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THE $\mathcal{L}_n^m$-PROPOSITIONAL CALCULUS

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(Received November 20, 2012)

Abstract. T. Almada and J. Vaz de Carvalho (2001) stated the problem to investigate if these Łukasiewicz algebras are algebras of some logic system. In this article an affirmative answer is given and the $\mathcal{L}_n^m$-propositional calculus, denoted by $\mathcal{L}_n^m$, is introduced in terms of the binary connectives $\to$ (implication), $\colon \to$ (standard implication), $\land$ (conjunction), $\lor$ (disjunction) and the unary ones $f$ (negation) and $D_i$, $1 \leq i \leq n - 1$ (generalized Moisil operators). It is proved that $\mathcal{L}_n^m$ belongs to the class of standard systems of implicative extensional propositional calculi. Besides, it is shown that the definitions of $L_n^m$-algebra and $\mathcal{L}_n^m$-algebra are equivalent. Finally, the completeness theorem for $\mathcal{L}_n^m$ is obtained.

Keywords: Łukasiewicz algebra of order $n$; $m$-generalized Łukasiewicz algebra of order $n$; equationally definable principal congruences; implicative extensional propositional calculus; completeness theorem

MSC 2010: 03G10, 06D99, 03B60

1. INTRODUCTION AND PRELIMINARIES

In 1977, generalizing De Morgan algebras by omitting the polarity condition (i.e. the law of double negation), J. Berman [2] began the study of what he called distributive lattices with an additional unary operation. Two years later, A. Urquhart in [11] named them Ockham lattices. These algebras are the algebraic counterpart of logics provided with a negation operator which satisfies De Morgan laws. Then, recall that an Ockham algebra is an algebra $\langle L, \land, \lor, f, 0, 1 \rangle$, where the reduct $\langle L, \land, \lor, 0, 1 \rangle$ is a bounded distributive lattice and $f$ is a unary operation satisfying the following conditions:

\begin{align*}
(O1) \quad & f0 = 1, \\
(O2) \quad & f1 = 0, \\
(O3) \quad & f(x \land y) = fx \lor fy, \\
(O4) \quad & f(x \lor y) = fx \land fy.
\end{align*}
Ockham algebras, which are more closely related to De Morgan algebras, are the ones that satisfy the identity \( f^{2m}x = x \) for some \( m \geq 1 \). The variety of these algebras will be denoted by \( \mathcal{K}_{m,0} \). More details on these algebras can be consulted in [3]. Furthermore, for the notions of universal algebra including De Morgan algebras and \( n \)-valued Lukasiewicz-Moisil algebras outlined in this paper we refer the reader to [4], [5].

On the other hand, in 2001, T. Almada and J. Vaz de Carvalho [1] generalized Lukasiewicz-Moisil algebras of order \( n \) by considering algebras of the same type which have a reduct in \( \mathcal{K}_{m,0} \) instead of a reduct which is a De Morgan algebra. Hence, they introduced the variety \( \mathcal{L}_n^m \) of \( m \)-generalized Lukasiewicz algebras of order \( n \) which were defined as follows:

An \( m \)-generalized Lukasiewicz algebra of order \( n \) (or \( L_n^m \)-algebra) is an algebra \( \langle A, \lor, \land, f, D_1, \ldots, D_{n-1}, 0, 1 \rangle \) of type \( (2, 2, 1, \ldots, 1, 0, 0) \) such that

\[
\begin{align*}
\text{(GL1)} & \quad (\langle A, \lor, \land, 0, 1 \rangle \text{ is a bounded distributive lattice for which } f \text{ is a dual endomorphism satisfying the identity } f^{2m}x = x, \\
\text{(GL2)} & \quad D_i\left(x \land \bigvee_{p=0}^{m-1} f^{2p}y\right) = D_i x \land D_i \left(\bigvee_{p=0}^{m-1} f^{2p}y\right), 1 \leq i \leq n-1, \\
\text{(GL3)} & \quad D_i x \lor D_j x = D_j x, 1 \leq i \leq j \leq n-1, \\
\text{(GL4)} & \quad D_i x \lor f D_i x = 1, 1 \leq i \leq n-1, \\
\text{(GL5)} & \quad D_i f \left(\bigvee_{p=0}^{m-1} f^{2p}x\right) = f D_{n-i} \left(\bigvee_{p=0}^{m-1} f^{2p}x\right), 1 \leq i \leq n-1, \\
\text{(GL6)} & \quad D_i D_j x = D_j x, 1 \leq i, j \leq n-1, \\
\text{(GL7)} & \quad x \lor D_1 x = D_1 x, \\
\text{(GL8)} & \quad D_i x = D_i \left(\bigvee_{p=0}^{m-1} f^{2p}x\right), 1 \leq i \leq n-1, \\
\text{(GL9)} & \quad (x \land f) \lor y \lor f y = y \lor f y, \\
\text{(GL10)} & \quad \bigvee_{p=0}^{m-1} f^{2p}x \leq \bigvee_{p=0}^{m-1} f^{2p}y \lor f D_i \left(\bigvee_{p=0}^{m-1} f^{2p}y\right) \lor D_{i+1} \left(\bigvee_{p=0}^{m-1} f^{2p}x\right), 1 \leq i \leq n-2.
\end{align*}
\]

From the definition it follows that the identities listed below are also verified.

**Proposition 1.1** ([1]). Let \( A \in \mathcal{L}_n^m \). Then

\[
\begin{align*}
\text{(GL11)} & \quad D_i (x \lor y) = D_i x \lor D_i y, 1 \leq i \leq n-1, \\
\text{(GL12)} & \quad f^{2m} x = D_i x, 1 \leq i \leq n-1, \\
\text{(GL13)} & \quad D_i x \land f D_i x = 0, 1 \leq i \leq n-1, \\
\text{(GL14)} & \quad f x \lor D_1 x = 1, f \left(\bigvee_{p=0}^{m-1} f^{2p}x\right) \land D_{n-1} \left(\bigvee_{p=0}^{m-1} f^{2p}x\right) = 0, \\
\text{(GL15)} & \quad \bigvee_{p=0}^{m-1} f^{2p}x \lor D_{n-1} \left(\bigvee_{p=0}^{m-1} f^{2p}x\right) = D_{n-1} \left(\bigvee_{p=0}^{m-1} f^{2p}x\right), \\
\text{(GL16)} & \quad D_i 0 = 0, D_i 1 = 1, 1 \leq i \leq n-1.
\end{align*}
\]
Let \( A \in \mathcal{L}_n^m \). The set \( S_A = \{ x \in A : f^2 x = x \} = \left\{ x \in A : \bigvee_{p=0}^{m-1} f^{2p} x = x \right\} \) plays an important role in the study of these algebras. In particular, as a direct consequence of (GL_8) it follows that in \( L_n^m \)-algebras the operations \( D_i, 1 \leq i \leq n-1 \) are determined by its restrictions to \( S_A \). Besides, \( S_A \) is a subalgebra of \( A \) and it is the greatest subalgebra of \( A \) that belongs to the variety of \( L_n \)-algebras ([1], Proposition 2.2).

In addition to the properties (GL_11) through (GL_16), we show other ones that will be useful throughout this paper.

**Proposition 1.2 ([7]).** Let \( A \in \mathcal{L}_n^m \). Then the following properties are verified:

\[(g_1)\] \( D_j f D_i x = f D_i x, 1 \leq i, j \leq n-1, \)

\[(g_2)\] \( f D_i x \) is the Boolean complement of \( D_i x, 1 \leq i \leq n-1, \)

\[(g_3)\] \( D_i x \leq D_i y \) if and only if \( f D_i x \lor D_i y = 1, 1 \leq i \leq n-1, \)

\[(g_4)\] \( D_j (D_i x \land D_i y) = D_i x \land D_i y, 1 \leq i, j \leq n-1, \)

\[(g_5)\] \( x \land f D_1 x = 0, \)

\[(g_6)\] \( (f D_i x \land f D_i y) \lor (D_i x \land D_i y) = (D_i x \lor f D_i y) \land (D_i y \lor f D_i x), 1 \leq i \leq n-1, \)

\[(g_7)\] \( z \in S_A \) implies \( D_i (x \land z) = D_i x \land D_i z, 1 \leq i \leq n-1. \)

Bearing in mind some unpublished results established by M. Sequeira in the context of congruences on algebras of certain subvarieties of Ockham algebras some of which are \( K_{m,0} \), J. Vaz de Carvalho considered certain elements which we will describe in what follows.

Let \( A \in \mathcal{L}_n^m \) and \( T = \{ 0, 1, \ldots, m-1 \} \). For each \( z \in A \) and \( s \in \{1, \ldots, m\} \) take

\[ q_s z = \bigwedge_{J \subseteq T, |J| = s} \bigvee_{j \in J} f^{2j} z. \]

The same author asserted that it is straightforward to see the following statements.

**Lemma 1.1 ([12]).** Let \( A \in \mathcal{L}_n^m \). Then

\[(i)\] \( f^2 q_s z = q_s z, s \in \{1, \ldots, m\}, \)

\[(ii)\] \( q_s z \leq q_{s+1} z, s \in \{1, \ldots, m-1\}, \)

\[(iii)\] \( q_1 z = \bigwedge_{p=0}^{m-1} f^{2p} z \) and \( \bigvee_{p=0}^{m-1} f^{2p} z = q_m z, \)

\[(iv)\] \( z \in S_A \) implies \( q_s z = z, s \in \{1, \ldots, m\}, \)

\[(v)\] \( x \leq z \) implies \( q_s x \leq q_s z, s \in \{1, \ldots, m\}. \)

On the other hand, in [7], we introduced a new binary operation \( \rightarrow \) on \( L_n^m \)-algebras, called weak implication, as follows:

\[ x \rightarrow y = D_1 f x \lor y. \]
The deductive systems associated with this implication enable us to establish
an isomorphism between the congruence lattice of an \( m \)-generalized \( \wedge \) Lukasiewicz
algebra \( A \) of order \( n \) and the lattice of all the deductive systems of \( A \). This result
turns out to be quite useful for characterizing the principal congruences on these
algebras. Furthermore, it is worth noting that from this operation the one considered
by R. Cignoli [6] for \( L_n \)-algebras is deduced.

