Małgorzata Wyrwas; Ewa Pawluszewicz; Ewa Girejko

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STABILITY OF NONLINEAR H-DIFFERENCE SYSTEMS WITH N FRACTIONAL ORDERS

Małgorzata Wyrwas, Ewa Pawluszewicz and Ewa Girejko

In the paper we study the subject of stability of systems with h-differences of Caputo–, Riemann–Liouville– and Grünwald–Letnikov–type with n fractional orders. The equivalent descriptions of fractional h-difference systems are presented. The sufficient conditions for asymptotic stability are given. Moreover, the Lyapunov direct method is used to analyze the stability of the considered systems with n-orders.

Keywords: fractional difference systems, difference operators, stability

Classification: 39A, 93D

1. INTRODUCTION

The fractional calculus is a field of mathematics that grows out of the traditional definitions of integrals, derivatives, difference operators and deals with fractional integrals, derivatives and differences of any order. Basic information on fractional calculus, concepts, ideas and their applications can be found for example in [16, 20, 32]. Dynamical systems are one of the most active areas, and several authors focused on the stability of fractional order systems, see for instance [6, 8, 9, 12, 14, 15, 17, 21, 22, 23, 24, 29, 30, 31, 33, 34, 35, 36]. Due to the lack of a geometric interpretation of the fractional derivatives and differences, it is difficult to find a valid tool to analyze the stability of fractional equations, and to our knowledge there are some works on the stability of solutions for either fractional differential equations, see [6,14,21,22,23,24,30,31,33,34,36] or fractional difference equations, see [8, 9, 12, 15, 17, 29, 35].

In the paper we focus on h-difference operators, so we are interested in the discrete fractional calculus that was initiated by Miller and Ross in [25]. Their work found its continuation in [1, 3, 4, 5, 7, 13, 17, 25, 26, 32] and others. The calculus of fractional h-differences was given for instance in [4, 5, 26, 28].

In this paper, the conditions for stability of the fractional h-difference systems are presented. The stability of systems defined by the fractional difference equations ($h = 1$) was studied for example in [8,9,15,35]. In [8,9] authors examine the asymptotic stability of nonlinear fractional difference equations with $h = 1$. They consider implicit discrete equations, so they have to prove the existence of solutions of the considered equations.

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Since we study explicit fractional difference systems with equilibrium points, solutions of the considered systems always exist and in the paper we show the recurrence formula for the solutions. Note that in [8, 9] the authors study the stability of difference equations while we formulate the conditions for the systems of h-difference equations with n fractional orders. There exist relations between cases $h > 0$ and $h = 1$ for fractional summation and differences, see [10], so we recall the transitions formulas between these operators. Later on, based on the given formulas and some results from [8, 9], we prove the sufficient conditions for the stability and asymptotic stability of the trivial solution of the considered systems. Additionally, the Lyapunov theorems presented in [35] are generalized in this paper for the considered systems. It is well known that in nonlinear systems, Lyapunov's direct method provides a way to analyze the stability of a system without explicitly solving equations. The method generalizes the idea that the system is stable if there exists a Lyapunov function, a candidate for the system. The Lyapunov stability of differential equations has been studied in [14] where the authors propose Lyapunov stability theorem for fractional systems without delay and extend the theorem for fractional systems with delay. The difference between proposed theory and the fractional Lyapunov direct one is that they take the integer derivative instead of the fractional derivative of the positive definite function V. The similar idea is used in [35] where the integer difference is taken instead of fractional one of the Lyapunov function V. In [35] we propose Lyapunov stability theorems for the Caputo–type difference systems with two fractional orders and with the step $h = 1$. The proposed method can be extended for studying the stability of solutions to the Caputo–, Riemann–Liouville–type as well as Grünwald–Letnikov–type h-difference (with arbitrary $h > 0$) fractional nonlinear systems with n orders. Therefore the facts presented in the paper are the generalization of results given in [35]. In [27] it is shown that the Riemann–Liouville–type fractional h-difference operator and the Caputo–type fractional h-difference operator are related to each other. Moreover, the Grünwald–Letnikov–type fractional h-difference operator can be expressed by the Riemann–Liouville–type fractional h-difference operator. So, systems with these operators can be studied simultaneously. The analysis of Lyapunov direct method to h-fractional systems is the same for all h-difference operators, so we decided to formulate the sufficient conditions for the stability of the fractional h-difference systems with n orders with the Caputo– and Riemann–Liouville–type operators as facts. In the paper we show that the positive number h that appears in the presented operators can be treated as a parameter.

The paper is organized as follows. In Section 2 we gather some definitions, notations and results needed in the sequel. Some of the presented results concerning h-factorial functions, where $h > 0$, are the generalizations of properties given in [8,9] (e. g. Lemmas 2.3 and 2.4). Section 3 contains the equivalent description of nonlinear fractional difference systems with n-orders. Since the Grünwald–Letnikov–type fractional h-difference operator can be directly expressed by the the Riemann–Liouville–type fractional h-difference operator, we consider only the Riemann–Liouville– and Caputo–type ones. The sufficient conditions for asymptotic stability of considered systems are given in Section 4. In Subsection 4.1 the analysis of Lyapunov direct method to nonlinear h-fractional systems is briefly presented. The proofs of results given in this part are the same as in [35] for the Caputo–type difference systems with two fractional orders, so we only state the conditions for stability where the Lyapunov function is used and for the proofs we refer the reader to [35]. Finally, Section 5 provides the conclusions.

2. PRELIMINARIES

Let $\mathcal{F}_D$ denote the set of real valued functions defined on D. Let $\alpha > 0$ and $h > 0$. For $a \in \mathbb{R}$ we define $(h\mathbb{N})_a := \{a, a+h, a+2h, ...\}$ and the *forward operator* $\sigma : (h\mathbb{N})_a \to (h\mathbb{N})_a$ is defined by $\sigma(t) := t+h$. The next definitions of h-difference operators were originally given in [5], here we propose a simpler notation.

Definition 2.1. For a function $x \in \mathcal{F}_{(h\mathbb{N})_a}$ the *forward h-difference operator* is defined as

$$(\Delta_h x)(t) := \frac{x(\sigma(t)) - x(t)}{h}, \quad t = a + nh,\, n \in \mathbb{N}_0\,,$$

while the *h-difference sum* is given by

$$\left({}_a\Delta_h^{-1} x\right)(t) := h \sum_{k=0}^{n} x(a+kh)\,,$$

where $t = a + (n+1)h$, $n \in \mathbb{N}_0$, and $\left({}_a\Delta_h^{-1} x\right)(a) := 0$.

Definition 2.2. For arbitrary $t, \alpha \in \mathbb{R}$ the *h-factorial function* is defined by

$$t_h^{(\alpha)} := h^\alpha \frac{\Gamma(\frac{t}{h} + 1)}{\Gamma(\frac{t}{h} + 1 - \alpha)}\,, \tag{1}$$

where Γ is the Euler gamma function, $\frac{t}{h} \notin \mathbb{Z}_- := \{-1, -2, -3, \ldots\}$, and we use the convention that division at a pole yields zero.

There is the following relation between the *h-factorial function* and a factorial function with $h = 1$

$$t_h^{(\alpha)} := h^\alpha \left(\frac{t}{h}\right)_1^{(\alpha)}\,. \tag{2}$$

Hereinafter, if $h = 1$, we will write $t^{(\alpha)}$ instead of $t_1^{(\alpha)}$. If we use the general binomial coefficient $\binom{a}{b} := \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}$, then (1) can be rewritten as

$$t_h^{(\alpha)} = h^\alpha \Gamma(\alpha+1) \binom{\frac{t}{h}}{\alpha}\,.$$

For the general binomial coefficient we use the same convention as for the h-factorial function, i. e. the division at a pole yields zero. Moreover, for $\tau \in (h\mathbb{N})_0$ one gets

$$\tau_h^{(\alpha)} = h^\alpha \Gamma(\alpha+1) \binom{n}{\alpha} = h^\alpha n^{(\alpha)} = h^\alpha \left(\frac{\tau}{h}\right)^{(\alpha)}\,,$$

where $\tau = nh$, $n \in \mathbb{N}_0$, h^α and $\Gamma(\alpha+1)$ are positive numbers.

Now, we will present some of the properties of the h-factorial function in order to show that they are very similar to the properties of a factorial function with $h = 1$.

In [2] the authors show the following formula for fractional functions with $h = 1$

$$t^{(\beta+\gamma)} = (t-\gamma)^{(\beta)} \cdot t^{(\gamma)}\,. \tag{3}$$

The formula (3) can be generalized for arbitrary h-factorial functions, where $h > 0$, as follows:

Lemma 2.3. Let $t_h^{(\beta+\gamma)}$, $(t-\gamma h)_h^{(\beta)}$ and $t_h^{(\gamma)}$ be well defined. Then

$$t_h^{(\beta+\gamma)} = (t-\gamma h)_h^{(\beta)} \cdot t_h^{(\gamma)}$$

or equivalently,

$$\binom{\frac{t}{h}}{\beta+\gamma} \cdot \binom{\beta+\gamma}{\beta} = \binom{\frac{t}{h}-\gamma}{\beta} \cdot \binom{\frac{t}{h}}{\gamma}. \tag{4}$$

P r o o f. By (2) and (3) we get

$$t_h^{(\beta+\gamma)} = h^{\beta+\gamma} \cdot \left(\frac{t}{h}\right)^{(\beta+\gamma)} = h^{\beta} \cdot \left(\frac{t}{h}-\gamma\right)^{(\beta)} \cdot h^{\gamma} \cdot \left(\frac{t}{h}\right)^{(\gamma)} = (t-\gamma h)_h^{(\beta)} \cdot t_h^{(\gamma)} .$$

□

Similarly, the following inequality

$$t^{(-\beta)} > (t+\alpha)_h^{(-\beta)}, \tag{5}$$

where $\alpha, \beta, t > 0$, presented in [9], can be generalized for $h > 0$ in the following way:

Lemma 2.4. Let $\gamma, \alpha, t > 0$ and $t_h^{(-\alpha)}$, $(t+\gamma h)_h^{(-\alpha)}$ be well defined. Then

$$t_h^{(-\alpha)} > (t+\gamma h)_h^{(-\alpha)} .$$

or equivalently,

$$\binom{\frac{t}{h}}{-\alpha} > \binom{\frac{t}{h}+\gamma}{-\alpha}. \tag{6}$$

P r o o f. By (2) and (5) we have

$$t_h^{(-\alpha)} = h^{-\alpha}\left(\frac{t}{h}\right)^{(-\alpha)} > h^{-\alpha}\left(\frac{t}{h}+\gamma\right)^{(-\alpha)} = (t+\gamma h)_h^{(-\alpha)} .$$

□

The h-factorial function is used to define the fractional h-sum of order $\alpha > 0$ for a real valued function.

