cdot\nabla\theta=-\int\nabla u\cdot\nabla\theta\cdot\nabla\theta. \tag{17}$$

First, let us consider the case $\alpha+\beta=3/2$. Then, we further bound (17) by

$$\|\nabla u\|_{L^{6/(3-2\alpha)}}\|\nabla\theta\|_{L^2}\|\nabla\theta\|_{L^{6/(3-2\beta)}}\leqslant\frac{1}{2}\|\Lambda^\beta\nabla\theta\|_{L^2}^2+c\|\Lambda^\alpha\nabla u\|_{L^2}^2\|\nabla\theta\|_{L^2}^2$$

due to Hölder's inequality, the Sobolev embeddings $\dot H^\alpha\hookrightarrow L^{6/(3-2\alpha)}$, $\dot H^\beta\hookrightarrow L^{6/(3-2\beta)}$ and Young's inequality. Absorbing the diffusive term and then making use of Proposition 3.5, we see that

$$\sup_{t\in[0,T]}\|\nabla\theta(t)\|_{L^2}^2+\int_0^T\|\Lambda^\beta\nabla\theta\|_{L^2}^2\,\mathrm{d}\tau\lesssim\|\nabla\theta_0\|_{L^2}^2\mathrm{e}^{\int_0^T\|\Lambda^\alpha\nabla u\|_{L^2}^2\,\mathrm{d}\tau}\lesssim 1.$$

Next, if $\alpha+\beta>3/2$, then we use Hölder's inequality, the Sobolev embedding of $\dot H^\alpha\hookrightarrow L^{6/(3-2\alpha)}$, the Gagliardo-Nirenberg and Young's inequalities to bound (17) by

$$\begin{aligned}\|\nabla u\|_{L^{6/(3-2\alpha)}}\|\nabla\theta\|_{L^2}\|\nabla\theta\|_{L^{3/\alpha}}&\lesssim\|\Lambda^\alpha\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}^{1+(2\beta+2\alpha-3)/(2\beta)}\|\Lambda^\beta\nabla\theta\|_{L^2}^{(3-2\alpha)/(2\beta)}\\&\leqslant\frac{1}{2}\|\Lambda^\beta\nabla\theta\|_{L^2}^2+c\|\Lambda^\alpha\nabla u\|_{L^2}^{4\beta/(4\beta+2\alpha-3)}\|\nabla\theta\|_{L^2}^2.\end{aligned}$$

Absorbing the diffusive term, Gronwall's inequality and Proposition 3.5 give

$$\sup_{t\in[0,T]}\|\nabla\theta(t)\|_{L^2}^2+\int_0^T\|\Lambda^\beta\nabla\theta\|_{L^2}^2\,\mathrm{d}\tau\leqslant c\|\nabla\theta_0\|_{L^2}^2\mathrm{e}^{\int_0^T\|\Lambda^\alpha\nabla u\|_{L^2}^{4\beta/(4\beta+2\alpha-3)}\,\mathrm{d}\tau}\lesssim 1.$$

3.8. The H^2-estimate. Next, we apply Δ on the first equation of (1), take an L^2-inner product with Δu to obtain

$$\begin{aligned}&\frac{1}{2}\partial_t\|\Delta u\|_{L^2}^2+\|\Lambda^\alpha\Delta u\|_{L^2}^2\\&\quad=-\int\Delta((u\cdot\nabla)u)\cdot\Delta u-(u\cdot\nabla)\Delta u\cdot\Delta u+\Lambda^{1+\beta}\theta e_3\cdot\Lambda^{1-\beta}\Delta u.\end{aligned} \tag{18}$$

Again, let us consider the case $\alpha+\beta=3/2$ first. We bound (18) by

$$\begin{aligned}&\|\Delta((u\cdot\nabla)u)-(u\cdot\nabla)\cdot\Delta u\|_{L^2}\|\Delta u\|_{L^2}+\|\Lambda^{1+\beta}\theta\|_{L^2}\|\Lambda^{3/2+\alpha}u\|_{L^2}\\&\quad\lesssim(\|\nabla u\|_{L^{3/\alpha}}\|\Lambda\nabla u\|_{L^{6/(3-2\alpha)}}+\|\Lambda^2u\|_{L^{6/(3-2\alpha)}}\|\nabla u\|_{L^{3/\alpha}})\|\Delta u\|_{L^2}\\&\qquad+\|\Lambda^{1+\beta}\theta\|_{L^2}\|\Lambda^{2+\alpha}u\|_{L^2}^{(1+2\alpha)/(2+2\alpha)}\|\nabla u\|_{L^2}^{1/(2+2\alpha)}\end{aligned} \tag{19}$$