**Proposition 1.3.** Let \( A \in \mathcal{L}_n^m \). Then the following statements hold:

(W\(_1\)) \( x \to 1 = 1 \),
(W\(_2\)) \( x \to x = 1 \),
(W\(_3\)) \( 1 \to x = x \),
(W\(_4\)) \( x \to (y \to x) = 1 \),
(W\(_5\)) \( x \leq y \) implies \( x \to y = 1 \),
(W\(_6\)) \( x \to (y \to z) = (x \to y) \to (x \to z) \),
(W\(_7\)) \( x \to (x \land y) = x \to y \),
(W\(_8\)) \( (x \to y) \to ((x \to z) \to (y \to (x \land z))) = 1 \),
(W\(_9\)) \( (x \land y) \to z = x \to (y \to z) \),
(W\(_10\)) \( D_i x \to D_i y = f D_i x \lor D_i y, 1 \leq i \leq n-1 \),
(W\(_11\)) \( D_i x \to D_i y = 1 \) if and only if \( D_i x \leq D_i y, 1 \leq i \leq n-1 \),
(W\(_12\)) \( D_i q_s(x \vee y) \to D_i q_s(x \land y) = 1 \) if and only if \( x = y, 1 \leq i \leq n-1, 1 \leq s \leq m \),
(W\(_13\)) \( ((x \land z) \to (y \land z)) \to (z \to (x \to y)) = 1 \),
(W\(_14\)) \( D_i q_s x \to D_1 x = 1, 1 \leq i \leq n-1, 1 \leq s \leq m \),
(W\(_15\)) \( D_i q_s(x \land f x) \to D_i q_s((x \land f x) \land (y \land f y)) = 1, 1 \leq i \leq n-1, 1 \leq s \leq m \),
(W\(_16\)) \( D_i q_s \left( \bigvee_{p=0}^{m-1} f^{2p} x \right) \to D_i q_s \left( \left( \bigvee_{p=0}^{m-1} f^{2p} x \right) \land \left( \bigvee_{p=0}^{m-1} f^{2p} y \right) \right) \lor \)
\( \bigvee_{p=0}^{m-1} f^{2p} x \right) = 1, 1 \leq i \leq n-1, 1 \leq s \leq m \),
(W\(_17\)) \( D_i q_s x \to D_i q_s(x \land f^{2m-1}(f x \land f y)) = 1, 1 \leq i \leq n-1, 1 \leq s \leq m \),
(W\(_18\)) \( D_i q_s(x \land f^{2m-1}(f y \land f z) \lor f^{2m-1}(f(z \land x) \land f(y \land x))) \to \)
\( D_i q_s(x \land f^{2m-1}(f y \land f z) \lor f^{2m-1}(f(z \land x) \land f(y \land x))) = 1, 1 \leq i \leq n-1, \)
\( 1 \leq s \leq m \),
(W\(_19\)) \( x \land \bigwedge_{s=1}^{m} \bigwedge_{i=1}^{n-1} (D_i q_s(x \lor y) \to D_i q_s(x \land y)) = y \land \bigwedge_{s=1}^{m} \bigwedge_{i=1}^{n-1} (D_i q_s(x \lor y) \to D_i q_s(x \land y)) \),
(W\(_20\)) \( D_j q_s x \land \bigwedge_{s=1}^{m} \bigwedge_{i=1}^{n-1} (D_i q_s(x \lor y) \to D_j q_s(x \land y)) = D_j q_k y \land \bigwedge_{s=1}^{m} \bigwedge_{i=1}^{n-1} (D_i q_s(x \lor y) \to \)
\( D_i q_s(x \land y)), 1 \leq j \leq n-1, 1 \leq k \leq m \),
(W\(_21\)) \( D_{n-1} q_1 x \land \bigwedge_{s=1}^{m} \bigwedge_{i=1}^{n-1} (D_i q_s(x \lor y) \to D_j q_s(x \land y)) = \)
\( D_{n-1} q_1 y \land \bigwedge_{s=1}^{m} \bigwedge_{i=1}^{n-1} (D_i q_s(x \lor y) \to D_i q_s(x \land y)). \)
Proof. We will only prove (W₁₂), (W₁₃), (W₁₄), (W₁₅), (W₁₈), (W₁₉) and (W₂₀) since the proof of the remaining properties is routine.

(W₁₂): It is a direct consequence of [12], Proposition 4.2.

(W₁₃): From (W₀) and (W₆) we have that \( ((x \land z) \to (y \land z)) \to (z \to (x \to y)) = ((x \land z) \to (y \land z)) \to ((x \land z) \to y) \). Hence, by (W₅) and (W₁) we conclude that \((x \land z) \to (y \land z)) \to (z \to (x \to y)) = (x \land z) \to 1 = 1.\)

(W₁₄): From (ii) in Lemma 1.1 and (GL₁₁) we infer that \( D_i q_s x \leq D_i q_m x, 1 \leq i \leq n - 1, 1 \leq s \leq m. \) On the other hand, by (iii) in Lemma 1.1, (GL₆) and (GL₃) we have that \( D_i q_m x = D_i x \leq D_1 x, 1 \leq i \leq n - 1 \) and so, by (W₁₁) we conclude that \( D_i q_s x \to D_1 x = 1, 1 \leq i \leq n - 1, 1 \leq s \leq m. \)

(W₁₅): From (GL₀) we have that \( x \land f x = (x \land f x) \land (y \lor f y) \) and so, \( D_i q_s (x \land f x) = D_i q_s ((x \land f x) \land (y \lor f y)), 1 \leq i \leq n - 1, 1 \leq s \leq m. \) Hence, by (W₂) we conclude the proof.

(W₁₈): It is a direct consequence of the fact that \( x \land f^{2m-1}(f y \land f z) = f^{2m-1}(f(z \land x)) \land f(y \land x) \) and (W₂).

(W₁₉): By virtue of (g₁) and the definition of the weak implication we have that \( D_i q_s (x \land y) \lor f D_i q_s (x \land y) = D_i q_s (x \land y) \lor D_1 f D_i q_s (x \lor y) = D_i q_s (x \land y) \rightarrow D_i q_s (x \land y), 1 \leq i \leq n - 1, 1 \leq s \leq m \) and so, by [12], Proposition 3.5, we conclude that \( x \land \bigwedge_{i=1}^{n-1} (D_i q_s (x \lor y) \rightarrow D_i q_s (x \land y)) = y \land \bigwedge_{s=1}^{m-1} (D_i q_s (x \lor y) \rightarrow D_i q_s (x \land y)). \)

(W₂₀): Following reasoning analogous to that in (W₁₉) we obtain the proof. □

Next, in order to simplify reading we will summarize the fundamental concepts we use on the class of standard systems of implicational extensional propositional calculi ([9], VIII).

Let \( L = (A^0, F) \) be a formalized language of zero order ([9], VIII 1). A system \( S = (L, C_L) \), where \( C_L \) is determined by a set \( A \) of logical axioms and by a set \( \{r_1, \ldots, r_k\} \) of rules of inference, belongs to the class \( S \) of standard systems of implicational extensional propositional calculi provided that the following conditions are satisfied:

(s1) the set \( A \) of logical axioms is closed under substitutions,

(s2) the rules of inference \( r_i, i = 1, \ldots, k, \) are invariant under substitutions,

(s3) for every formula \( \alpha \in F, \alpha \Rightarrow \alpha \in C_L(\emptyset), \)

(s4) for all formulas \( \alpha, \beta \in F \) and for every set \( H \subseteq F, \) if \( \alpha, \alpha \Rightarrow \beta \in C_L(H) \), then \( \beta \in C_L(H), \)

(s5) for all formulas \( \alpha, \beta, \gamma \in F \) and for every set \( H \subseteq F, \) if \( \alpha \Rightarrow \beta, \beta \Rightarrow \gamma \in C_L(H), \)

(s6) for every formula \( \alpha \in F \) and for every set \( H \subseteq F \) the condition \( \alpha \in C_L(H) \)

implies that for every formula \( \beta \in F, \beta \Rightarrow \alpha \in C_L(H), \)
(s7) for all formulas $\alpha, \beta \in F$ and for every set $H \subseteq F$ the condition $\alpha \Rightarrow \beta, \beta \Rightarrow \alpha \in C_L(H)$ implies that for each unary connective $\circ$ of $L$, $\circ \alpha \Rightarrow \circ \beta \in C_L(H)$,

(s8) for all formulas $\alpha, \beta, \gamma, \delta \in F$ and for every set $H \subseteq F$ the condition $\alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow \delta, \delta \Rightarrow \gamma \in C_L(H)$ implies that for each binary connective $\ast$ of $L$, $(\alpha \ast \gamma) \Rightarrow (\beta \ast \delta) \in C_L(H)$.

If $S$ is a system in $S$ and there exists a formula $\alpha$ of $L$ such that $\alpha \notin C_L(\emptyset)$ we will say that $S$ is consistent.

On the other hand, any system $S \in S$ determines a class of algebras called $S$-algebras in the following way: an algebra $U = \langle A, \Rightarrow, \ast_1, \ldots, \ast_k, \circ_1, \ldots, \circ_l, o_1, \ldots, e_m, \lor \rangle$ associated with the formalized language $L$ ([9], VIII 1) is an $S$-algebra provided that

(a1) if a formula $\alpha$ of $L$ belongs to the set $A$ of logical axioms of $S$, then $v(\alpha) = \lor$ for every valuation $v$ of $L$ in $U$,

(a2) if a rule of inference $r$ in $S$ assigns to the premises $\alpha_1, \ldots, \alpha_n$ the conclusion $\beta$, then for every valuation $v$ of $L$ in $U$ the condition $v(\alpha_1) = \ldots = v(\alpha_n) = \lor$ implies $v(\beta) = \lor$,

(a3) for all $a, b, c \in A$, if $a \Rightarrow b = \lor$ and $b \Rightarrow c = \lor$, then $a \Rightarrow c = \lor$,

(a4) for all $a, b \in A$, if $a \Rightarrow b = \lor$ and $b \Rightarrow a = \lor$, then $a = b$.