Definition 2.5. For a function $x \in \mathcal{F}_{(h\mathbb{N})_a}$ the *fractional h-sum of order* $\alpha > 0$ is given by

$$\begin{aligned}\left({}_a\Delta_h^{-\alpha} x\right)(t) &:= \frac{h}{\Gamma(\alpha)} \sum_{k=0}^{n} (t-\sigma(a+kh))_h^{(\alpha-1)} x(a+kh) \\ &= h^{\alpha} \sum_{k=0}^{n} \binom{\frac{t-a}{h}-k-1}{\alpha-1} x(a+kh),\end{aligned}$$

where $t = a+(\alpha+n)h$, $\sigma(a+sh) = a+sh+h$ and $\left({}_a\Delta_h^{0} x\right)(t) := x(t)$.

Note that ${}_a\Delta_h^{-\alpha} : \mathcal{F}_{(h\mathbb{N})_a} \to \mathcal{F}_{(h\mathbb{N})_{a+\alpha h}}$. Accordingly to the definition of the h-factorial function we can write that for $t = a + (\alpha+n)h$, $n \in \mathbb{N}_0$:

$$\left({}_a\Delta_h^{-\alpha}x\right)(t) = h^\alpha \sum_{k=0}^{n} \frac{\Gamma(\alpha+n-k)}{\Gamma(\alpha)\Gamma(n-k+1)} x(a+kh) = h^\alpha \sum_{k=0}^{n} \binom{n-k+\alpha-1}{n-k} x(a+kh)\,.$$

Moreover, for $p \neq 0$ the following relation:

$$\Delta_h (t-a)_h^{(p)} = h^p \Gamma(p+1) \Delta_h \binom{\frac{t-a}{h}}{p} = h^{p-1}\Gamma(p+1)\binom{\frac{t-a}{h}}{p-1} = p(t-a)_h^{(p-1)}$$

holds. In [10] it is shown that if $\psi(r) = (r-a+\mu h)_h^{(\mu)}$, $r \in (h\mathbb{N})_a$, $t \in (h\mathbb{N})_{a+\alpha h}$, then the following power rule formula

$$\left({}_a\Delta_h^{-\alpha}\psi\right)(t) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a+\mu h)_h^{(\mu+\alpha)} = h^{\mu+\alpha}\Gamma(\mu+1)\binom{\frac{t-a}{h}+\mu}{\mu+\alpha} \tag{7}$$

holds. Then if $\psi \equiv 1$, for $\mu = 0$ and $t = nh + a + \alpha h$ we have

$$\left({}_a\Delta_h^{-\alpha}1\right)(t) = \frac{1}{\Gamma(\alpha+1)}(t-a)_h^{(\alpha)} = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)} h^\alpha = \binom{n+\alpha}{n} h^\alpha\,.$$

From the application of the power rule it follows the rule for composing two fractional h-sums. The proof for the case $h = 1$ one can find in [13] and for $h > 0$ in [26].

Proposition 2.6. (Mozyrska and Girejko [26]) Let x be a real valued function defined on $(h\mathbb{N})_a$, where $a, h \in \mathbb{R}, h > 0$. For $\alpha, \beta > 0$ the following equalities hold:

$$\left({}_{a+\beta h}\Delta_h^{-\alpha}\left({}_a\Delta_h^{-\beta}x\right)\right)(t) = \left({}_a\Delta_h^{-(\alpha+\beta)}x\right)(t) = \left({}_{a+\alpha h}\Delta_h^{-\beta}\left({}_a\Delta_h^{-\alpha}x\right)\right)(t)\,, \tag{8}$$

where $t \in (h\mathbb{N})_{a+(\alpha+\beta)h}$.

The fractional h-sum of order $\alpha > 0$ is characterized by the following property and later on used in the proofs of stability theorems.

Lemma 2.7. Let $a \in \mathbb{R}$, $x_i : (h\mathbb{N})_a \to \mathbb{R}$, $i = 1, 2$, $x_1(a) = x_2(a)$ and $\alpha > 0$. If $x_1(t) \le x_2(t)$ for all $t \in (h\mathbb{N})_a$, then

$$\left({}_a\Delta_h^{-\alpha}x_1\right)(t) \le \left({}_a\Delta_h^{-\alpha}x_2\right)(t)$$

for arbitrary $t \in (h\mathbb{N})_{a+\alpha h}$.

P r o o f. Assume that $x_1(t) \le x_2(t)$ for all $t \in (h\mathbb{N})_a$. Then $(x_2 - x_1)(t) \ge 0$ and

$$\left({}_a\Delta_h^{-\alpha}(x_2-x_1)\right)(t+\alpha h) = \frac{h}{\Gamma(\alpha)} \sum_{k=0}^{n} (t-\sigma(a+kh))_h^{(\alpha-1)}(x_2-x_1)(a+kh)\,.$$

Since $h>0$, $\Gamma(\alpha)>0$ and both $(t-\sigma(a+kh))_h^{(\alpha-1)}>0$ and $(x_2-x_1)(t)\geq 0$,

$$\left({}_a\Delta_h^{-\alpha}(x_2-x_1)\right)(t+\alpha h)\geq 0\,.$$

Therefore $\left({}_a\Delta_h^{-\alpha}x_1\right)(t)\leq\left({}_a\Delta_h^{-\alpha}x_2\right)(t)$ for arbitrary $t\in(h\mathbb{N})_{a+\alpha h}$. □

As the first we present the Riemann–Liouville–type fractional h-difference operator. The definition of this operator can be found, for example, in [2] (for $h=1$) or in [5] (for $h>0$). Later on, the Caputo– and the Grünwald–Letnikov–type fractional h-difference operator is given.

Definition 2.8. Let $\alpha\in(0,1]$. The *Riemann–Liouville–type fractional h-difference operator* ${}_a\Delta_h^\alpha$ of order α for a function $x\in\mathcal{F}_{(h\mathbb{N})_a}$ is defined by

$$\left({}_a\Delta_h^\alpha x\right)(t)=\left(\Delta_h\left({}_a\Delta_h^{-(1-\alpha)}x\right)\right)(t),\ \ t\in(h\mathbb{N})_{a+(1-\alpha)h}\,.$$

Note that ${}_a\Delta_h^\alpha:\mathcal{F}_{(h\mathbb{N})_a}\to\mathcal{F}_{(h\mathbb{N})_{a+(1-\alpha)h}}$ for $\alpha\in(0,1]$.

Definition 2.9. (Mozyrska and Girejko [26]) Let $\alpha\in(0,1]$. The *Caputo–type h-difference operator* ${}_a\Delta_{h,*}^\alpha$ of order α for a function $x\in\mathcal{F}_{(h\mathbb{N})_a}$ is defined by

$$\left({}_a\Delta_{h,*}^\alpha x\right)(t)=\left({}_a\Delta_h^{-(1-\alpha)}\left(\Delta_h x\right)\right)(t),\ \ t\in(h\mathbb{N})_{a+(1-\alpha)h}\,.$$

Note that: ${}_a\Delta_{h,*}^\alpha:\mathcal{F}_{(h\mathbb{N})_a}\to\mathcal{F}_{(h\mathbb{N})_{a+(1-\alpha)h}}$ for $\alpha\in(0,1]$.

For $\alpha\in(0,1]$ one gets

$$\left({}_a\Delta_{h,*}^\alpha x\right)(t)=\left({}_a\Delta_h^\alpha x\right)(t)-\frac{x(a)\cdot(t-a)_h^{(-\alpha)}}{\Gamma(1-\alpha)}=\left({}_a\Delta_h^\alpha x\right)(t)-\frac{x(a)}{h^\alpha}\binom{\frac{t-a}{h}}{-\alpha},\tag{9}$$

for $t\in(h\mathbb{N})_{a+(1-\alpha)h}$.

The last operator that we take under our consideration is the fractional h-difference Grünwald–Letnikov–type operator, see for example [18,19,32] for $h=1$ and also for the general case $h>0$.

Definition 2.10. Let $\alpha\in\mathbb{R}$. The *Grünwald–Letnikov–type h-difference operator* ${}_a\widetilde{\Delta}_h^\alpha$ of order α for a function $x\in\mathcal{F}_{(h\mathbb{N})_a}$ is defined by

$$\left({}_a\widetilde{\Delta}_h^\alpha x\right)(t)=\sum_{s=0}^{\frac{t-a}{h}}a_s^{(\alpha)}x(t-sh)$$

where

$$a_s^{(\alpha)}=(-1)^s\binom{\alpha}{s}\frac{1}{h^\alpha}$$

with

$$\binom{\alpha}{s}=\begin{cases}1 & \text{for } s=0\\ \frac{\alpha(\alpha-1)\ldots(\alpha-s+1)}{s!} & \text{for } s\in\mathbb{N}.\end{cases}$$

Note that ${}_a\widetilde{\Delta}_h^\alpha:\mathcal{F}_{(h\mathbb{N})_a}\to\mathcal{F}_{(h\mathbb{N})_a}$.