due to the Hölder's inequality, Lemma 2.3 and the Gagliardo-Nirenberg inequality. Furthermore, we bound (19) by

$$
\begin{aligned}
(20)\qquad & c\|\Lambda^\alpha \nabla u\|_{L^2}^{(3-2\alpha)/(2\alpha)}\|\nabla u\|_{L^2}^{(4\alpha-3)/(2\alpha)}\|\Lambda^{2+\alpha}u\|_{L^2}\|\Delta u\|_{L^2} \\
& + c\|\Lambda^{1+\beta}\theta\|_{L^2}^{2(2+2\alpha)/(3+2\alpha)} + \frac{1}{4}\|\Lambda^{2+\alpha}u\|_{L^2}^2
\end{aligned}
$$

due to Sobolev embedding of $\dot{H}^\alpha \hookrightarrow L^{6/(3-2\alpha)}$, the Gagliardo-Nirenberg and Young's inequalities. Proposition 3.5, Young's and Gronwall's inequalities give

$$\sup_{t\in[0,T]}\|\Delta u(t)\|_{L^2}^2 + \int_0^T \|\Lambda^\alpha \Delta u\|_{L^2}^2 \, d\tau \lesssim 1.$$

Next, we consider the case $\alpha+\beta > 3/2$. We continue from (18) and obtain

$$\|\Delta((u\cdot\nabla)u) - (u\cdot\nabla)\Delta u\|_{L^2}\|\Delta u\|_{L^2} + \|\Lambda^{1+\beta}\theta\|_{L^2}\|\Lambda^{1-\beta}\Delta u\|_{L^2}.$$

The estimate on the first term is the same as in (19)–(20). For the second term,

$$\|\Lambda^{1+\beta}\theta\|_{L^2}\|\Lambda^{1-\beta}\Delta u\|_{L^2} \lesssim \|\Lambda^{1+\beta}\theta\|_{L^2}\|\Lambda^{2+\alpha}u\|_{L^2}^{(2-\beta)/(1+\alpha)}\|\Lambda u\|_{L^2}^{(\alpha+\beta-1)/(1+\alpha)}$$

by the Gagliardo-Nirenberg inequality. Young's inequality further leads to

$$\|\Lambda^{1+\beta}\theta\|_{L^2}\|\Lambda^{1-\beta}\Delta u\|_{L^2} \leqslant \frac{1}{4}\|\Lambda^{2+\alpha}u\|_{L^2}^2 + c(\|\Lambda^{1+\beta}\theta\|_{L^2}^2 + \|\Lambda u\|_{L^2}^2).$$

Combined, after absorbing the dissipative term, using Proposition 3.5 and the H^1-estimate we obtain

$$\sup_{t\in[0,T]}\|\Delta u(t)\|_{L^2}^2 + \int_0^T \|\Lambda^\alpha \Delta u\|_{L^2}^2 \, d\tau \lesssim 1.$$

Next, we apply Δ on the second equation of (1), take an L^2-inner product with $\Delta\theta$ to estimate by Lemma 2.3

$$
\begin{aligned}
(21)\qquad & \frac{1}{2}\partial_t\|\Delta\theta\|_{L^2}^2 + \|\Lambda^{2+\beta}\theta\|_{L^2}^2 \\
& \lesssim (\|\nabla u\|_{L^{3/\beta}}\|\Lambda\nabla\theta\|_{L^{6/(3-2\beta)}} + \|\Lambda^2 u\|_{L^{6/(3-2\alpha)}}\|\nabla\theta\|_{L^{3/\alpha}})\|\Delta\theta\|_{L^2}.
\end{aligned}
$$

We bound (21) by a constant multiple of

$$
\begin{aligned}
& (\|\Lambda^{2+\alpha}u\|_{L^2}^{(1-2\beta)/(2\alpha)}\|\Delta u\|_{L^2}^{(2\alpha-1+2\beta)/(2\alpha)}\|\Lambda^{2+\beta}\theta\|_{L^2} \\
& \quad + \|\Lambda^{2+\alpha}u\|_{L^2}\|\nabla\theta\|_{L^2}^{(2(\alpha+1+\beta)-3)/(2(1+\beta))}\|\Lambda^{2+\beta}\theta\|_{L^2}^{1-(2(\alpha+1+\beta)-3)/(2(1+\beta))})\|\Delta\theta\|_{L^2}
\end{aligned}
$$

due to the Gagliardo-Nirenberg inequality and the Sobolev embeddings of $\dot{H}^\beta \hookrightarrow L^{6/(3-2\beta)}$ and $\dot{H}^\alpha \hookrightarrow L^{6/(3-2\alpha)}$. Using the H^1-estimate and Young's inequality lead to