Let $S = (L, C_L)$ be a consistent system in $S$. A formula $\alpha \in L$ is valid in an algebra $U$ associated with $L$ provided that $v(\alpha) = \lor$ for every valuation $v$ of $L$ in $U$. Furthermore, $\alpha$ is $S$-valid if it is valid in every $S$-algebra. Taking into account that if $\alpha$ is derivable in $S$ ([9], VIII 5), then $v(\alpha) = \lor$ for every valuation $v$ of $L$ in every $S$-algebra $U$ ([9], VIII 6.1), every formula derivable in $S$ is $S$-valid. The converse statement is also true and this equivalence is known as the completeness theorem for propositional calculi in the class $S$ ([9], VIII 7.2).

2. The standard implication

In order to establish an implicative extensional propositional calculus (see [9]) which has $L_n^m$-algebras as the algebraic counterpart, we introduce another implication operation $\to$ on these algebras by means of the formula

$$x \to y = D_{n-1}q_1y \lor \bigwedge_{s=1}^m \bigwedge_{i=1}^{n-1} (D_iq_s(x \lor y) \to D_iq_s(x \land y))$$

and we call it standard implication. Furthermore, this implication allows us to obtain a new description of the congruence lattice $\text{Con}(A)$ of an $L_n^m$-algebra $A$ which plays an important role in what follows.
Proposition 2.1. Let $A \in \mathcal{L}^m_n$. Then the following statements hold:

(S1) $x \rightarrow 1 = 1$,
(S2) $x \rightarrow x = 1$,
(S3) $1 \rightarrow x = D_{n-1}q_1x$,
(S4) $D_{n-1}q_1x \land (x \rightarrow y) = D_{n-1}q_1y \land (y \rightarrow x)$,
(S5) $x \land (x \rightarrow y) \land (y \rightarrow x) = y \land (x \rightarrow y) \land (y \rightarrow x)$,
(S6) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$,
(S7) $(D_{n-1}q_1x \land (x \rightarrow y)) \rightarrow y = 1$,
(S8) $f^2(x \rightarrow y) = x \rightarrow y$,
(S9) $D_i(x \rightarrow y) = x \rightarrow y$, $1 \leq i \leq n - 1$.

Proof. We will only prove (S4), (S5), (S6) and (S9), since the proof of the others is straightforward.

(S4): Taking into account the definition of the standard implication and (W_{21}) we have that $D_{n-1}q_1x \land (x \rightarrow y) = (D_{n-1}q_1x \land D_{n-1}q_1y) \lor \left(D_{n-1}q_1x \land \bigwedge_{i=1}^{m-1} (D_iq_s(x \lor y) \rightarrow D_iq_s(x \land y))\right) = (D_{n-1}q_1x \land D_{n-1}q_1y) \lor \left(D_{n-1}q_1y \land \bigwedge_{s=1}^{m-1} (D_iq_s(x \lor y) \rightarrow D_iq_s(x \land y))\right)$

(S5): Taking into account (i) and (iii) in Lemma 1.1 we infer that $D_{n-1}q_1x \leq q_1x \leq x$ and so we have that $x \land (x \rightarrow y) \land (y \rightarrow x) = x \land \left(D_{n-1}q_1y \lor \bigwedge_{i=1}^{m-1} (D_iq_s(x \lor y) \rightarrow D_iq_s(x \land y))\right)$

Analogously, we have that $y \land ((x \rightarrow y) \land (y \rightarrow x)) = (D_{n-1}q_1y \land D_{n-1}q_1x) \lor \left(D_{n-1}q_1x \land y \lor \bigwedge_{i=1}^{m-1} (D_iq_s(x \lor y) \rightarrow D_iq_s(x \land y))\right) \lor \left(y \land \bigwedge_{s=1}^{m-1} (D_iq_s(x \lor y) \rightarrow D_iq_s(x \land y))\right)$. Hence, taking into account (W_{19}) and (W_{21}) we infer that $x \land (x \rightarrow y) \land (y \rightarrow x) = y \land ((x \rightarrow y) \land (y \rightarrow x))$. 

17
(S6): Let \( A \) be a subdirectly irreducible \( L_n^m \)-algebra, then by ([1], Proposition 4.1) we have that the set of Boolean elements of \( S_A \) is \( \{0, 1\} \). Hence, by (i) in Lemma 1.1 and \((g_2)\) we have that \( D_i q_s(a \lor b) \rightarrow D_i q_s(a \land b) \in \{0, 1\} \) for all \( a, b \in A \). Suppose now that there are \( x, y \in A \) such that \( D_i q_s(x \lor y) \rightarrow D_i q_s(x \land y) = 1 \). Hence, by \((W_{12})\) it follows that \( x = y \) and so, by \((W_2)\) we have that \( (x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = (x \rightarrow x) \rightarrow ((x \rightarrow z) \rightarrow (x \rightarrow z)) = 1 \). On the other hand, if we suppose that there are \( x, y, z \in A \) such that \( D_i q_s(y \land z) \rightarrow D_i q_s(y \lor z) = 1 \) or \( D_i q_s(x \lor z) \rightarrow D_i q_s(x \land z) = 1 \), following an analogously reasoning we prove \((S6)\).

Finally, if \( D_i q_s(x \lor y) \rightarrow D_i q_s(x \land y) = D_i q_s(y \lor z) \rightarrow D_i q_s(y \land z) = D_i q_s(x \lor z) \rightarrow D_i q_s(x \land z) = 0 \), then \( y \rightarrow z = D_{n-1} q_1 z = x \rightarrow z \) and so, by \((W_2)\) and \((W_1)\) we conclude the proof.

(S9): It follows as a consequence of \((g_1)\), \((GL_{11})\), \((GL_{12})\), \((g_7)\) and \((GL_6)\).

For any \( A \in \mathcal{L}_n^m \) we will denote by \( D(A) \) the set of all deductive systems of \( A \) associated with \( \rightarrow \), which are defined as usual ([7]).

**Lemma 2.1.** Let \( A \in \mathcal{L}_n^m \) and \( F \in D(A) \). Then the following conditions are equivalent for all \( x, y \in A \):

(i) there is \( u \in F \) such that \( D_{n-1} u \rightarrow f x = D_{n-1} u \rightarrow f y \),

(ii) there is \( w \in F \) such that \( x \land D_{n-1} w = y \land D_{n-1} w \),

(iii) \( x \rightarrow y, y \rightarrow x \in F \).

**Proof.** Taking into account [7], Remark 2.11, we will only prove the equivalence between (ii) and (iii).

(ii) \(\Rightarrow\) (iii): From the hypothesis and [7], Theorem 2.14, we have that \( (x, y) \in R_F = \{(a, b) \in A^2 \colon \text{there is } w \in F \text{ such that } a \land D_{n-1} w = b \land D_{n-1} w\} \) and so, \( (D_i q_s(x \lor y), D_i q_s(x \land y)) \in R_F \). Hence, \( (D_i q_s(x \lor y) \rightarrow D_i q_s(x \land y), 1) \in R_F \) which implies that \( \left( D_{n-1} q_1 y \lor \bigwedge_{i=1}^{n-1} \bigwedge_{s=1}^m (D_i q_s(x \lor y) \rightarrow q_s(x \land y)), 1 \right) \in R_F \). Therefore, \( x \rightarrow y \in F \). Similarly, we get that \( y \rightarrow x \in F \).

(iii) \(\Rightarrow\) (ii): From the hypothesis and \((S8)\) we have that \( w = (x \rightarrow y) \land (y \rightarrow x) \in F \cap S_A \) and taking into account that \( S_A \) is an \( L_n \)-algebra we have that \( D_{n-1} w \leq w \). Hence, by \((S5)\) we conclude that \( x \land D_{n-1} w = y \land D_{n-1} w \). \(\square\)

From now on, for any \( A \in \mathcal{L}_n^m \) we will denote by \( A/R \) the quotient algebra of \( A \) by \( R \) for any \( R \in \text{Con}(A) \). Besides, for \( x \in A \) the equivalence class of \( x \) modulo \( R \) will be denoted by \([x]_R\).
Theorem 2.1. Let $A \in \mathcal{L}_n^m$. Then the following statements hold:

(i) $\text{Con}(A) = \{ R(F) : F \in \mathcal{D}(A) \}$ where $R(F) = \{(x, y) \in A^2 : x \rightarrow y, y \rightarrow x \in F \}$,

(ii) the lattices $\text{Con}(A)$ and $\mathcal{D}(A)$ are isomorphic considering the applications $\theta \mapsto [1]_\theta$ and $F \mapsto R(F)$ which are inverse to each other.

Proof. It is a direct consequence of Lemma 2.1 and [7], Theorem 2.14. \qed

Let $A \in \mathcal{L}_n^m$ and $z \in A$. We will denote by $[z]$ the principal filter of $A$ generated by $z$ (i.e., $[z] = \{ x \in A : z \leq x \}$).

Lemma 2.2. Let $A \in \mathcal{L}_n^m$ and $a, b \in A$. If $w = (a \rightarrow b) \land (b \rightarrow a)$, then $[w]$ is a deductive system of $A$.