Since the Grünwald–Letnikov–type h-difference operator can be expressed by the Riemann–Liouville–type fractional h-difference operator, see [27], we restrict our consideration only to the Caputo– and Riemann–Liouville–type fractional h-difference operators.

Hereinafter, if $h = 1$, then we use ${}_{n_0}\Delta^{-\alpha}$, ${}_{a}\Delta^{\alpha}_{*}$ and ${}_{a}\Delta^{\alpha}$ instead of ${}_{n_0}\Delta^{-\alpha}_{1}$, ${}_{a}\Delta^{\alpha}_{1,*}$ and ${}_{a}\Delta^{\alpha}_{1}$, respectively.

The next propositions give useful identities of transforming fractional difference equations into fractional summations. These properties are the generalization of the results given in [11].

Proposition 2.11. Let $\alpha \in (0,1]$, $h > 0$, $a = (\alpha-1)h$, $n_0 \in \mathbb{N}_0$, $t_0 = a + n_0 h \in (h\mathbb{N})_a$ and x be a real valued function defined on $(h\mathbb{N})_{t_0}$. Then

$$\left({}_{n_0h}\Delta_h^{-\alpha}\left({}_{t_0}\Delta_h^{\alpha}x\right)\right)(t) = x(t) - x(t_0)\cdot\frac{h^{1-\alpha}}{\Gamma(\alpha)}(t-n_0h)_h^{(\alpha-1)} = x(t) - x(t_0)\cdot\binom{\frac{t-n_0h}{h}}{\alpha-1},$$

for $t \in (h\mathbb{N})_{\alpha h + n_0 h}$.

Proof. Let us define the function $\tilde{x} : (h\mathbb{N})_a \to \mathbb{R}$ as follows: $\tilde{x}(t) := x(n_0h + t)$, where $t \in (h\mathbb{N})_a$. Using Definition 2.5 it is easy to see that for the functions $y : (h\mathbb{N})_b \to \mathbb{R}$ and $\tilde{y} : (h\mathbb{N})_0 \to \mathbb{R}$ such that $\tilde{y}(\tau) = y(\tau - b)$, $\tau \in (h\mathbb{N})_0$, we have

$$\left({}_{b}\Delta_h^{-\beta}y\right)(t) = \left({}_{0}\Delta_h^{-\beta}\tilde{y}\right)(t-b), \tag{10}$$

where $b \in \mathbb{R}$, $\beta > 0$, $t \in (h\mathbb{N})_{b+\beta h}$.

Let $\psi(\tau_1) := \left({}_{t_0}\Delta_h^{\alpha}x\right)(\tau_1)$, $\tau_1 \in (h\mathbb{N})_{n_0h}$ and $\tilde{\psi}(\tau_2) := \left({}_{a}\Delta_h^{\alpha}\tilde{x}\right)(\tau_2)$, $\tau_2 \in (h\mathbb{N})_0$. Then, by (10), for $n \in \mathbb{N}_0$ one gets:

$$\begin{aligned}\left({}_{n_0h}\Delta_h^{-\alpha}\left({}_{t_0}\Delta_h^{\alpha}x\right)\right)(n_0h+\alpha h+nh) &= \left({}_{n_0h}\Delta_h^{-\alpha}\psi\right)(n_0h+\alpha h+nh)\\ &= \left({}_{0}\Delta_h^{-\alpha}\tilde{\psi}\right)(\alpha h+nh)\\ &= \left({}_{0}\Delta_h^{-\alpha}\left({}_{a}\Delta_h^{\alpha}\tilde{x}\right)\right)(\alpha h+nh).\end{aligned}$$

In [26, Proposition 4.4] the following relation

$$\left({}_{0}\Delta_h^{-\alpha}\left({}_{a}\Delta_h^{\alpha}\tilde{x}\right)\right)(t) = \tilde{x}(t) - \tilde{x}(a)\cdot\frac{h^{1-\alpha}}{\Gamma(\alpha)}(t)_h^{(\alpha-1)} = \tilde{x}(t) - \tilde{x}(a)\cdot\binom{\frac{t}{h}}{\alpha-1} \tag{11}$$

was showed. Hence, by (11), one gets

$$\begin{aligned}\left({}_{0}\Delta_h^{-\alpha}\left({}_{a}\Delta_h^{\alpha}\tilde{x}\right)\right)(\alpha h+nh) &= \tilde{x}(\alpha h+nh) - \tilde{x}(a)\cdot\frac{h^{1-\alpha}}{\Gamma(\alpha)}(\alpha h+nh)_h^{(\alpha-1)}\\ &= x(n_0h+\alpha h+nh) - x(t_0)\cdot\frac{h^{1-\alpha}}{\Gamma(\alpha)}(\alpha h+nh)_h^{(\alpha-1)}\\ &= x(n_0h+\alpha h+nh) - x(t_0)\cdot\binom{n+\alpha}{\alpha-1}.\end{aligned}$$

Then for $t = n_0h+\alpha h+nh \in (h\mathbb{N})_{\alpha h+n_0h}$ one gets $\alpha h+nh = t-n_0h$ and consequently, the thesis holds. □

Corollary 2.12. Let $\alpha \in (0,1]$, $a = \alpha - 1$, $n_0 \in \mathbb{N}_0$, $t_0 = a + n_0$ and x be a real valued function defined on $\mathbb{N}_{t_0}$. Then for $h = 1$ we have

$$\left({}_{n_0}\Delta^{-\alpha}\left({}_{t_0}\Delta^{\alpha}x\right)\right)(t) = x(t) - x(t_0)\cdot\binom{t-n_0}{\alpha-1}$$
$$= x(t) - x(t_0)\cdot\binom{n+\alpha}{\alpha-1} = x(t) - x(t_0)\cdot\binom{n+\alpha}{n+1},$$

where $t_0 = \alpha - 1 + n_0$ and $t = \alpha + n_0 + n \in \mathbb{N}_{\alpha+n_0}$, $n \in \mathbb{N}_0$.

Proposition 2.13. Let $\alpha \in (0,1]$, $h > 0$, $a = (\alpha-1)h$, $t_0 = a + n_0h \in (h\mathbb{N})_a$ and x be a real valued function defined on $(h\mathbb{N})_{t_0}$. Then

$$\left({}_{n_0h}\Delta_h^{-\alpha}\left({}_{t_0}\Delta_{h,*}^{\alpha}x\right)\right)(t) = x(t) - x(t_0), \quad t \in (h\mathbb{N})_{\alpha h + n_0h}.$$

P r o o f. By Definition 2.9 and the formula (8) we have

$$\left({}_{n_0h}\Delta_h^{-\alpha}\left({}_{t_0}\Delta_{h,*}^{\alpha}x\right)\right)(t) = \left({}_{n_0h}\Delta_h^{-\alpha}\left({}_{t_0}\Delta_h^{-(1-\alpha)}\left(\Delta_h x\right)\right)\right)(t)$$
$$= \left({}_{t_0}\Delta_h^{-1}\left(\Delta_h x\right)\right)(t).$$

Then by the definition of the fractional h-sum of order 1, see Definition 2.5, we get

$$\left({}_{t_0}\Delta_h^{-1}\left(\Delta_h x\right)\right)(t) = \sum_{k=0}^{n}\left(t - \sigma(t_0 + kh)\right)_h^{(0)}\left[x(t_0 + (k+1)h) - x(t_0 + kh)\right]$$
$$= x(t_0 + (n+1)h) - x(t_0) = x(\alpha h + n_0h + nh) - x(t_0) = x(t) - x(t_0)$$

for $t = \alpha h + n_0h + nh \in (h\mathbb{N})_{\alpha h + n_0h}$. □

Corollary 2.14. Let $\alpha \in (0,1]$, $a = \alpha - 1$, $n_0 \in \mathbb{N}_0$, $t_0 = a + n_0$ and x be a real valued function defined on $\mathbb{N}_{t_0}$. Then

$$\left({}_{n_0}\Delta^{-\alpha}\left({}_{t_0}\Delta_*^{\alpha}x\right)\right)(t) = x(t) - x(t_0), \quad t = \alpha + n_0 + n \in \mathbb{N}_{\alpha+n_0}.$$

Let us now prove some properties of fractional operators that are used in the study of stability of considered systems.

Lemma 2.15. Let $\beta \in (0,1]$, $h > 0$, $b = (\beta-1)h$, $t_0 = b + n_0h \in (h\mathbb{N})_b$, x and y be real valued functions defined on $(h\mathbb{N})_b$. If $\left({}_{t_0}\Delta_{h,*}^{\beta}x\right)(nh) \geq \left({}_{t_0}\Delta_{h,*}^{\beta}y\right)(nh)$ for $n \geq n_0$ and $x(t_0) = y(t_0)$, then $x(t) \geq y(t)$ for $t \in (h\mathbb{N})_{t_0}$.