$$\begin{aligned}&\frac{1}{2}\partial_t\|\Delta\theta\|_{L^2}^2 + \|\Lambda^{2+\beta}\theta\|_{L^2}^2\\ &\qquad \leqslant \frac{1}{2}\|\Lambda^{2+\beta}\theta\|_{L^2}^2 + c\big((\|\Lambda^{2+\alpha}u\|_{L^2}+1)\|\Delta\theta\|_{L^2}^2 + (\|\Lambda^{2+\alpha}u\|_{L^2}^2+1)(\|\Delta\theta\|_{L^2}^2+1)\big).\end{aligned}$$

Hence, Gronwall's inequality completes the H^2-estimate.

3.9. The H^3-estimate. Now we apply Λ^3 on the first equation of (1), take an L^2-inner product with $\Lambda^3 u$ and estimate by Lemma 2.3 and Hölder's inequality

$$\begin{aligned}(22)\qquad \partial_t\|\Lambda^3 u\|_{L^2}^2 + \|\Lambda^{3+\alpha}u\|_{L^2}^2 &\lesssim (\|\nabla u\|_{L^{3/\alpha}}\|\Lambda^2\nabla u\|_{L^2}\\ &\quad + \|\Lambda^3 u\|_{L^2}\|\nabla u\|_{L^{3/\alpha}})\|\Lambda^3 u\|_{L^{6/(3-2\alpha)}} + \|\Lambda^{3-\alpha}\theta\|_{L^2}\|\Lambda^{3+\alpha}u\|_{L^2}.\end{aligned}$$

Using the Sobolev embedding of $\dot{H}^\alpha \hookrightarrow L^{6/(3-2\alpha)}$, the Gagliardo-Nirenberg and Young's inequalities we further bound (22) by

$$\begin{aligned}&c\|\nabla u\|_{L^{3/\alpha}}\|\Lambda^3 u\|_{L^2}\|\Lambda^{3+\alpha}u\|_{L^2} + c\|\Lambda^{2+\beta}\theta\|_{L^2}^{(2-\alpha)/(1+\beta)}\|\nabla\theta\|_{L^2}^{(\alpha+\beta-1)/(1+\beta)}\|\Lambda^{3+\alpha}u\|_{L^2}\\ &\qquad \leqslant \frac{1}{2}\|\Lambda^{3+\alpha}u\|_{L^2}^2 + c\|\nabla u\|_{L^{3/\alpha}}^2\|\Lambda^3 u\|_{L^2}^2 + c(\|\Lambda^{2+\beta}\theta\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2).\end{aligned}$$

Absorbing the dissipative term, by Gronwall's inequality we obtain

$$\sup_{t\in[0,T]}\|\Lambda^3 u(t)\|_{L^2}^2 + \int_0^T \|\Lambda^{3+\alpha}u\|_{L^2}^2\,\mathrm{d}\tau \lesssim 1.$$

Finally, we apply Λ^3 on the second equation of (1), take an L^2-inner product with $\Lambda^3\theta$ to estimate

$$\begin{aligned}(23)\qquad &\partial_t\|\Lambda^3\theta\|_{L^2}^2 + \|\Lambda^{3+\beta}\theta\|_{L^2}^2\\ &\qquad \leqslant c\big(\|\nabla u\|_{L^{3/\beta}}\|\Lambda^2\nabla\theta\|_{L^2} + \|\Lambda^3 u\|_{L^{3/\beta}}\|\nabla\theta\|_{L^2}\big)\|\Lambda^3\theta\|_{L^{6/(3-2\beta)}}.\end{aligned}$$

We consider the case $\alpha+\beta = 3/2$ first. Then, we bound (23) by

$$\begin{aligned}&c(\|\Lambda^{2+\alpha}u\|_{L^2}^{(1-2\beta)/(2\alpha)}\|\Delta u\|_{L^2}^{(2\alpha-1+2\beta)/(2\alpha)}\|\Lambda^3\theta\|_{L^2} + \|\Lambda^{3+\alpha}u\|_{L^2}\|\nabla\theta\|_{L^2})\|\Lambda^{3+\beta}\theta\|_{L^2}\\ &\lesssim \big((\|\Lambda^{2+\alpha}u\|_{L^2} + \|\Delta u\|_{L^2})\|\Lambda^3\theta\|_{L^2} + \|\Lambda^{3+\alpha}u\|_{L^2}\|\nabla\theta\|_{L^2}\big)\|\Lambda^{3+\beta}\theta\|_{L^2}\\ &\leqslant \frac{1}{2}\|\Lambda^{3+\beta}\theta\|_{L^2}^2 + c\big(\|\Lambda^{2+\alpha}u\|_{L^2}^2+1\big)\|\Lambda^3\theta\|_{L^2}^2 + c\|\Lambda^{3+\alpha}u\|_{L^2}^2\end{aligned}$$