Proof. Taking into account [7], Proposition 2.6, it only remains to prove that $fD_{n-1}fx \in [w]$ for all $x \in [w]$. By (S8), (GL$_5$), (g$_7$) and (S9) we have that $fD_{n-1}fw = fD_{n-1}f((a \rightarrow b) \land (b \rightarrow a)) = f^2D_1((a \rightarrow b) \land (b \rightarrow a)) = f^2(D_1(a \rightarrow b) \land D_1(b \rightarrow a)) = f^2((a \rightarrow b) \land (b \rightarrow a)) = w$. From this assertion the proof is straightforward. \qed

Taking into account the above results we obtain a characterization of the principal congruences on $\mathcal{L}_n^m$-algebras. For any $A \in \mathcal{L}_n^m$ and $a, b \in A$ we will denote by $\theta(a, b)$ the principal congruence of $A$ generated by $(a, b)$.

Theorem 2.2. Let $A \in \mathcal{L}_n^m$ and $a, b \in A$. Then $\theta(a, b) = \{(x, y) \in A^2 : x \land ((a \rightarrow b) \land (b \rightarrow a)) = y \land ((a \rightarrow b) \land (b \rightarrow a)) \}$.

Proof. Let $T = \{(x, y) \in A^2 : x \land ((a \rightarrow b) \land (b \rightarrow a)) = y \land ((a \rightarrow b) \land (b \rightarrow a)) \}$. By (S5) we have that $(a, b) \in T$. Besides, by (S9) and (S8) it follows that $T = \{(x, y) \in A^2 : x \land D_{n-1}((a \rightarrow b) \land (b \rightarrow a)) = y \land D_{n-1}((a \rightarrow b) \land (b \rightarrow a)) \}$ and so, by Lemma 2.2, Lemma 2.1 and [7], Theorem 2.14, we conclude that $T \in \text{Con}(A)$.

On the other hand, let $R \in \text{Con}(A)$ be such that $(a, b) \in R$ and suppose that $(x, y) \in T$. Hence, we have that $((a \rightarrow b) \land (b \rightarrow a), 1) \in R$ and so, $(x \land (a \rightarrow b) \land (b \rightarrow a), x) \in R$ and $(y \land (a \rightarrow b) \land (b \rightarrow a), y) \in R$. From these last assertions and the fact that $(x, y) \in T$ we conclude that $(x, y) \in R$. Therefore, $T = \theta(a, b)$. \qed

Example 2.1. Let us consider the $L_3^2$-algebra $A$ shown in Figure 1, where the operations $f, D_i, 1 \leq i \leq 2$ and $q_i, 1 \leq i \leq 2$ are defined as follows:

If $w = (a \rightarrow b) \land (b \rightarrow a) = h$, by Lemma 2.2 we have that $F = \{ h \} = \{ h, i, j, k, m, n, 1 \}$ is a deductive system of $A$. Hence, by Theorem 2.1 we have that $A/R(F) = \{ [0]_{R(F)}, [1]_{R(F)} \}$ where $[1]_{R(F)} = F$ and $[0]_{R(F)} = \{ 0, a, b, c, d, e, g \}$.
On the other hand, by (S1) and (S3) we have that \( g \rightarrow 1 = 1 \) and \( 1 \rightarrow g = g \). Then, taking into account Theorem 2.2 we obtain that
\[
\theta(g, 1) = \{ (x, y) \in A^2 : x \land g = y \land g \} = \text{Id}_A \cup \{(g, 1), (1, g), (d, m), (n, e), (e, n), (k, c), (a, i), (i, a), (b, j), (j, b), (0, h), (h, 0)\}.
\]

From Theorem 2.2 it is easy to verify Proposition 2.2, which will be quite useful in the development of the \( L^m_n \)-propositional calculus.

**Proposition 2.2.** Let \( A \in L^m_n \). Then the following statements hold:

(S10) \( D_i x \land (x \rightarrow y) \land (y \rightarrow x) = D_i y \land (x \rightarrow y) \land (y \rightarrow x), 1 \leq i \leq n - 1, \)
(S11) \( D_i q_s(f x \lor f y) \land (x \rightarrow y) \land (y \rightarrow x) = D_i q_s(f x \land f y) \land (x \rightarrow y) \land (y \rightarrow x), 1 \leq i \leq n - 1, 1 \leq s \leq m, \)
(S12) \( D_i q_s((x \land z) \lor (y \land z)) \land (x \rightarrow y) \land (y \rightarrow x) = D_i q_s((x \land z) \land (y \land z)) \land (x \rightarrow y) \land (y \rightarrow x), 1 \leq i \leq n - 1, 1 \leq s \leq m, \)
(S13) \( D_i q_s((x \rightarrow z) \lor (y \rightarrow z)) \land (x \rightarrow y) \land (y \rightarrow x) = D_i q_s((x \rightarrow z) \land (y \rightarrow z)) \land (x \rightarrow y) \land (y \rightarrow x), 1 \leq i \leq n - 1, 1 \leq s \leq m, \)
(S14) \( D_i q_s((x \rightarrow z) \lor (y \rightarrow z)) \land (x \rightarrow y) \land (y \rightarrow x) = D_i q_s((x \rightarrow z) \land (y \rightarrow z)) \land (x \rightarrow y) \land (y \rightarrow x), 1 \leq i \leq n - 1, 1 \leq s \leq m. \)

**Proof.** It is routine. \( \square \)
A characterization of $k_{2m}$-lattices

The goal of this section is to find an equivalent formulation to $(GL_1)$ with a simpler proof than the previous one. To this end, we take into account the results established in [8].

Definition 3.1. A $k_{2m}$-lattice, $m \in \mathbb{N}$, is an algebra $\langle A, \vee, \wedge, f \rangle$ such that $\langle A, \vee, \wedge \rangle$ is a distributive lattice and $f$ is a unary operation on $A$ verifying the following conditions:

\[(r1) \quad f^{2m} x = x,\]
\[(r2) \quad f(x \lor y) = f(x) \land f(y).\]

Theorem 3.2 enables us to characterize $k_{2m}$-lattices by means of the operations of infimum $\land$ and the dual endomorphism $f$. This characterization results easier by the use of Sholander’s characterization of distributive lattices as follows:

Theorem 3.1 ([10]). An algebra $\langle A, \land, \lor \rangle$ of type $(2, 2)$ is a distributive lattice if and only if it verifies the conditions

\[(l1) \quad a = a \land (a \lor b),\]
\[(l2) \quad a \lor (b \lor c) = (c \land a) \lor (b \land a).\]

Theorem 3.2. Let $\langle A, \land, f \rangle$ be an algebra of type $(2, 1)$. Define $(s): a \lor b = f^{2m-1}(fa \land fb)$ for all $a, b \in A$. Then $\langle A, \land, \lor, f \rangle$ is a $k_{2m}$-lattice, $m \in \mathbb{N}$, if and only if the following conditions are verified:

\[(m1) \quad a = a \land f^{2m-1}(fa \land fb),\]
\[(m2) \quad a \land f^{2m-1}(fb \land fc) = f^{2m-1}(f(c \land a) \land f(b \land a)).\]

Proof. From (l1), (l2) and taking into account the definition of $\lor$ we have that (m1) and (m2) immediately follow. In order to prove the converse we will first show that $A$ is a distributive lattice, which is a consequence of the fact that (l1) and (l2) hold. Indeed, from (m1), (m2) and (s) we have (l1): $a \land (a \lor b) = f^{2m-1}(fa \land fb) \land a = a$ and (l2): $(c \land a) \lor (b \land a) = f^{2m-1}(f(c \land a) \land f(b \land a)) = a \land f^{2m-1}(fb \land fc) = a \land (b \lor c)$. Hence, by (m1) and (m2) we obtain (r1): $a = a \land f^{2m-1}(fa \land fa) = f^{2m-1}(fa \land a) \land f(a \land a)) = f^{2m-1}(fa \land fa) = f^{2m-1}fa$. Finally, from (r1) and (s) we get (r2): $f(a \lor b) = ff^{2m-1}(fa \land fb) = fa \land fb$. \qed
4. The $L_n^m$-propositional calculus

In this section, which is the core of this paper, we describe a propositional calculus and show that it has $L_n^m$-algebras as the algebraic counterpart. We are interested in finding a calculus which belongs to the class of standard systems of implicative propositional calculi. The complexity of the standard implication together with the fact that $L_n^m$-algebras do not verify Moisil’s determination principle and that the operators $D_i$ are not $\land$-homomorphisms have made that in this calculus the number of axioms and inference rules are greater than in $n$-valued Lukasiewicz propositional calculus ([4]). The terminology and symbols used here coincide with those used in [9].

Let $L = (A^0, F)$ be a formalized language of zero order where in the alphabet $A^0 = (V, L_0, L_1, L_2, U)$ the set

(i) $V$ of propositional variables is countable;
(ii) $L_0$ is empty;
(iii) $L_1$ contains $n$ elements denoted by $f, D_i$ for $1 \leq i \leq n - 1$, called negation sign and generalized Moisil operators signs, respectively;
(iv) $L_2$ contains four elements denoted by $\land, \lor, \rightarrow$ and $\rightarrow$ called conjunction sign, disjunction sign, weak implication sign and standard implication sign, respectively;
(v) $U$ contains two elements denoted by $(, )$.

In what follows, for any $\alpha_1, \ldots, \alpha_k$ in the set $F$ of all formulas over $A^0$, $\bigvee_{p=0}^{k} \alpha_p$ and $\bigwedge_{p=0}^{k} \alpha_p$ will mean $\alpha_0 \lor (\ldots \lor (\alpha_{k-1} \lor \alpha_k) \ldots)$ and $\alpha_0 \land (\ldots \land (\alpha_{k-1} \land \alpha_k) \ldots)$, respectively. Besides, for any $\alpha$ in $F$, $f^t \alpha$ is the result of applying $f$ $t$ times to $\alpha$ if $t > 0$, or $\alpha$ if $t = 0$. Furthermore, for any $\alpha, \beta$ in $F$, we will write for brevity $\alpha \leftrightarrow \beta$, $\alpha \leftrightarrow \beta$ and $q_s \alpha$ instead of $(\alpha \rightarrow \beta) \land (\beta \rightarrow \alpha)$, $(\alpha \rightarrow \beta) \land (\beta \rightarrow \alpha)$ and $\bigwedge_{J \subseteq T, |J| = s} \bigvee_{j \in J} f^{2j} \alpha$, where $T = \{0, 1, \ldots, m - 1\}$ and $s \in \{1, \ldots, m\}$, respectively.