P r o o f. Assume that $\left({}_{t_0}\Delta_{h,*}^{\beta}x\right)(nh) \geq \left({}_{t_0}\Delta_{h,*}^{\beta}y\right)(nh)$ for $n \geq n_0$ and $x(t_0) = y(t_0)$. Then by Lemma 2.7 we get

$$\left({}_{n_0h}\Delta_h^{-\beta}\left({}_{t_0}\Delta_{h,*}^{\beta}x\right)\right)(t) \geq \left({}_{n_0h}\Delta_h^{-\beta}\left({}_{t_0}\Delta_{h,*}^{\beta}y\right)\right)(t), \tag{12}$$

for all $t \in (h\mathbb{N})_{n_0 h+\beta h}$. From Proposition 2.13 the relation (12) can be rewritten as

$$x(nh+\beta h) - x(t_0) \geq y(nh+\beta h) - y(t_0)\,,$$

for all $n \geq n_0$. Since $x(t_0) = y(t_0)$, we have $x(nh+\beta h) \geq y(nh+\beta h)$. Consequently, $x(t) \geq y(t)$ for $t \in (h\mathbb{N})_{t_0}$. □

Lemma 2.16. Let $\beta \in (0,1]$, $h > 0$, $b = (\beta-1)h$, $t_0 = b + n_0 h \in (h\mathbb{N})_b$, x and y be real valued functions defined on $(h\mathbb{N})_b$. If $\left({}_{t_0}\Delta_h^{\beta} x\right)(nh) \geq \left({}_{t_0}\Delta_h^{\beta} y\right)(nh)$ for $n \geq n_0$ and $x(t_0) = y(t_0)$, then $x(t) \geq y(t)$ for $t \in (h\mathbb{N})_{t_0}$.

P r o o f. Assume that $\left({}_{t_0}\Delta_h^{\beta} x\right)(nh) \geq \left({}_{t_0}\Delta_h^{\beta} y\right)(nh)$ for $n \geq n_0$ and $x(t_0) = y(t_0)$. Then by Lemma 2.7 we get

$$\left({}_{n_0 h}\Delta_h^{-\beta}\left({}_{t_0}\Delta_h^{\beta} x\right)\right)(t) \geq \left({}_{n_0 h}\Delta_h^{-\beta}\left({}_{t_0}\Delta_h^{\beta} y\right)\right)(t)\,, \tag{13}$$

for all $t \in (h\mathbb{N})_{n_0 h+\beta h}$. From Proposition 2.11 the relation (13) can be rewritten as follows

$$x(nh+\beta h) - x(t_0)\cdot\binom{n+\alpha}{\alpha-1} \geq y(nh+\beta h) - y(t_0)\cdot\binom{n+\alpha}{\alpha-1},$$

for all $n \geq n_0$. Since $x(t_0) = y(t_0)$, we have $x(nh+\beta h) \geq y(nh+\beta h)$. Consequently, $x(t) \geq y(t)$ for $t \in (h\mathbb{N})_{t_0}$. □

There exist relations between fractional summation operators for any $h > 0$ and $h = 1$ and between fractional difference operators for any $h > 0$ and $h = 1$. In [10] the following properties that give transition between these operators are proved.

Lemma 2.17. (Ferreira and Torres [10]) Let $x : (h\mathbb{N})_a \to \mathbb{R}$ and $\alpha > 0$. Then

$$\left({}_a\Delta_h^{-\alpha} x\right)(t) = h^{\alpha}\left({}_{\frac{a}{h}}\Delta^{-\alpha}\bar{x}\right)\left(\frac{t}{h}\right),$$

$$\left({}_a\Delta_{h,*}^{\alpha} x\right)(t) = h^{-\alpha}\left({}_{\frac{a}{h}}\Delta_*^{\alpha}\bar{x}\right)\left(\frac{t}{h}\right) \text{ and } \left({}_a\Delta_h^{\alpha} x\right)(t) = h^{-\alpha}\left({}_{\frac{a}{h}}\Delta^{\alpha}\bar{x}\right)\left(\frac{t}{h}\right),$$

where $t \in (h\mathbb{N})_{a+\alpha h}$ and $\bar{x}(s) := x(sh)$.

Lemma 2.17 shows that the positive number h can be treated as a parameter and its values do not influence on the property of the h-difference operators.

3. EQUIVALENT DESCRIPTIONS OF FRACTIONAL H-DIFFERENCE SYSTEMS WITH N ORDERS

Let $i = 1,\ldots,n$ and $0 < \alpha_i \leq 1$. Let us consider the following fractional Caputo h-difference system with n orders $\alpha_1,\ldots,\alpha_n$:

$$\left({}_{t_{0i}}\Delta_{h,*}^{\alpha_i} x_i\right)(t) = f_i(t, x_1(a_1+t), x_2(a_2+t), \ldots, x_n(a_n+t))\,, \tag{14}$$

with initial values

$$x_i(t_{0i}) = x_{0i} \in \mathbb{R}\,, \tag{15}$$

where $a_i = (\alpha_i - 1)h \in (-h, 0] \subset \mathbb{R}$, $t_{0i} = a_i + n_0 h \in (h\mathbb{N})_{a_i}$, $n_0 \in \mathbb{N}_0$, $t \in (h\mathbb{N})_{n_0 h}$, $f_i : (h\mathbb{N})_0 \times \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, n$, are continuous. By Lemma 2.17 system (14) can be rewritten as

$$\left({}_{\frac{t_{0i}}{h}}\Delta_*^{\alpha_i} \bar{x}_i\right)\left(\frac{t}{h}\right) = h^{\alpha_i} f_i\left(t, \bar{x}_1\left(\alpha_1 - 1 + \frac{t}{h}\right), \dots, \bar{x}_n\left(\alpha_n - 1 + \frac{t}{h}\right)\right),$$

where $\bar{x}_i(\tau) := x(\tau h)$. Note that $\bar{x}_i \in \mathcal{F}_{\mathbb{N}_{\alpha_i - 1 + n_0}}$, i.e. $\bar{x}_i : \mathbb{N}_{\alpha_i - 1 + n_0} \to \mathbb{R}$. Obviously, for $\tilde{x}_i(k) := \bar{x}_i\left(\frac{t_{0i}}{h} + k\right) = x_i(t_{0i} + kh)$, by (10), system (14) can also be rewritten as

$$\left({}_0\Delta_*^{\alpha_i} \tilde{x}_i\right)(k + 1 - \alpha_i) = h^{\alpha_i} f_i\left(kh + n_0 h, \tilde{x}_1(k), \dots, \tilde{x}_n(k)\right), \tag{16}$$

where $k \in \mathbb{N}_0$ and $\tilde{x}_i \in \mathcal{F}_{\mathbb{N}_0}$, i.e. $\tilde{x}_i : \mathbb{N}_0 \to \mathbb{R}$.

Remark 3.1. If the Riemann–Liouville–type fractional h-difference operator ${}_{t_{0i}}\Delta_h^{\alpha_i}$ is used instead of the Caputo–type h-difference operator ${}_{t_{0i}}\Delta_{h,*}^{\alpha_i}$ in (14), then one gets the fractional Riemann–Liouville h-difference system with n orders, i.e.

$$\left({}_{t_{0i}}\Delta_h^{\alpha_i} x_i\right)(t) = f_i(t, x_1(a_1 + t), x_2(a_2 + t), \dots, x_n(a_n + t))\,. \tag{17}$$

By Lemma 2.17 system (17) can be rewritten as

$$\left({}_{\frac{t_{0i}}{h}}\Delta^{\alpha_i} \bar{x}_i\right)\left(\frac{t}{h}\right) = h^{\alpha_i} f_i\left(t, \bar{x}_1\left(\alpha_1 - 1 + \frac{t}{h}\right), \dots, \bar{x}_n\left(\alpha_n - 1 + \frac{t}{h}\right)\right),$$

where $\bar{x}_i(\tau) := x(\tau h)$. Note that $\bar{x}_i \in \mathcal{F}_{\mathbb{N}_{\alpha_i - 1 + n_0}}$, i.e. $\bar{x}_i : \mathbb{N}_{\alpha_i - 1 + n_0} \to \mathbb{R}$. Obviously, for $\tilde{x}_i(k) := \bar{x}_i\left(\frac{t_{0i}}{h} + k\right) = x_i(t_{0i} + kh)$, by (10), system (17) can also be rewritten as

$$\left({}_0\Delta^{\alpha_i} \tilde{x}_i\right)(k + 1 - \alpha_i) = h^{\alpha_i} f_i\left(kh + n_0 h, \tilde{x}_1(k), \dots, \tilde{x}_n(k)\right), \tag{18}$$

where $k \in \mathbb{N}_0$ and $\tilde{x}_i \in \mathcal{F}_{\mathbb{N}_0}$, i.e. $\tilde{x}_i : \mathbb{N}_0 \to \mathbb{R}$. Moreover, by (9) one has

$$\left({}_0\Delta^{\alpha_i} \tilde{x}_i\right)(k + 1 - \alpha_i) = \left({}_0\Delta_*^{\alpha_i} \tilde{x}_i\right)(k + 1 - \alpha_i) + \tilde{x}_i(0)\binom{k + 1 - \alpha_i}{-\alpha_i}.$$

We restrict our consideration to the fractional Caputo and Riemann–Liouville h-difference systems with n orders, because the Grünwald–Letnikov–type fractional h-difference operator ${}_{t_{0i}}\widetilde{\Delta}_h^{\alpha_i}$ can be expressed using the Riemann–Liouville–type fractional h-difference operator ${}_{t_{0i}}\Delta_h^{\alpha_i}$, see [27].