due to Hölder's, the Gagliardo-Nirenberg and Young's inequalities. Absorbing the diffusive term, Gronwall's inequality implies the desired result. On the other hand, if $\alpha + \beta > 3/2$, then we estimate (23) by

$$\begin{aligned} &c\big(\|\Lambda^{\alpha}\nabla u\|_{L^2}^{(3-2\beta)/(2\alpha)}\|\nabla u\|_{L^2}^{(2\alpha+2\beta-3)/(2\alpha)}\|\Lambda^3\theta\|_{L^2} \\ &\qquad + \|\Lambda^{3+\alpha}u\|_{L^2}^{(3-2\beta)/(2\alpha)}\|\Lambda^3 u\|_{L^2}^{(2\alpha+2\beta-3)/(2\alpha)}\big)\|\Lambda^{3+\beta}\theta\|_{L^2} \\ &\qquad \leqslant \frac{1}{2}\|\Lambda^{3+\beta}\theta\|_{L^2}^2 + c\big(\|\Lambda^{\alpha}\nabla u\|_{L^2}^2 + 1\big)\|\Lambda^3\theta\|_{L^2}^2 + \|\Lambda^{3+\alpha}u\|_{L^2}^2 + \|\Lambda^3 u\|_{L^2}^2 \end{aligned}$$

due to the Gagliardo-Nirenberg inequality, the Sobolev embedding of $\dot{H}^{\beta} \hookrightarrow L^{6/(3-2\beta)}$ and Young's inequality. Absorbing the diffusive term, Gronwall's inequality implies the desired result.

In dimension three, by the Sobolev embedding this implies any higher regularity.

4. Appendix

Now we sketch the proof of Theorem 1.4 in the case $\alpha = 1, \beta = 1/2$ because it is similar. First we can prove an analogous version of Proposition 3.5:

Proposition 4.1. *Let $N = 3$, $\nu > 0$, $\eta \geqslant 0$, and $\alpha = 1$, $\beta \geqslant 0$. Suppose (u, θ) solves (1) in $[0, T]$ and satisfies (6) or (7). Then*

$$\sup_{t\in[0,T]} \|\nabla u(t)\|_{L^2}^2 + \int_0^T \|\Delta u\|_{L^2}^2 \,\mathrm{d}\tau < \infty.$$

4.0.1. The $\|\nabla_h u\|_{L^2}^2$-estimate. Taking an L^2-inner product of the first equation with $-\Delta_h u$ and using Lemma 2.5, one can estimate

$$\int |u_3||\nabla u||\nabla\nabla_h u| \leqslant \frac{1}{4}\|\nabla\nabla_h u\|_{L^2}^2 + c\|\partial_3 u_3\|_{L^{2/(3-\gamma)}}^{2/(\gamma-2)}\|\nabla u\|_{L^2}^2. \tag{24}$$

This leads to

$$\sup_{t\in[0,T]} \|\nabla_h u(t)\|_{L^2}^2 + \int_0^T \|\nabla\nabla_h u\|_{L^2}^2 \,\mathrm{d}\tau \lesssim \int_0^T \|\partial_3 u_3\|_{L^{2/(3-\gamma)}}^{2/(\gamma-2)}\|\nabla u\|_{L^2}^2 \,\mathrm{d}\tau + 1. \tag{25}$$

This completes the first $\|\nabla_h u\|_{L^2}^2$-estimate. The second estimate is similar:

$$\frac{1}{2}\partial_t\|\nabla_h u\|_{L^2}^2 + \|\nabla\nabla_h u\|_{L^2}^2 \leqslant c\|u_3\|_{L^\infty}^2\|\nabla u\|_{L^2}^2 + \frac{1}{2}\|\nabla\nabla_h u\|_{L^2}^2 + c. \tag{26}$$

Absorbing the dissipative term, integrating over $[0,t]$, we obtain

$$\sup_{t\in[0,T]} \|\nabla_h u(t)\|_{L^2}^2 + \int_0^T \|\nabla\nabla_h u\|_{L^2}^2 \,\mathrm{d}\tau \lesssim \int_0^T \|u_3\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2 \,\mathrm{d}\tau + 1. \tag{27}$$