We assume that the set $A_I$ of logical axioms consists of all formulas of the following form, where $\alpha, \beta, \gamma$ are any formulas in $F$:

(A1) $\alpha \rightarrow (\beta \rightarrow \alpha)$,
(A2) $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$,
(A3) $\alpha \rightarrow (\alpha \lor \beta)$,
(A4) $\beta \rightarrow (\alpha \lor \beta)$,
(A5) $(\alpha \land \beta) \rightarrow \alpha$,
(A6) $(\alpha \land \beta) \rightarrow \beta$,
(A7) $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \gamma) \rightarrow (\alpha \rightarrow (\beta \land \gamma)))$,
(A8) $\alpha \rightarrow D_i \alpha$,
(A9) $D_j D_i \alpha \leftrightarrow D_i \alpha, 1 \leq i, j \leq n - 1$.
(A10) $D_i \bigvee_{p=0}^{m-1} f^{2p} \alpha \leftrightarrow D_i \alpha, 1 \leq i \leq n - 1,$

(A11) $((\alpha \wedge \gamma) \rightarrow (\beta \wedge \gamma)) \rightarrow (\gamma \rightarrow (\alpha \rightarrow \beta)),$

(A12) $D_i \alpha \lor f D_i \alpha, 1 \leq i \leq n - 1,$

(A13) $D_i q_s(\alpha \lor \alpha) \rightarrow D_i q_s(\alpha \wedge \alpha), 1 \leq i \leq n - 1, 1 \leq s \leq m,$

(A14) $D_i q_s \alpha \rightarrow D_i q_s(\alpha \wedge D_i \alpha), 1 \leq i \leq n - 1, 1 \leq s \leq m,$

(A15) $D_i q_s(f^2 D_i \alpha \lor D_i \alpha) \rightarrow D_i q_s(f^2 D_i \alpha \wedge D_i \alpha), 1 \leq i \leq n - 1, 1 \leq s \leq m,$

(A16) $D_i q_s \left(D_i \left(\alpha \wedge \bigvee_{p=0}^{m-1} f^{2p} \beta \right) \lor (D_i \alpha \wedge D_i \beta)\right) \rightarrow$

$$D_i q_s \left(D_i \left(\alpha \wedge \bigvee_{p=0}^{m-1} f^{2p} \beta \right) \land (D_i \alpha \wedge D_i \beta)\right), 1 \leq i \leq n - 1, 1 \leq s \leq m,$$

(A17) $D_i q_s(D_j \alpha \lor (D_i \alpha \wedge D_j \alpha)) \rightarrow D_i q_s(D_j \alpha \wedge (D_i \alpha \wedge D_j \alpha)), 1 \leq i \leq j \leq n - 1, 1 \leq s \leq m,$

(A18) $D_i q_s \left(D_i f \left(\bigvee_{p=0}^{m-1} f^{2p} \alpha \right) \lor f D_{n-i} \left(\bigvee_{p=0}^{m-1} f^{2p} \alpha \right)\right) \rightarrow$

$$D_i q_s \left(D_i f \left(\bigvee_{p=0}^{m-1} f^{2p} \alpha \right) \land f D_{n-i} \left(\bigvee_{p=0}^{m-1} f^{2p} \alpha \right)\right), 1 \leq i \leq n - 1, 1 \leq s \leq m,$$

(A19) $D_i q_s(\alpha \wedge \alpha) \rightarrow D_i q_s((\alpha \wedge \alpha) \land (\beta \lor \beta)), 1 \leq i \leq n - 1, 1 \leq s \leq m,$

(A20) $D_i q_s \left(\bigvee_{p=0}^{m-1} f^{2p} \alpha \right) \rightarrow$

$$D_i q_s \left(\bigvee_{p=0}^{m-1} f^{2p} \alpha \wedge f D_i \left(\bigvee_{p=0}^{m-1} f^{2p} \beta \right) \lor D_{i-1} \left(\bigvee_{p=0}^{m-1} f^{2p} \alpha \right)\right), 1 \leq i \leq n - 1, 1 \leq s \leq m,$$

(A21) $D_i q_s \alpha \rightarrow D_i q_s(\alpha \land f^{2m-1}(f \alpha \land f \beta)), 1 \leq i \leq n - 1, 1 \leq s \leq m,$

(A22) $D_i q_s((\alpha \land f^{2m-1}(f \beta \land f \gamma)) \lor f^{2m-1}(f (\gamma \land \alpha) \land f (\beta \land \alpha))) \rightarrow D_i q_s(\alpha \land f^{2m-1}(f \beta \land f \gamma) \land f^{2m-1}(f (\gamma \land \alpha) \land f (\beta \land \alpha))'), 1 \leq i \leq n - 1, 1 \leq s \leq m,$

(A23) $\alpha \rightarrow \beta \leftrightarrow D_{n-1} q_{1, \beta} \lor \bigwedge_{i=1}^{n-m} \bigvee_{s=1}^{m} (D_i q_s(\alpha \lor \beta) \rightarrow D_i q_s(\alpha \land \beta)),$

(A24) $(D_i q_s(f \alpha \lor f \beta) \land (\alpha \leftrightarrow \beta)) \rightarrow (D_i q_s(f \alpha \lor f \beta) \land (\alpha \leftrightarrow \beta)), 1 \leq i \leq n - 1, 1 \leq s \leq m,$

(A25) $(D_i \alpha \land (\alpha \leftrightarrow \beta)) \rightarrow (D_i \beta \land (\alpha \leftrightarrow \beta)), 1 \leq i \leq n - 1,$

(A26) $(D_i q_s((\alpha \land \gamma) \lor (\beta \land \gamma)) \land (\alpha \leftrightarrow \beta)) \rightarrow (D_i q_s((\alpha \land \gamma) \land (\beta \land \gamma)) \land (\alpha \leftrightarrow \beta)), 1 \leq i \leq n - 1, 1 \leq s \leq m,$

(A27) $(\alpha \land \gamma) \rightarrow (\gamma \land \alpha),$

(A28) $(D_i q_s((\gamma \rightarrow \alpha) \lor (\gamma \rightarrow \beta)) \land (\alpha \leftrightarrow \beta)) \rightarrow (D_i q_s((\gamma \rightarrow \alpha) \land (\gamma \rightarrow \beta)) \land (\alpha \leftrightarrow \beta)), 1 \leq i \leq n - 1, 1 \leq s \leq m,$

(A29) $(D_i q_s((\alpha \rightarrow \gamma) \lor (\beta \rightarrow \gamma)) \land (\alpha \leftrightarrow \beta)) \rightarrow (D_i q_s((\alpha \rightarrow \gamma) \land (\beta \rightarrow \gamma)) \land (\alpha \leftrightarrow \beta)), 1 \leq i \leq n - 1, 1 \leq s \leq m,$

(A30) $(D_i q_s((\gamma \rightarrow \alpha) \lor (\gamma \rightarrow \beta)) \land (\alpha \leftrightarrow \beta)) \rightarrow (D_i q_s((\gamma \rightarrow \alpha) \land (\gamma \rightarrow \beta)) \land (\alpha \leftrightarrow \beta)), 1 \leq i \leq n - 1, 1 \leq s \leq m,$
(A31) \((D_i q_s((\alpha \to \gamma) \vee (\beta \to \gamma)) \land (\alpha \leftrightarrow \beta)) \to (D_i q_s((\alpha \to \gamma) \land (\beta \to \gamma) \land (\alpha \leftrightarrow \beta))),\)
\[1 \leq i \leq n - 1, 1 \leq s \leq m,\]

(A32) \((\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma)),\)

(A33) \((f^{2m-1}(f \alpha \land f \beta) \to (\alpha \lor \beta)) \land ((\alpha \lor \beta) \to f^{2m-1}(f \alpha \land f \beta)),\)

(A34) \((D_{n-1} q_1 \alpha \land (\alpha \to \beta)) \to \beta.\)

The consequence operation \(C_L\) in \(\mathcal{L} = (A^0, F)\) is determined by \(A_t\) and by the following rules of inference:

(R1) \[\frac{\alpha, \alpha \to \beta}{\beta},\]

(R2) \[\frac{D_i \alpha \to D_j \beta, D_j \beta \to D_i \alpha}{D_i \alpha \to D_j \beta}, 1 \leq i, j \leq n - 1,\]

(R3) \[\frac{D_i \alpha \to D_j \beta}{D_i q_s(\alpha \lor (\alpha \land \beta)) \to D_j q_s(\alpha \land (\alpha \land \beta))}, 1 \leq i \leq n - 1, 1 \leq s \leq m,\]

(R4) \[\frac{D_{n-1} q_1 \alpha}{D_i q_s(\alpha \lor \beta) \to D_i q_s(\alpha \land \beta)}, 1 \leq i \leq n - 1, 1 \leq s \leq m.\]

The system \(\ell^m_n = (\mathcal{L}, C_L)\) thus obtained will be called the \(L^m_n\)-propositional calculus. It is worth mentioning that the above connectives are not independent, however, we consider them for simplicity. We will denote by \(T\) the set of all formulas derivable in \(\ell^m_n\). If \(\alpha \in T\), we will write \(\vdash \alpha\).

Lemma 4.1 summarizes the most important rules and theorems necessary for the further development.