The constant vector $X^{\mathrm{e}} := (x_1^{\mathrm{e}}, x_2^{\mathrm{e}}, \dots, x_n^{\mathrm{e}})^{\mathrm{T}}$ is an *equilibrium point* from time $t_0 = n_0 h$ of fractional difference system (14) if and only if

$$\left({}_{t_{0i}}\Delta_{h,*}^{\alpha_i} x_i^{\mathrm{e}}\right)(t) = f_i\left(t, x_1^{\mathrm{e}}, x_2^{\mathrm{e}}, \dots, x_n^{\mathrm{e}}\right)$$

(and $\left({}_{t_{0i}}\Delta_h^{\alpha_i} x_i^{\mathrm{e}}\right)(t) = f_i\left(t, x_1^{\mathrm{e}}, x_2^{\mathrm{e}}, \dots, x_n^{\mathrm{e}}\right)$ in the case of the Riemann–Liouville h-difference systems), $i = 1, \dots, n$ for all $t \in (h\mathbb{N})_{n_0 h}$.

Remark 3.2. For the Caputo h-difference system $\left({}_{t_{0i}}\Delta_{h,*}^{\alpha_i} x_i^{\mathrm{e}}\right)(t) \equiv 0$, so the constant vector $X^{\mathrm{e}} = (x_1^{\mathrm{e}}, x_2^{\mathrm{e}}, \ldots, x_n^{\mathrm{e}})^{\mathrm{T}}$ is an equilibrium point from time $t_0 = n_0 h$ of the Caputo fractional h-difference system (14) if and only if $f_i(t, x_1^{\mathrm{e}}, \ldots, x_n^{\mathrm{e}}) = 0$, $i = 1, \ldots, n$ for all $t \in (h\mathbb{N})_{n_0 h}$.

For simplicity, we state all definitions and theorems for the case when the equilibrium point is the origin of $\mathbb{R}^n$, i.e. $x_i^{\mathrm{e}} = 0$, $i = 1, \ldots, n$. There is no loss of generality in doing so because any equilibrium point can be shifted to the origin via certain change of variables. Assume that the equilibrium point for (14) or (17) is $X^{\mathrm{e}} \neq 0$. Then for $j \in \{1, \ldots, n\}$ such that $x_j^{\mathrm{e}} \neq 0$ we consider the change of variable $y_j(t) = x_j(t) - x_j^{\mathrm{e}}$. The α_jth order Caputo h-difference of y_j is given by

$$\begin{aligned}\left({}_{t_{0j}}\Delta_{h,*}^{\alpha_j} y_j\right)(t) &= \left({}_{t_{0j}}\Delta_{h,*}^{\alpha_j}(x_j - x_j^{\mathrm{e}})\right)(t) = f(t, x_1(t), \ldots, x_n(t)) \\ &= f_j(t, y_1(t) + x_1^{\mathrm{e}}, \ldots, y_n(t) + x_n^{\mathrm{e}}) = g_j^1(t, y_1(t), \ldots, y_n(t))\end{aligned}$$

and similarly α_jth order Riemann–Liouville h-difference of y_j is given by

$$\begin{aligned}\left({}_{t_{0j}}\Delta_h^{\alpha_j} y_j\right)(t) &= \left({}_{t_{0j}}\Delta_h^{\alpha_j}(x_j - x_j^{\mathrm{e}})\right)(t) = f(t, x_1(t), \ldots, x_n(t)) - \frac{x_j^{\mathrm{e}}}{h^{\alpha_j}} \cdot \binom{\frac{t-t_{0j}}{h}}{-\alpha_j} \\ &= f_j(t, y_1(t) + x_1^{\mathrm{e}}, \ldots, y_n(t) + x_n^{\mathrm{e}}) - \frac{x_j^{\mathrm{e}}}{h^{\alpha_j}} \cdot \binom{\frac{t-t_{0j}}{h}}{-\alpha_j} \\ &= g_j^2(t, y_1(t), \ldots, y_n(t)),\end{aligned}$$

where $g_j^\ell(t, 0) = 0$, $\ell = 1, 2$, and in the new variables y_j, $j = 1, \ldots, n$, the fractional h-difference systems have equilibriums at the origin.

Without loss of generality, let $x_i^{\mathrm{e}} = 0$, $i = 1, \ldots, n$, be the equilibrium point. Then for the Riemann–Liouville h-difference system we get $({}_{t_{0i}}\Delta_h^{\alpha_i} 0)(t) \equiv 0$.

Let $\tilde{f}_i : (h\mathbb{N})_{n_0 h} \to \mathbb{R}$ be defined as $\tilde{f}_i(kh) := f_i(kh, x_1(a_1 + kh), \ldots, x_n(a_n + kh))$ for $i = 1, \ldots, n$ and $k \in \mathbb{N}_{n_0}$. We apply the operator ${}_{n_0 h}\Delta_h^{-\alpha_i}$ to equations of (14) and from Proposition 2.13 we get the Caputo recurrence formula of the following form

$$x_i(\alpha_i h + n_0 h + kh) = x_{0i} + h^{\alpha_i} \sum_{j=0}^{k} \binom{k - j + \alpha_i - 1}{k - j} \tilde{f}_i(n_0 h + jh),$$

for $k \in \mathbb{N}_0$. Note that $a_i = (\alpha_i - 1)h$, , $t_{0i} = a_i + n_0 h$, $i = 1, 2, \ldots, n$, so

$$\begin{aligned}x_i(t_{0i} + (k+1)h) &= x_{0i} + h^{\alpha_i} \sum_{j=0}^{k} \binom{k - j + \alpha_i - 1}{k - j} \cdot \tilde{f}_i(n_0 h + jh) \\ &= x_{0i} + h^{\alpha_i} \sum_{j=0}^{k} \binom{k - j - n_0 + \frac{t_{0i}}{h}}{k - j} \cdot \\ &\quad \cdot f_i(n_0 h + kh, x_1(t_{01} + kh), \ldots, x_n(t_{0n} + kh)).\end{aligned}$$

Since $\binom{j+\alpha_i-1}{j} = (-1)^j\binom{-\alpha_i}{j}$ for $j \geq 1$ and $\binom{-\alpha_i}{0} = 1$, for all $k \in \mathbb{N}_0$ and $i = 1, \ldots, n$ one gets:

$$\begin{aligned} x_i(\alpha_i h + (n_0+k)h) =& x_{0i} + h^{\alpha_i}\sum_{j=0}^{k}(-1)^j\binom{-\alpha_i}{j}\tilde{f}_i(n_0h+(k-j)h) \\ =& x_{0i} + h^{\alpha_i}\sum_{j=0}^{k}(-1)^j\binom{-\alpha_i}{j}\cdot f_i(n_0h+(k-j)h, \\ & x_1(\alpha_1+n_0h+(k-j-1)h), \ldots, x_n(\alpha_n+n_0h+(k-j-1)h))\,. \end{aligned}$$

Applying the operator ${}_{n_0h}\Delta_h^{-\alpha_i}$ to the Riemann–Liouville h-difference system (17) and using Proposition 2.11 one gets for $k \in \mathbb{N}_0$ the following equivalent recursive formulas describing the system:

$$x_i(t_{0i}+(k+1)h) = x_{0i}\cdot\binom{k+\alpha_i}{k+1} + h^{\alpha_i}\sum_{j=0}^{k}\binom{k-j+\alpha_i-1}{k-j}\tilde{f}_i(n_0h+jh)\,,$$

or

$$\begin{aligned} x_i(t_{0i}+(k+1)h) = & x_{0i}\cdot\binom{k+\alpha_i}{k+1} + h^{\alpha_i}\sum_{j=0}^{k}(-1)^j\binom{-\alpha_i}{j}\tilde{f}_i(n_0h+(k-j)h) \\ =& x_{0i}\cdot\binom{k+\alpha_i}{k+1} + h^{\alpha_i}\sum_{j=0}^{k}(-1)^j\binom{-\alpha_i}{j}\cdot f_i(n_0h+(k-j)h, \\ & x_1(\alpha_1+n_0h+(k-j-1)h), \ldots, x_n(\alpha_n+n_0h+(k-j-1)h))\,. \end{aligned}$$

Note that if $x_i(t_{0i}) = x_i^{\mathrm{e}} = 0$, then $x_i(\alpha_i + n_0h + kh) \equiv x_i^{\mathrm{e}} = 0$, $i = 1, \ldots, n$ for all $k \in \mathbb{N}_0$.

For $k \in \mathbb{N}_{n_0}$, $n_0 \in \mathbb{N}_0$, let us define

$$\left({}_{n_0}\Delta^{(\alpha)}X\right)(k) := \begin{bmatrix} \left({}_{t_{01}}\Delta_{h,*}^{\alpha_1}x_1\right)(kh) \\ \vdots \\ \left({}_{t_{0n}}\Delta_{h,*}^{\alpha_n}x_n\right)(kh) \end{bmatrix} \tag{19a}$$

or

$$\left({}_{n_0}\Delta^{(\alpha)}X\right)(k) := \begin{bmatrix} \left({}_{t_{01}}\Delta_{h}^{\alpha_1}x_1\right)(kh) \\ \vdots \\ \left({}_{t_{0n}}\Delta_{h}^{\alpha_n}x_n\right)(kh) \end{bmatrix} \tag{19b}$$

for the Caputo or Riemann–Liouville h-difference systems, respectively, and

$$F(k, X(k)) := \begin{bmatrix} f_1(kh, x_1(a_1+kh), x_2(a_2+kh), \ldots, x_n(a_n+kh)) \\ \vdots \\ f_n(kh, x_1(a_1+kh), x_2(a_2+kh), \ldots, x_n(a_n+kh)) \end{bmatrix}. \tag{20}$$