4.0.2. The $\|\nabla u\|_{L^2}^2$-estimate. Next, taking an L^2-inner product of the first equation of (1) with $-\Delta u$, using (16), (24), and (25), we can obtain

$$\|\nabla u(t)\|_{L^2}^2 + \int_0^T \|\Delta u\|_{L^2}^2 \,\mathrm{d}\tau \lesssim \int_0^T \|\partial_3 u_3\|_{L^2}^{2/(\gamma-2)} \|\nabla u\|_{L^2}^2 \,\mathrm{d}\tau + \int_0^T \|\partial_3 u_3\|_{L^{2/(3-\gamma)}}^{8/(3(\gamma-2))} \|\nabla u\|_{L^2}^2 \,\mathrm{d}\tau + 1.$$

The case of (7) may be done similarly using (26) and (27). Thus, Gronwall's inequality completes the proof of Proposition 4.1. Extension to higher regularity may be done similarly as before, taking the H^1, H^2 and H^3-estimates.

4.1. Proof of Lemma 2.7. We recall the notion of Besov spaces (cf. [10]). We denote by $\mathcal{S}(\mathbb{R}^N)$ the Schwartz class functions and by $\mathcal{S}'(\mathbb{R}^N)$ its dual. We define

$$\mathcal{S}_0 = \left\{\varphi \in \mathcal{S}, \int_{\mathbb{R}^N} \varphi(x) x^\gamma \,\mathrm{d}x = 0, \quad |\gamma| = 0,1,2,\dots\right\}.$$

Its dual $\mathcal{S}_0'$ is given by $\mathcal{S}_0' = \mathcal{S}/\mathcal{S}_0^\perp = \mathcal{S}'/\mathcal{P}$, where $\mathcal{P}$ is the space of polynomials. For $j \in \mathbb{Z}$ we define $A_j = \{\xi \in \mathbb{R}^N\colon\ 2^{j-1} < |\xi| < 2^{j+1}\}$, $\{\Phi_j\} \in \mathcal{S}(\mathbb{R}^N)$ such that $\operatorname{supp}\hat{\Phi}_j \subset A_j$, $\hat{\Phi}_j(\xi) = \hat{\Phi}_0(2^{-j}\xi)$ or $\Phi_j(x) = 2^{jN}\Phi_0(2^j x)$ and $\Psi \in C_0^\infty(\mathbb{R}^N)$ is even such that

$$1 = \hat{\Psi}(\xi) + \sum_{k=0}^\infty \hat{\Phi}_k(\xi), \quad \Psi * f + \sum_{k=0}^\infty \Phi_k * f = f \tag{28}$$

for any $f \in \mathcal{S}'$. With that, we set $\Delta_j f = 0$ if $j \leqslant -2$ and otherwise

$$\Delta_{-1} f = \Psi * f, \quad \Delta_j f = \Phi_j * f, \quad \text{if } j = 0,1,2,\dots,$$

and define for $s \in \mathbb{R}$, $1 \leqslant p,\ q \leqslant \infty$, the inhomogeneous Besov space $B_{p,q}^s = \{f \in \mathcal{S}'\colon\ \|f\|_{B_{p,q}^s} < \infty\}$, where

$$\|f\|_{B_{p,q}^s} = \left(\sum_{j=-1}^\infty (2^{js}\|\Delta_j f\|_{L^p})^q\right)^{1/q}, \quad \text{if } q < \infty,$$

with standard modification if $q = \infty$. The following embeddings hold if $1 < p < \infty$:

$$B^s_{p,\min\{p,2\}} \subset W^{s,p} \subset B^s_{p,\max\{p,2\}}, \quad \dot{B}^s_{p,\min\{p,2\}} \subset \dot{W}^{s,p} \subset \dot{B}^s_{p,\max\{p,2\}}. \tag{29}$$

We estimate with $n \in \mathbb{N}$ large to be determined afterwards: writing f as in (28),

$$\begin{aligned} \|f\|_{L^\infty} &\leqslant \|\Delta_{-1} f\|_{L^\infty} + \sum_{j=0}^{n-1} \|\Delta_j f\|_{L^\infty} + \sum_{j=n}^{\infty} \|\Delta_j f\|_{L^\infty} \\ &\lesssim \|f\|_{L^2} + n\|f\|_{B^{N/2}_{2,\infty}} + \|f\|_{B^s_{p,\infty}} 2^{n(N/p-s)} \end{aligned}$$

by Young's and Bernstein's inequalities (cf. [10]). Setting $n := (s - N/p)^{-1} \log_2(2 + \|f\|_{B^s_{p,\infty}})$ and using the embedding from (29) imply the desired result.

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