**Lemma 4.1.** In \(\ell^m_n\) the following rules and theorems hold:

(R6) \[\frac{\alpha}{\beta \to \alpha},\]

(R7) \[\frac{\alpha \to (\beta \to \gamma)}{(\alpha \to \beta) \to (\alpha \to \gamma)},\]

(T1) \[\vdash \alpha \to \alpha,\]

(T2) \[\vdash (\alpha \to \beta) \to ((\gamma \to \alpha) \to (\gamma \to \beta)),\]

(R8) \[\frac{(\gamma \to \alpha) \to (\gamma \to \beta)}{(\alpha \to \beta) \to (\alpha \to \gamma)},\]

(R9) \[\frac{(\alpha \to \beta) \to (\alpha \to \gamma)}{\beta \to (\alpha \to \gamma)},\]

(R10) \[\frac{\alpha \to (\beta \to \gamma)}{\beta \to (\alpha \to \gamma)},\]

(T3) \[\vdash (\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta),\]

(T4) \[\vdash (\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma)),\]

(R11) \[\frac{\alpha \to \beta, \beta \to \gamma}{\alpha \to \gamma}.\]
(R12) \[
\alpha \rightarrow \beta
\]
\[
(\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma)
\]

(R13) \[
\alpha, \beta
\]
\[
\alpha \land \beta, \beta
\]

(T5) \[
\vdash \alpha \rightarrow \alpha,
\]

(R14) \[
D_i q_s (\alpha \lor \beta) \rightarrow D_i q_s (\alpha \land \beta), 1 \leq i \leq n-1, 1 \leq s \leq m,
\]

(R15) \[
\alpha, \alpha \rightarrow \beta
\]
\[
\beta
\]

(R16) \[
\alpha \rightarrow \beta, \beta \rightarrow \gamma
\]
\[
\alpha \rightarrow \gamma
\]

(R17) \[
\alpha \rightarrow \beta
\]
\[
\beta
\]

(R18) \[
\alpha \rightarrow \beta, \beta \rightarrow \alpha
\]
\[
f \alpha \rightarrow f \beta
\]

(R19) \[
D_i \alpha \rightarrow D_i \beta, 1 \leq i \leq n-1,
\]

(R20) \[
(\alpha \land \gamma) \rightarrow (\beta \land \gamma)
\]

(R21) \[
(\gamma \land \alpha) \rightarrow (\gamma \land \beta)
\]

(R22) \[
(\alpha \lor \gamma) \rightarrow (\beta \lor \gamma)
\]

(R23) \[
(\gamma \lor \alpha) \rightarrow (\gamma \lor \beta)
\]

(R24) \[
(\alpha \rightarrow \gamma) \rightarrow (\gamma \rightarrow \beta)
\]

(R25) \[
(\alpha \rightarrow \gamma) \rightarrow (\beta \rightarrow \gamma)
\]

(R26) \[
(\alpha \rightarrow \gamma) \rightarrow (\gamma \rightarrow \beta)
\]

(R27) \[
(\alpha \rightarrow \gamma) \rightarrow (\beta \rightarrow \gamma)
\]

**Proof.** The proof of (R6) to (R13) is routine.

(T5):

1. \[
D_i q_s (\alpha \lor \alpha) \rightarrow D_i q_s (\alpha \land \alpha), \quad [(A13)]
\]
2. \[
\bigwedge_{i=1}^{n-1} \bigwedge_{s=1}^{m} (D_i q_s (\alpha \lor \alpha) \rightarrow D_i q_s (\alpha \land \alpha)), \quad [(1), (R13)]
\]
3. \[
\alpha \rightarrow \alpha. \quad [(2), (A4), (R1), (A23)]
\]

(R14):

1. \[
D_i q_s (\alpha \lor \beta) \rightarrow D_i q_s (\alpha \land \beta), 1 \leq i \leq n-1, 1 \leq s \leq m,
\]
2. \[
\bigwedge_{i=1}^{n-1} \bigwedge_{s=1}^{m} (D_i q_s (\alpha \lor \beta) \rightarrow D_i q_s (\alpha \land \beta)), \quad [(1), (R13)]
\]
(3) \( n^{-1} m \sum_{i=1}^{n} (D_iq_s(\alpha \lor \beta) \rightarrow D_iq_s(\alpha \land \beta)) \rightarrow \\
\left( D_{i-1}q_1 \beta \lor \sum_{i=1}^{n-1} m \sum_{s=1}^{m} (D_iq_s(\alpha \lor \beta) \rightarrow D_iq_s(\alpha \land \beta)) \right) \), \[ (A4) \]

(4) \( D_{n-1}q_1 \beta \lor \sum_{i=1}^{n-1} m \sum_{s=1}^{m} (D_iq_s(\alpha \lor \beta) \rightarrow D_iq_s(\alpha \land \beta)) \), \[ (2, 3, (R1)) \]

(5) \( \alpha \rightarrow \beta. \) \[ (4, (A23)) \]

(R15): It is a consequence of (R4), (R13), (A34) and (R1).

(R16): It is routine.

(R17): It follows as a consequence of (R4), (A3), (R1) and (A23).

(R18):

(1) \( \alpha \rightarrow \beta, \)

(2) \( \beta \rightarrow \alpha, \)

(3) \( \alpha \leftrightarrow \beta, \) \[ (1, 2, (R13)) \]

(4) \( (D_iq_s(f\alpha \lor f\beta \land (\alpha \leftrightarrow \beta)) \rightarrow (D_iq_s(f\alpha \land f\beta \land (\alpha \leftrightarrow \beta)), 1 \leq i \leq n - 1, 1 \leq s \leq m, \]

(5) \( (\alpha \leftrightarrow \beta) \rightarrow (D_iq_s(f\alpha \lor f\beta \rightarrow D_iq_s(f\alpha \land f\beta)), 1 \leq i \leq n - 1, 1 \leq s \leq m, \) \[ (A11, 4, (R1)) \]

(6) \( D_iq_s(f\alpha \lor f\beta \rightarrow D_iq_s(f\alpha \land f\beta), 1 \leq i \leq n - 1, 1 \leq s \leq m, \) \[ (3, 5, (R1)) \]

(7) \( f\alpha \rightarrow f\beta. \) \[ (6, (R14)) \]

(R19):

(1) \( \alpha \rightarrow \beta, \)

(2) \( \beta \rightarrow \alpha, \)

(3) \( \alpha \leftrightarrow \beta, \) \[ (1, 2, (R13)) \]

(4) \( (D_i\alpha \land (\alpha \leftrightarrow \beta)) \rightarrow (D_i\beta \land (\alpha \leftrightarrow \beta), 1 \leq i \leq n - 1, \) \[ (A25) \]

(5) \( (\alpha \leftrightarrow \beta) \rightarrow (D_i\alpha \rightarrow D_i\beta), 1 \leq i \leq n - 1, \) \[ (A11, 4, (R1)) \]

(6) \( D_i\alpha \rightarrow D_i\beta, 1 \leq i \leq n - 1, \) \[ (3, 5, (R1)) \]

(7) \( (D_i\beta \land (\beta \leftrightarrow \alpha)) \rightarrow (D_i\alpha \land (\beta \leftrightarrow \alpha), 1 \leq i \leq n - 1, \) \[ (A25) \]

(8) \( \beta \leftrightarrow \alpha, \) \[ (2, 1, (R13)) \]

(9) \( (\beta \leftrightarrow \alpha) \rightarrow (D_i\beta \rightarrow D_i\alpha), 1 \leq i \leq n - 1, \) \[ (A11, 7, (R1)) \]

(10) \( D_i\beta \rightarrow D_i\alpha, 1 \leq i \leq n - 1, \) \[ (8, 9, (R1)) \]

(11) \( D_i\alpha \rightarrow D_i\beta, 1 \leq i \leq n - 1. \) \[ (6, 10, (R2)) \]

(R20):

(1) \( \alpha \rightarrow \beta, \)

(2) \( \beta \rightarrow \alpha, \)

(3) \( \alpha \leftrightarrow \beta, \) \[ (1, 2, (R13)) \]

(4) \( (D_iq_s((\alpha \land \gamma) \lor (\beta \land \gamma)) \land (\alpha \leftrightarrow \beta)) \rightarrow (D_iq_s((\alpha \land \gamma) \lor (\beta \land \gamma)) \land (\alpha \leftrightarrow \beta), 1 \leq i \leq n - 1, 1 \leq s \leq m, \) \[ (A26) \]
(5) \((\alpha \leftrightarrow \beta) \rightarrow (D_iq_s((\alpha \land \gamma) \lor (\beta \land \gamma)) \rightarrow D_iq_s((\alpha \land \gamma) \lor (\beta \land \gamma))), 1 \leq i \leq n - 1, 1 \leq s \leq m,\) 
\[\text{(A11), (4), (R1)}\]

(6) \(D_iq_s((\alpha \land \gamma) \lor (\beta \land \gamma)) \rightarrow D_iq_s((\alpha \land \gamma) \lor (\beta \land \gamma)), 1 \leq i \leq n - 1, 1 \leq s \leq m,\) 
\[\text{(3), (5), (R1)}\]

(7) \((\alpha \land \gamma) \rightarrow (\beta \land \gamma).\)
\[\text{(6), (R14)}\]

(R22):
(1) \(\alpha \rightarrow \beta, \beta \rightarrow \alpha,\) 
\[\text{[hip.]}\]
(2) \(f\alpha \rightarrow f\beta, f\beta \rightarrow f\alpha,\) 
\[\text{[(1), (R18)]}\]
(3) \((f\alpha \land f\gamma) \rightarrow (f\beta \land f\gamma),\) 
\[\text{[(2), (R20)]}\]
(4) \((f\beta \land f\gamma) \rightarrow (f\alpha \land f\gamma),\) 
\[\text{[(2), (R20)]}\]
(5) \(f^{2m-1}(f\alpha \land f\gamma) \rightarrow f^{2m-1}(f\beta \land f\gamma),\) 
\[\text{[(3), (4), (R18)]}\]
(6) \(f^{2m-1}(f\beta \land f\gamma) \rightarrow (\beta \lor \gamma),\) 
\[\text{[(A5), (A33), (R1)]}\]
(7) \((\alpha \lor \gamma) \rightarrow f^{2m-1}(f\alpha \land f\gamma),\) 
\[\text{[(A6), (A33), (R1)]}\]
(8) \((\alpha \lor \gamma) \rightarrow (\beta \lor \gamma).\)
\[\text{[(7), (5), (6), (R16)]}\]