Applying (19a) and (20) to (14) (or applying (19b) and (20) to (17)) one can rewrite the considered fractional Caputo (or Riemann–Liouville) h-difference system with n orders, i. e. system (14) (or (17)), as follows

$$\left({}_{n_0}\Delta^{(\alpha)} X\right)(k) = F(k, X(k)) \,, \tag{21}$$

where $F : \mathbb{N}_0 \times \mathbb{R}^n \to \mathbb{R}^n$. Then the initial value corresponding to (21) has the form

$$X_0 := \begin{bmatrix} x_1(a_1 + n_0 h) \\ \vdots \\ x_n(a_n + n_0 h) \end{bmatrix} = \begin{bmatrix} x_{01} \\ \vdots \\ x_{0n} \end{bmatrix} \in \mathbb{R}^n \,. \tag{22}$$

Let us define inductively the sequence of mappings

$$S^k : \mathbb{R}^n \to \mathbb{R}^n$$

by

$$\begin{aligned} S^0(X) &= X \\ S^{k+1}(X) &= I_k \cdot X + \sum_{j=0}^{k} (-1)^j \cdot \Lambda_j \cdot F(n_0 + k - j, S^{k-j}(X)) \,, \end{aligned}$$

where $\Lambda_j = \begin{bmatrix} \binom{-\alpha_1}{j} & 0 & \dots & 0 \\ 0 & \binom{-\alpha_2}{j} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \binom{-\alpha_n}{j} \end{bmatrix} \in \mathbb{R}^{n\times n}$, and

$$\mathbb{R}^{n\times n} \ni I_k = \begin{cases} \begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix}, & \text{for Caputo } h\text{-difference systems} \\ \begin{bmatrix} \binom{k+\alpha_1}{k+1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \binom{k+\alpha_n}{k+1} \end{bmatrix}, & \text{for Riemann–Liouville } h\text{-difference systems.} \end{cases}$$

Note that for $k \geq 0$ $X(k) = S^k(X_0)$ is the solution of system (21) that is uniquely defined by the initial state $X_0 \in \mathbb{R}^n$. Therefore

$$S^k(X_0) = \begin{bmatrix} x_1(a_1 + n_0 h + kh) \\ x_2(a_2 + n_0 h + kh) \\ \vdots \\ x_n(a_n + n_0 h + kh) \end{bmatrix} = \begin{bmatrix} \bar{x}_1(\frac{a_1}{h} + n_0 + h) \\ \bar{x}_2(\frac{a_2}{h} + n_0 + h) \\ \vdots \\ \bar{x}_n(\frac{a_n}{h} + n_0 + h) \end{bmatrix} = \begin{bmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \\ \vdots \\ \tilde{x}_n(k) \end{bmatrix} .$$

For each $p \in \mathbb{R}^n$, let us denote by $S^k(p)$ the value at time k of the solution of (21) starting at p. Therefore the relation

$$X(k) = S^k(X_0) = [\tilde{x}_1(k), \dots, \tilde{x}_n(k)]^{\mathrm{T}}$$

gives us the equivalent description of the system (21) with the initial condition X_0. Note that corresponding to the equilibrium point $X^{\mathrm{e}} = (x_1^{\mathrm{e}}, \dots, x_n^{\mathrm{e}})^{\mathrm{T}} = (0, \dots, 0)^{\mathrm{T}}$, the system (21) has a constant solution $S^k(X^{\mathrm{e}}) \equiv X^{\mathrm{e}}$, $k \in \mathbb{N}_0$.

Let $f_i(t, 0, \dots, 0) = 0$, $i = 1, \dots, n$, for all $t \in \mathbb{N}_{n_0}$, so that the system (14) (or the system (17) with Riemann–Liouville–type h-difference operator instead of Caputo one) admits the trivial solution, i. e.

$$\begin{bmatrix} x_1(a_1 + t) \\ x_2(a_2 + t) \\ \vdots \\ x_n(a_n + t) \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n,$$

for all $t \in \mathbb{N}_{n_0 h}$. Then $F(k, 0) = 0$ for all $k \in \mathbb{N}_{n_0}$ and consequently system (21) admits the trivial solution $S^k(0) \equiv 0$, $k \in \mathbb{N}_0$.

Remark 3.3. Note that by (19a), (20) and (22) the solutions of systems (14) and (21) coincide. Of course, from (19b), (20) and (22) the solutions of systems (17) and (21) are the same. Moreover, system (14) (or (17)) has the trivial solutions if and only if system (21) has the trivial solution.

4. STABILITY

Let $X(\cdot, X_0)$ denote the solution of (21) with initial condition (22). Then

$$X(k, X_0) = \begin{bmatrix} x_1(a_1 + n_0 h + kh) \\ x_2(a_2 + n_0 h + kh) \\ \vdots \\ x_n(a_n + n_0 h + kh) \end{bmatrix} = \begin{bmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \\ \vdots \\ \tilde{x}_n(k) \end{bmatrix} = S^k(X_0)\,,$$

where $k \in \mathbb{N}_0$ and $x_i : (h\mathbb{N})_{(\alpha_i - 1)h} \to \mathbb{R}$, $i = 1, \dots, n$, are the solutions of (14) (or (17)) with the initial conditions (15). Therefore the stability of the system with both the Caputo– and Riemann–Liouville–type h-difference operators is studied simultaneously.

Let $\|\cdot\|$ denote a vector norm.

Definition 4.1. The trivial solution of (21) ((14) or (17)) is said to be

(i) *stable* if, for each $\epsilon > 0$ and $n_0 \in \mathbb{N}_0$, there exists $\delta = \delta(\epsilon, n_0) > 0$ such that $\|X_0\| < \delta$ implies $\|S^k(X_0)\| < \epsilon$, for all $k \in \mathbb{N}_{n_0}$.

(ii) *uniformly stable* if it is stable and δ depends solely on ϵ, i. e. for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $\|X_0\| < \delta$ implies $\|S^k(X_0)\| < \epsilon$, for all $k \in \mathbb{N}_{n_0}$.

(iii) *attractive* if there exists $\delta(t_0) > 0$ such that $\|X_0\| < \delta$ implies

$$\lim_{k \to \infty} X(k, X_0) = 0\,.$$

(iv) *asymptotically stable* if it is stable and attractive.

(v) *uniformly asymptotically stable* if it is uniformly stable and, for each $\epsilon > 0$, there exists $T = T(\epsilon) \in \mathbb{N}_0$ and $\delta_0 > 0$ such that $\|X_0\| < \delta_0$ implies $\|S^k(X_0)\| < \epsilon$ for all $k \in \mathbb{N}_{n_0+T}$ and for all $n_0 \in \mathbb{N}_0$.

(vi) *globally asymptotically stable* if it is asymptotically stable for all $X_0 \in \mathbb{R}^n$.

(vii) *globally uniformly asymptotically stable* if it is uniformly asymptotically stable for all $X_0 \in \mathbb{R}^n$.

In order to show the stability of the fractional-order system the comparison principle given in Lemma 2.15 can be used. The following example shows the conditions (i. e. negative values of functions f_i, $i = 1, 2, \ldots, n$) that guarantee the stability of the equilibrium point.

Example 4.2. Consider the fractional-order system defined by the following equations:

$$\left({}_{t_0}\Delta^{\alpha_i}_{h,*} x_i\right)(t) = f_i(t, x_1((\alpha_1 - 1)h + t), \ldots, x_n((\alpha_n - 1)h + t)), \tag{23}$$

where $i = 1, 2, \ldots, n$, $\alpha_i \in (0, 1]$, $t_{0i} = (\alpha_i - 1)h + n_0 h$, $t \in (h\mathbb{N})_{n_0 h}$, $x_i(t_{0i}) = x_{0i} \geq 0$, $f_i : (h\mathbb{N})_{n_0 h} \times \mathbb{R}^n \to \mathbb{R}$, $x^{\mathrm{e}} = (x^{\mathrm{e}}_1, \ldots, x^{\mathrm{e}}_n) = (0, \ldots, 0)$ is the equilibrium point of (23) and $f_i(t, x_1((\alpha_1 - 1)h + t), \ldots, x_n((\alpha_n - 1)h + t)) < 0$. Note that $\left({}_{t_0}\Delta^{\alpha_i}_{h,*} x^{\mathrm{e}}_i\right)(t) = 0 > f_i(t, x_1((\alpha_1 - 1)h + t), \ldots, x_n((\alpha_n - 1)h + t)) = \left({}_{t_0}\Delta^{\alpha_i}_{h,*} x_i\right)(t)$. Hence, by Lemma 2.15 we get $x_i(t) \leq x_{0i}$ for $t \in (h\mathbb{N})_{t_0}$. Since $x = 0$ is the equilibrium point of (23) and $0 \leq x_i \leq x^{\mathrm{e}}_i$, the equilibrium point $x^{\mathrm{e}} = 0$ is stable.

Moreover, if one replaces the Caputo–type difference operator ${}_{t_0}\Delta^{\alpha_i}_{h,*}$ by the Riemann–Liouville–type operator ${}_{t_0}\Delta^{\alpha_i}_{h}$ in the left hand side of (23), then by Lemma 2.16 one gets $x_i(t) \leq x_{0i}$ for $t \in (h\mathbb{N})_{t_0}$ and similarly as for the Caputo difference systems we have $0 \leq x_i \leq x^{\mathrm{e}}$, so the equilibrium point $x^{\mathrm{e}} = 0$ is stable.

In many cases the stability of systems can be checked on the bases on the right hand side of the equations. Note that the considered systems can be transformed to the forms (16) and (18), where the Caputo– and Riemann–Liouville–type difference operators, i.e ${}_0\Delta^{\alpha}_*$ and ${}_0\Delta^{\alpha}$, are used, respectively. Therefore basing on the results from [9] we get the sufficient conditions that guarantee the stability of the considered systems.