(R24):
(1) \(\alpha \rightarrow \beta,\) 
\[\text{[hip.]}\]
(2) \(\beta \rightarrow \alpha,\) 
\[\text{[hip.]}\]
(3) \(\alpha \leftrightarrow \beta,\) 
\[\text{[(1), (2), (R13)]}\]
(4) \((D_iq_s((\gamma \rightarrow \alpha) \lor (\gamma \rightarrow \beta)) \land (\alpha \leftrightarrow \beta)) \rightarrow (D_iq_s((\gamma \rightarrow \alpha) \land (\gamma \rightarrow \beta)), \land (\alpha \leftrightarrow \beta))\)
\[\text{[A28]}\]
(5) \((\alpha \leftrightarrow \beta) \rightarrow ((D_iq_s((\gamma \rightarrow \alpha) \lor (\gamma \rightarrow \beta))) \rightarrow (D_iq_s((\gamma \rightarrow \alpha) \land (\gamma \rightarrow \beta)))),\) 
\[\text{[(A11), (4), (R1)]}\]
(6) \(D_iq_s((\gamma \rightarrow \alpha) \lor (\gamma \rightarrow \beta)) \rightarrow D_iq_s((\gamma \rightarrow \alpha) \land (\gamma \rightarrow \beta)),\) 
\[\text{[(3), (5), (R1)]}\]
(7) \((\gamma \rightarrow \alpha) \rightarrow (\gamma \rightarrow \beta).\)
\[\text{[(6), (R14)]}\]

(R26):
(1) \(\alpha \rightarrow \beta,\) 
\[\text{[(1), (2), (R13)]}\]
(2) \(\beta \rightarrow \alpha,\) 
\[\text{[(1), (2), (R13)]}\]
(3) \(\alpha \leftrightarrow \beta,\) 
\[\text{[(1), (2), (R13)]}\]
(4) \((D_iq_s((\gamma \rightarrow \alpha) \lor (\gamma \rightarrow \beta)) \land (\alpha \leftrightarrow \beta)) \rightarrow (D_iq_s((\gamma \rightarrow \alpha) \land (\gamma \rightarrow \beta)) \land (\alpha \leftrightarrow \beta)), 1 \leq i \leq n - 1, 1 \leq s \leq m,\) 
\[\text{[(A30)]}\]
(5) \((\alpha \leftrightarrow \beta) \rightarrow ((D_iq_s((\gamma \rightarrow \alpha) \lor (\gamma \rightarrow \beta))) \rightarrow (D_iq_s((\gamma \rightarrow \alpha) \land (\gamma \rightarrow \beta)))), 1 \leq i \leq n - 1, 1 \leq s \leq m,\) 
\[\text{[(A11), (4), (R1)]}\]
(6) \(D_iq_s((\gamma \rightarrow \alpha) \lor (\gamma \rightarrow \beta)) \rightarrow D_iq_s((\gamma \rightarrow \alpha) \land (\gamma \rightarrow \beta)), 1 \leq i \leq n - 1, 1 \leq s \leq m,\) 
\[\text{[(3), (5), (R1)]}\]
(7) \((\gamma \rightarrow \alpha) \rightarrow (\gamma \rightarrow \beta).\)
\[\text{[(6), (R14)]}\]

Using a reasoning similar to that for (R20), (R22), (R24) and (R26) we infer (R21), (R23), (R25) and (R27), respectively.
Theorem 4.1. The propositional calculus \( \ell_m^n \) belongs to the class of standard systems of implicative extensional propositional calculi.

Proof. We have to prove that conditions (s1) to (s8) in Section 1 are verified. Clearly, (s1) and (s2) hold. Besides, (s3), (s4), (s5) and (s6) follow from (T12), (R15), (R16) and (R17), respectively. On the other hand, taking into account (R18) and (R19), we have that (s7) is satisfied. Finally, if \( \alpha \rightarrow \beta, \beta \rightarrow \alpha, \delta \rightarrow \gamma, \gamma \rightarrow \delta \in C_L(H) \) for every subset \( H \) of formulas, then by (R20) we have that \( (\alpha \land \delta) \rightarrow (\beta \land \delta) \in C_L(H) \). Besides, by (R21) we get \( (\beta \land \delta) \rightarrow (\beta \land \gamma) \in C_L(H) \). Hence, by (R16) we infer that \( (\alpha \land \delta) \rightarrow (\beta \land \gamma) \in C_L(H) \). In an analogous manner, from (R22), (R23), (R25), (R26) and (R27) we conclude the proof of (s8). □

In what follows, our attention is focused on establishing the relationship between \( L_m^n \)-algebras and \( \ell_m^n \)-algebras which are the class of algebras determined by the system \( \ell_m^n \). To this aim, Lemma 4.2 will be fundamental.

Lemma 4.2. In \( \ell_m^n \) the following theorems hold:

\[
\begin{align*}
(T6) & \vdash (\alpha \land f^{2m-1}(f\alpha \land f\beta)) \rightarrow \alpha, \\
(T7) & \vdash \alpha \rightarrow (\alpha \land f^{2m-1}(f\alpha \land f\beta)), \\
(T8) & \vdash (\alpha \land f^{2m-1}(f\beta \land f\gamma)) \rightarrow f^{2m-1}(f(\gamma \land \alpha) \land f(\beta \land \alpha)), \\
(T9) & \vdash f^{2m-1}(f(\gamma \land \alpha) \land f(\beta \land \alpha)) \rightarrow (\alpha \land f^{2m-1}(f\beta \land f\gamma)), \\
(T10) & \vdash D_1(\alpha \rightarrow \alpha), \\
(T11) & \vdash f^2D_1\alpha \rightarrow D_1\alpha, \\
(T12) & \vdash D_1\alpha \rightarrow f^2D_1\alpha, \\
(T13) & \vdash D_1\left(\alpha \land \bigvee_{p=0}^{m-1} f^{2p}\beta \right) \rightarrow (D_1\alpha \land D_1\beta), 1 \leq i \leq n - 1, \\
(T14) & \vdash (D_1\alpha \land D_1\beta) \rightarrow D_1\left(\alpha \land \bigvee_{p=0}^{m-1} f^{2p}\beta \right), 1 \leq i \leq n - 1, \\
(T15) & \vdash D_j\alpha \rightarrow (D_1\alpha \land D_j\alpha), 1 \leq i \leq j \leq n - 1, \\
(T16) & \vdash (D_1\alpha \land D_j\alpha) \rightarrow D_j\alpha, 1 \leq i \leq j \leq n - 1, \\
(T17) & \vdash D_1f\left(\bigvee_{p=0}^{m-1} f^{2p}\alpha \right) \rightarrow fD_{n-1}\left(\bigvee_{p=0}^{m-1} f^{2p}\alpha \right), 1 \leq i \leq n - 1, \\
(T18) & \vdash fD_{n-1}\left(\bigvee_{p=0}^{m-1} f^{2p}\alpha \right) \rightarrow D_1f\left(\bigvee_{p=0}^{m-1} f^{2p}\alpha \right), 1 \leq i \leq n - 1, \\
(T19) & \vdash D_1\alpha \rightarrow D_jD_1\alpha, 1 \leq i, j \leq n - 1, \\
(T20) & \vdash D_jD_1\alpha \rightarrow D_1\alpha, 1 \leq i, j \leq n - 1, \\
(T21) & \vdash (\alpha \land D_1\alpha) \rightarrow \alpha, \\
(T22) & \vdash \alpha \rightarrow (\alpha \land D_1\alpha), \\
(T23) & \vdash D_1\left(\bigvee_{p=0}^{m-1} f^{2p}\alpha \right) \rightarrow D_1\alpha, 1 \leq i \leq n - 1,
\end{align*}
\]
\[ \frac{T24}{D_i \alpha \rightarrow D_i \left( \bigvee_{p=0}^{m-1} f_{2p} \alpha \right), \ 1 \leq i \leq n - 1,} \]
\[ \frac{T25}{((\alpha \land f \alpha) \land (\beta \lor f \beta)) \rightarrow (\alpha \land f \alpha),} \]
\[ \frac{T26}{((\alpha \land f \alpha) \lor (((\alpha \land f \alpha) \land (\beta \lor f \beta)) \lor D_i f_{2p} \alpha)) \lor D_i f_{2p} \alpha, \ 1 \leq i \leq n - 1,} \]
\[ \frac{T27}{((\alpha \land f \alpha) \land (\beta \lor f \beta)) \lor D_i f_{2p} \alpha, \ 1 \leq i \leq n - 1,} \]
\[ \frac{T28}{D_i f_{2p} \alpha, \ 1 \leq i \leq n - 1,} \]

**Proof.** The proofs of (T6) through (T18) are routine.

\[(T19): \]
\[1. \quad (D_i \alpha \rightarrow D_j D_i \alpha) \land (D_j D_i \alpha \rightarrow D_i \alpha), \ 1 \leq i, j \leq n - 1, \quad [[A9]] \]
\[2. \quad ((D_i \alpha \rightarrow D_j D_i \alpha) \land (D_j D_i \alpha \rightarrow D_i \alpha)) \rightarrow (D_i \alpha \rightarrow D_j D_i \alpha), \ 1 \leq i, j \leq n - 1, \quad [[A5]] \]
\[3. \quad D_i \alpha \rightarrow D_j D_i \alpha, \ 1 \leq i, j \leq n - 1, \quad [[1], (2), (R1)] \]
\[4. \quad ((D_i \alpha \rightarrow D_j D_i \alpha) \land (D_j D_i \alpha \rightarrow D_i \alpha)) \rightarrow (D_j D_i \alpha \rightarrow D_i \alpha), \ 1 \leq i, j \leq n - 1, \quad [[A6]] \]
\[5. \quad D_j D_i \alpha \rightarrow D_i \alpha, \ 1 \leq i, j \leq n - 1, \quad [[1], (4), (R1)] \]
\[6. \quad D_i \alpha \rightarrow D_j D_i \alpha, \ 1 \leq i, j \leq n - 1. \quad [[(3), (5), (R2)] \]

\[(T21): \text{It follows as a consequence of (A14), (R3), (R5) and (R14).} \]

\[(T23): \]
\[1. \quad D_i \left( \bigvee_{p=0}^{m-1} f_{2p} \alpha \right) \rightarrow D_i f_{2p} \alpha, \ 1 \leq i \leq n - 1, \quad [[A10]] \]
\[2. \quad D_i \left( \bigvee_{p=0}^{m-1} f_{2p} \alpha \right) \rightarrow D_i \alpha, \ 1 \leq i \leq n - 1, \quad [[A5], (1), (R1)] \]
\[3. \quad D_i \alpha \rightarrow D_i \left( \bigvee_{p=0}^{m-1} f_{2p} \alpha \right), \ 1 \leq i \leq n - 1, \quad [[A6], (1), (R1)] \]
\[4. \quad D_i \left( \bigvee_{p=0}^{m-1} f_{2p} \alpha \right) \rightarrow D_i \alpha, \ 1 \leq i \leq n - 1. \quad [[2], (3), (R2)] \]

\[(T25): \]
\[1. \quad D_i \alpha \rightarrow D_i \alpha \land f \alpha, \ 1 \leq i \leq n - 1, 1 \leq s \leq m, \quad [[A19]] \]
\[2. \quad D_i \alpha \rightarrow (\alpha \land f \alpha) \lor (\alpha \land f \alpha) \lor (\beta \lor f \beta)), \ 1 \leq i \leq n - 1, 1 \leq s \leq m, \quad [[1], (R3)] \]
\[3. \quad D_i \alpha \rightarrow (\alpha \land f \alpha) \lor (\alpha \land f \alpha) \lor (\alpha \land f \alpha), \ 1 \leq i \leq n - 1, 1 \leq s \leq m, \quad [[2], (R5)] \]
Proposition 4.1. If \( \alpha \) is a formula derivable in \( \mathcal{L}_n^m \), then \( v(\alpha) = 1 \) for every valuation \( v \) of \( \mathcal{L} \) in every \( \mathcal{L}_n^m \)-algebra \( \mathcal{U} \).

Proof. Since \( \alpha \) is a formula derivable in \( \mathcal{L}_n^m \) if and only if \( \vdash \alpha \), then by (a1) and (a2) we conclude that \( v(\alpha) = 1 \) for every valuation \( v \) of \( \mathcal{L} \) in every \( \mathcal{L}_n^m \)-algebra \( \mathcal{U} \). \( \square \)

Proposition 4.2. Let \( \langle L, \lor, \land, f, D_1, \ldots, D_{n-1}, 0, 1 \rangle \in \mathcal{L}_n^m \). Then \( \langle L, \rightarrow, \rightarrow, \lor, \land, f, D_1, \ldots, D_{n-1}, 1 \rangle \) is an \( \mathcal{L}_n^m \)-algebra, where \( \rightarrow \) and \( \rightarrow \) are defined as in Section 1 and Section 2, respectively.

Proof. We will prove that conditions (a1) to (a4) in Section 1 hold. Indeed, taking into account the definitions of \( \rightarrow \) and \( \rightarrow \) we have that (a1) and (a2) are satisfied. On the other hand, let \( a, b \in L \) be such that \( a \rightarrow b = b \rightarrow c = 1 \). Then, by (S6) and (W3) we conclude (a3). Besides, if \( a \rightarrow b = b \rightarrow a = 1 \), hence (S5) allows us to infer (a4). \( \square \)
Proposition 4.3. Let \( \langle A, \rightarrow, \rightarrow, \vee, \wedge, f, D_1, \ldots, D_{n-1}, 1 \rangle \) be an \( \ell_m \)-algebra. Then \( \langle A, \vee, \wedge, f, D_1, \ldots, D_{n-1}, 0, 1 \rangle \in \mathcal{L}_m^m \), where \( 0 = f1 \).

Proof. From (T11), (T12) and (a4) we infer that \( f^2D_1(\alpha \rightarrow \alpha) = D_1(\alpha \rightarrow \alpha) \). Besides, from (T10) we have that \( D_1(\alpha \rightarrow \alpha) = 1 \) and so, we conclude that \( f^21 = 1 \).

This assertion and the fact that \( f1 = 0 \) imply that \( f0 = 1 \). Moreover, from (T6), (T7), (T8) and (T9) we have that conditions (m1) and (m2) in Theorem 3.2 hold. Therefore, \( (GL_1) \) is verified. Besides, by (a4) and taking into account (T13) through (T28) we infer \( (GL_2), (GL_3) \) and \( (GL_5) \) through \( (GL_{10}) \). Furthermore, from (A12) and (a1) in Section 1 we get \( (GL_4) \) and so, the proof is complete. \( \square \)

From Propositions 4.2 and 4.3 we conclude:

Theorem 4.2. The notions of the \( \ell_m \)-algebra and the \( \mathcal{L}_n^m \)-algebra are equivalent.

Let \( \equiv \) be the binary relation on \( F \) defined as follows:

\[ \alpha \equiv \beta \text{ if and only if } \vdash \alpha \rightarrow \beta \text{ and } \vdash \beta \rightarrow \alpha \text{ in } \ell_m^m. \]

Then \( \equiv \) is a congruence relation on \( \langle F, \rightarrow, \rightarrow, \wedge, \vee, f, D_1, \ldots, D_{n-1} \rangle \) and \( \mathcal{T} \) determines an equivalence class. On the other hand, it is easy to verify that the relation \( \leq \) defined on \( F/\equiv \) by

\[ [\alpha] \leq [\beta] \text{ if and only if } \vdash \alpha \rightarrow \beta, \]

is a preorder on \( F/\equiv \).

Proposition 4.4. \( \mathcal{F} = \langle F/\equiv, \rightarrow, \rightarrow, \wedge, \vee, f, D_1, \ldots, D_{n-1}, 1 \rangle \) is an \( \ell_m^m \)-algebra, and \( 1 = \mathcal{T} \).

Proof. Let \( v \) be a valuation of \( \mathcal{L} \) in \( \mathcal{F} \) and let \( \varrho \) be a substitution from \( \mathcal{L} \) into \( \mathcal{L} \) such that \( v(x) = [\varrho(x)] \) for every propositional variable \( x \) in \( \mathcal{L} \) and so, we have that

\[ (1) \quad v(\alpha) = [\varrho(\alpha)] \text{ for every formula } \alpha \text{ in } \mathcal{L}. \]

Hence, conditions (a1)–(a4) are verified. Indeed, if \( \alpha \in \mathcal{A} \), then by (s1), \( \varrho(\alpha) \in \mathcal{A} \). Thus, \( [\varrho(\alpha)] = 1 \) and consequently (a1) holds.

Suppose that a rule of inference (r) assigns to premises \( \alpha_1, \ldots, \alpha_n \) a formula \( \beta \) as the conclusion and let \( v(\alpha_i) = 1 \) for all \( i, 1 \leq i \leq n \). Thus, by (1), \( [\varrho(\alpha_i)] = 1 \) for all \( i, 1 \leq i \leq n \). Hence, by (s2) it follows that \( [\varrho(\beta)] = 1 \) and so, by (1) we have that \( v(\beta) = 1 \), which proves that (a2) holds.

Taking into account that \( \vdash \alpha \rightarrow \beta \) if and only if \( 1 = [\alpha \rightarrow \beta] = [\alpha] \rightarrow [\beta] \) we obtain that \( [\alpha] \leq [\beta] \) if and only if \( [\alpha] \rightarrow [\beta] = 1 \). From this last assertion the proofs of (a3) and (a4) are straightforward. \( \square \)

From Proposition 4.3 and Proposition 4.4 we conclude the following theorem.
Theorem 4.3. $\mathcal{F} = \langle F/\equiv, \land, \lor, f, D_1, \ldots, D_{n-1}, 0, 1 \rangle \in \mathcal{L}_n^m$.

On the other hand, since $\ell_n^m$ is consistent, from [9], VIII 7, and Theorem 4.2 we have that the completeness theorem for $\ell_n^m$ holds, which is included in Theorem 4.4.

Theorem 4.4. Let $\alpha$ be a formula of $\ell_n^m$. Then the following conditions are equivalent:

(i) $\alpha$ is derivable in $\ell_n^m$,
(ii) $\alpha$ is valid in every $L_n^m$-algebra,
(iii) $v_0(\alpha) = 1$, where $v_0$ is the canonical valuation ([9], VIII 3.4), in the algebra $\mathcal{F}$.

Proof. (i) $\Rightarrow$ (ii): It follows from the assertions given in Section 1.
(ii) $\Rightarrow$ (iii): It is straightforward.
(iii) $\Rightarrow$ (i): From the hypothesis we have that $[\alpha] = 1 = T$. Hence, $\alpha$ is derivable in $\ell_n^m$. $\square$

Remark 4.1. In case $m = 1$, we conclude that the propositional calculus $\ell_1^n$ has $n$-valued Lukasiewicz-Moisil algebras as the algebraic counterpart.

5. Conclusions

In this paper we have presented new results about the congruence lattice of $L_n^m$-algebras as well as the principal congruences by means of the standard implication. Furthermore, we have established a characterization of $k_{2m}$-lattices which has provided an easy way to prove that $L_n^m$-algebras are the algebraic counterpart of a propositional calculus. Finally, we have described a standard implicative extensional propositional calculus $\ell_n^m$ and proved that $L_n^m$-algebras and $\ell_n^m$-algebras are equivalent.

On the other hand, it would be interesting to find a sequent calculus, along with a proper notion of validity, sound and complete with respect to $L_n^m$-algebras, which has the desirable property of cut-elimination. Another interesting problem would be to present a Gentzen-style system using the tool of hypersequents.

Acknowledgement. The authors are truly thankful to the referee for his/her helpful remarks on this paper.
References


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