Let $g_{h,\alpha_i} : \mathbb{N}_1 \to \mathbb{R}$ be defined as follows

$$g_{h,\alpha_i}(k) := \begin{cases} 0, & \text{for the Riemann–Liouville difference systems,} \\ h^{-\alpha_i} \cdot \binom{k}{-\alpha_i}, & \text{for the Caputo difference systems.} \end{cases}$$

Theorem 4.3. Let $i = 1, \ldots, n$. If there exist constants $\beta_{1i} \in (\alpha_i, 1)$ and $L_{1i} \geq 0$ such that for all $k \in \mathbb{N}_1$ we have

$$\begin{aligned} &|f_i(n_0 h + kh, x_1(t_{01} + kh), \ldots, x_n(t_{01} + kh)) + x_i(t_{0i}) \cdot g_{h,\alpha_i}(k)| \\ &\qquad \leq L_{1i} \cdot h^{-\alpha_i} \cdot \binom{n_0 + \alpha_i + k}{-\beta_{1i}}, \end{aligned} \tag{24}$$

then the solution $X(\cdot, X_0)$ is attractive.

Proof. For the Riemann–Liouville difference systems by (24) and using the results from [9, Theorem 3.3], the solutions of (17) are in the set

$$\mathcal{S}_1 = \{X(k) = (x_1(t_{01}+kh), \dots, x_n(t_{0n}+kh)) : \\ |x_i(t_{0i}+kh)| \leqslant \binom{\alpha_i + k}{-\gamma_{1i}} \text{ for } k \in \mathbb{N}_{n_0} \Big\},$$

where $\gamma_{1i} = \frac{1}{2}(\beta_{1i} - \alpha_i) > 0$ and $n_0 \in \mathbb{N}_0$ such that

$$\Gamma(1-\gamma_{1i}) \cdot \left[\frac{|x_0|}{\Gamma(\alpha_i)} (n_1 + \alpha_i + \gamma_{1i})^{(\frac{1}{2}(\alpha_i + \beta_{1i}) - 1)} + L_{1i} \frac{(n_1 + \alpha_i + \gamma_{1i})^{(-\gamma_{1i})}}{\Gamma(1 + \alpha_i - \beta_{1i})} \right] \leq 1 .$$

From (9) the Caputo difference system (14) can be rewritten as the Riemann–Liouville one as follows:

$$\left({}_{t_{0i}}\Delta_h^{\alpha_i} x_i\right)(t) = f_i(t, x_1(a_1+t), x_2(a_2+t), \dots, x_n(a_n+t)) + x_i(t_{0i}) \cdot g_{h,\alpha_i}(k) . \quad (25)$$

Note that the solutions of (25) coincide with the solutions of (14) and due to (24) they are in the set $\mathcal{S}_1$.

Since

$$\binom{\alpha_i + k}{-\gamma_{1i}} = \frac{\Gamma(\alpha_i + k + 1)}{\Gamma(1-\gamma_{1i})\Gamma(\alpha_i + k + \gamma_{1i} + 1)} = \frac{1}{\Gamma(1-\gamma_{1i})} (\alpha_i + k)^{-\gamma_{1i}} \left[1 + O\left(\frac{1}{\alpha_i + k} \right) \right]$$

tends to 0 as $k \to \infty$, the vector function $X(k) = (x_1(t_{01}+kh), \dots, x_n(t_{0n}+kh))$ that belongs to $\mathcal{S}_1$ tends to zero as $k \to \infty$ as well. Hence $X(\cdot, X_0) = X(k)$ is attractive.

□

Theorem 4.4. Let $i = 1, \dots, n$ and $\|x_i\| = \sup_{k \in \mathbb{N}_0} |x_i(t_{0i}+kh)|$. If there exist constants $\beta_{2i} \in (\alpha_i, 1)$ and $L_{2i} \geq 0$ such that

$$|f_i(n_0 h + kh, x_1(t_{01}+kh), \dots, x_n(t_{01}+kh)| \leq L_{2i} \cdot h^{-\alpha_i} \cdot \binom{n_0 + \alpha_i + k}{-\beta_{2i}} \cdot \|x_i\| , \quad (26)$$

for $k \in \mathbb{N}_0$, then the trivial solution of (21) is stable provided that

$$c_i := L_{2i} \binom{\alpha_i - 1}{\beta_{2i} - 1} = \frac{L_{2i}\Gamma(\alpha_i)}{\Gamma(\beta_{2i})\Gamma(1 + \alpha_i - \beta_{2i})} < 1 , \quad (27)$$

for $i = 1, \dots, n$.

Proof. Let $X(k) = S^k(X_0)$ be the solution of (21) satisfying the initial condition $X(0) = X_0$ and

$$\iota_i^k = \begin{cases} 1, & \text{for the Caputo difference systems,} \\ \binom{k+\alpha_i-1}{k} = \binom{k+\alpha_i-1}{\alpha_i-1}, & \text{for the Riemann–Liouville difference systems.} \end{cases}$$

Since $\alpha_i - 1 < 0$, by (6) for $k \in \mathbb{N}_1$ we have $\binom{k+\alpha_i-1}{\alpha_i-1} \leq \binom{\alpha_i-1}{\alpha_i-1} = 1$ and consequently, $0 < \iota_i^k \leq 1$. Then $X(k) = (x_1(t_{01}+kh), \dots, x_n(t_{0n}+kh)) \in \mathbb{R}^n$ and using (6), (7), Definition 2.5 and assumption (26) one gets

$$\begin{aligned}
|x_i(t_{0i}+kh)| &\leq |x_{0i}|\iota_i^k + h^{\alpha_i} \sum_{j=0}^{k-1} \binom{k+\alpha_i-j-2}{k-j-1} \cdot |\tilde{f}_i(n_0h+jh)| \\
&\leq |x_{0i}|\iota_i^k + L_{2i} \cdot \sum_{j=0}^{k-1} \binom{k+\alpha_i-j-2}{k-j-1} \cdot \binom{n_0+\alpha_i+j}{-\beta_{2i}} \cdot \|x_i\| \\
&= |x_{0i}|\iota_i^k + L_{2i} \cdot {}_0\Delta^{-\alpha_i} \binom{n_0+\alpha_i+k-1}{-\beta_{2i}} \cdot \|x_i\| \\
&= |x_{0i}|\iota_i^k + L_{2i} \cdot \binom{n_0+\alpha_i+k-1}{\alpha_i-\beta_{2i}} \cdot \|x_i\| \\
&\leq |x_{0i}| + L_{2i}\binom{\alpha_i-1}{\alpha_i-\beta_{2i}}\|x_i\| = |x_{0i}| + c_i\|x_i\|\,,
\end{aligned}$$

for $k \geq 1$. Therefore

$$\|x_i\| \leq \frac{1}{1-c_i}|x_{0i}|\,.$$

Let $C = \max_i c_i$. For any given $\varepsilon > 0$ let $\delta = \frac{1-C}{n}\varepsilon$ and $\|X_0\|_{\mathrm{E}} = \sqrt{\sum_{i=1}^n x_{0i}^2} < \delta$. Then we get

$$\begin{aligned}
\|X(k)\|_{\mathrm{E}} = \|S^k(X_0)\|_{\mathrm{E}} &= \sqrt{\sum_{i=1}^n (x_i(t_{0i}+kh))^2} \leq \sum_{i=1}^n |x_i(t_{0i}+kh)| \\
&\leq \sum_{i=1}^n \|x_i\| \leq \sum_{i=1}^n \frac{1}{1-c_i}|x_{0i}| \leq \frac{1}{1-C}\sum_{i=1}^n |x_{0i}| \leq \frac{n}{1-C}\max_i |x_{0i}| \\
&\leq \frac{n}{1-C}\sqrt{\sum_{i=1}^n x_{0i}^2} = \frac{n}{1-C}\|X_0\|_{\mathrm{E}} < \frac{n}{1-C}\delta = \varepsilon\,.
\end{aligned}$$

Hence the trivial solution of (21) is stable. □

As a simple consequence of Theorems 4.3 and 4.4 we get the following result:

Corollary 4.5. If the conditions (24) and (26) hold, then the solution $X(\cdot, X_0)$ is asymptotically stable provided that (27) holds.

Additionally, Theorem 3.6 given in [9] can be generalized for h-difference systems as follows:

Theorem 4.6. Let $i = 1, \dots, n$. If there exist constants $\beta_{3i} \in \left(\alpha_i, \frac{1}{2}(1+\alpha_i)\right)$, $\gamma_{2i} = \frac{1}{2}(1-\alpha_i)$ and $L_{3i} \geq 0$ such that for all $k \in \mathbb{N}_0$ we have

$$|f_i(n_0h+kh, x_1(t_{01}+kh),\dots,x_n(t_{01}+kh)+x(t_{0i})\cdot g_{h,\alpha_i}(k)| \\ \le L_{3i}\cdot h^{-\alpha_i}\cdot\binom{n_0+\alpha_i+\gamma_{2i}+k}{-\beta_{3i}}|x_i(t_{0i}+kh)|\,, \quad (28)$$

then the solution $X(\cdot,X_0)$ is attractive, i. e. it is in the following set

$$\mathcal{S}_2=\Big\{X(k)=(x_1(t_{01}+kh),\dots,x_n(t_{0n}+kh)): \\ |x_i(t_{0i}+kh)|\leqslant\binom{\alpha_i+k}{-\gamma_{2i}} \text{ for } k\in\mathbb{N}_{n_2}\Big\}\,,$$

where $n_2\in\mathbb{N}_0$ such that

$$\Gamma(1-\gamma_{2i})\cdot\left[\frac{|x_0|}{\Gamma(\alpha_i)}(n_2+\alpha_i+\gamma_{2i})^{(-\gamma_{2i})}+L_{3i}\frac{(n_2+\alpha_i+\gamma_{2i})^{(\alpha_i-\beta_{3i})}}{\Gamma(1+\alpha_i-\beta_{3i}-\gamma_{2i})}\right]\le 1\,. \quad (29)$$

P r o o f. Note that by (28) we have

$$\begin{aligned}
|x_i(t_{0i}+kh)| \le & |x_{0i}|\cdot\binom{k+\alpha_i-1}{k} \\
& + h^{\alpha_i}\sum_{j=0}^{k-1}\binom{k+\alpha_i-j-2}{k-j-1}|\tilde f_i(n_0h+jh)+x(t_{0i})\cdot g_{h,\alpha_i}(k)| \\
\le & |x_{0i}|\cdot\binom{k+\alpha_i-1}{k} \\
& + L_{3i}\sum_{j=0}^{k-1}\binom{k+\alpha_i-j-2}{\alpha_i-1}\binom{n_0+\alpha_i+\gamma_{2i}+j}{-\beta_{3i}}|x_i(t_{0i}+jh)| \\
\le & |x_{0i}|\cdot\binom{k+\alpha_i-1}{k}+L_{3i}\sum_{j=0}^{k-1}\binom{k+\alpha_i-j-2}{\alpha_i-1}\binom{\alpha_i+\gamma_{2i}+j}{-\beta_{3i}}\binom{\alpha_i+j}{-\gamma_{2i}} \\
= & |x_{0i}|\cdot\binom{k+\alpha_i-1}{k}+L_{3i}\sum_{j=0}^{k-1}\binom{k+\alpha_i-j-2}{\alpha_i-1}\binom{\alpha_i+j}{-\gamma_{2i}-\beta_{3i}} \\
= & |x_{0i}|\cdot\binom{k+\alpha_i-1}{k}+L_{3i0}\Delta^{-\alpha_i}\binom{\alpha_i+k-1}{-\gamma_{2i}-\beta_{3i}} \\
= & |x_{0i}|\cdot\binom{k+\alpha_i-1}{k}+L_{3i}\binom{\alpha_i+k-1}{\alpha_i-\gamma_{2i}-\beta_{3i}}\,.
\end{aligned}$$

Then by (29) for $k\ge n_2$ one gets

$$\begin{aligned}
|x_i(t_{0i}+kh)| \le & |x_{0i}|\binom{k+\alpha_i-1}{\alpha_i-1}+L_{3i}\binom{\alpha_i+k-1}{\alpha_i-\gamma_{2i}-\beta_{3i}} \\
\le & \Gamma(1-\gamma_{2i})\cdot\left[\frac{|x_0|}{\Gamma(\alpha_i)}(k+\alpha_i+\gamma_{2i})^{(-\gamma_{2i})}\right. \\
& \left.+L_{3i}\frac{(k+\alpha_i+\gamma_{2i})^{(\alpha_i-\beta_{3i})}}{\Gamma(1+\alpha_i-\beta_{3i}-\gamma_{2i})}\right]\binom{\alpha_i+k}{-\gamma_{2i}}\le\binom{\alpha_i+k}{-\gamma_{2i}}\,.
\end{aligned}$$

Hence the solutions of (21) are in $\mathcal{S}_2$. Since

$$\binom{\alpha_i+k}{-\gamma_{2i}} = \frac{\Gamma(\alpha_i+k+1)}{\Gamma(1-\gamma_{2i})\Gamma(\alpha_i+k+\gamma_{2i}+1)} = \frac{1}{\Gamma(1-\gamma_{2i})}(\alpha_i+k)^{-\gamma_{2i}}\left[1+O\left(\frac{1}{\alpha_i+k}\right)\right]$$

tends to 0 as $k\to\infty$, the vector function $X(k)=(x_1(t_{01}+kh),\ldots,x_n(t_{0n}+kh))$ that belongs to $\mathcal{S}_2$, tends to zero as $k\to\infty$. Hence $X(\cdot,X_0)=X(k)$ is attractive. □

Theorem 4.7. If the conditions (26) and (28) hold, then the solution $X(\cdot,X_0)$ is asymptotically stable provided that (27) holds.

Proof. From Theorem 4.4 the condition (26) gives the stability of the trivial solution provided that (27) holds. Assumption (28) implies that the solution $X(\cdot,X_0)=X(k)$ is attractive. Hence $X(\cdot,X_0)$ is asymptotically stable. □

Example 4.8. Let $h>0$, $0.5<\beta_1<1$ and $0.25<\beta_2<1$. Let us consider the following system of equations:

$$\left({}_{t_{01}}\Delta_h^{0.5}x_1\right)(kh) \quad = \tbinom{k+0.5}{-\beta_1}\sin(x_2(kh-0.75h)) \tag{30a}$$

$$\left({}_{t_{02}}\Delta_h^{0.25}x_2\right)(kh) \quad = \tbinom{k+0.25}{-\beta_2}\cdot x_1(kh-0.5h)\,, \tag{30b}$$

where $t_{01}=-0.5h$ and $t_{02}=-0.75h$, $k\in\mathbb{N}_0$, with initial conditions $x_1(-0.5h)=x_{01}$ and $x_2(-0.75h)=x_{02}$. Since

$$|f_1(kh,x_2(kh-0.75h))| = \left|\binom{k+0.5}{-\beta_1}\sin(x_2(kh-0.75h))\right| \le \binom{k+0.5}{-\beta_1},$$

there exists $L_{11}=h^{0.5}>0$ and by Theorem 4.3 we get

$$|x_1(kh-0.5h)| \le \binom{k+0.5}{\frac{1}{2}\,(0.5-\beta_1)},$$

for all $k\in\mathbb{N}_1$. Moreover, since $\binom{k+0.5}{\frac{1}{2}(0.5-\beta_1)}<1$ for all $k\in\mathbb{N}_0$, $|f_2(kh),x_1(kh-0.5h)| = \left|\binom{k+0.25}{-\beta_2}x_1(kh-0.5h)\right| \le \binom{k+0.25}{-\beta_2}$. Consequently, there exists $L_{12}=h^{0.75}>0$ and by Theorem 4.3 we get

$$|x_2(kh-0.75h)| \le \binom{k+0.25}{\frac{1}{2}\,(0.25-\beta_2)},$$

for all $k\in\mathbb{N}_1$. Therefore the solution is attractive, i. e.

$$\lim_{k\to\infty}(x_1(kh-0.5h),x_2(kh-0.75h))=(0,0)\,.$$

Note that

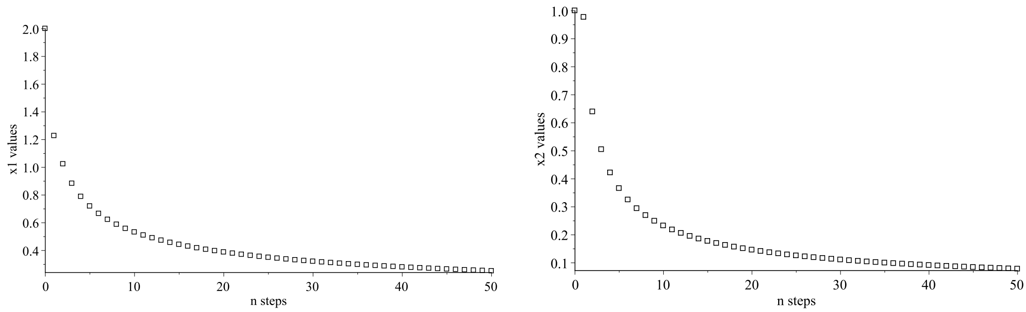
$$\|x_1\| = \sup_{k\in\mathbb{N}_0}|x_1(kh-0.5h)| < \max\{1,|x_{01}|\}$$

and

$$\|x_2\| = \sup_{k\in\mathbb{N}_0}|x_2(kh-0.75h)| < \max\{1,|x_{01}|\}\ .$$

Then if $h < (\Gamma(0.25))^{-4}$, then condition (27) is satisfied and by Theorem 4.4 the considered system is stable. Note that the point $(0,0)$ is the equilibrium point of the considered system.

Let $\beta_1 = \beta_2 = 0.6$. Then the values of x_1 and x_2 for $n = 1,\ldots,50$ are displayed in Figures 1(a) and 1(b), respectively.



(a) the graph of x_1 (b) the graph of x_2

Fig. 1. The graphs for Example 4.8.

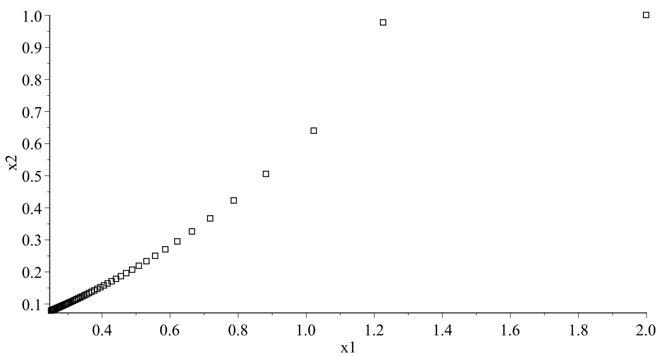


Fig. 2. The phase portrait for Example 4.8.

Figure 2 shows the phase portrait of (x_1, x_2) for $n = 0,\ldots,50$, at which we see that the trajectory is tending to the equilibrium point $(0,0)$.

4.1. Lyapunov stability

The facts presented in this section are the generalization of the results given in [35], where the Caputo–type fractional difference systems with two orders and $h = 1$ were studied. The class-$\mathcal{K}$ functions are applied to the analysis of fractional Lyapunov direct method.

Definition 4.9. A continuous function $\phi : [0, \rho] \to [0, \infty)$ is said to *belong to class-$\mathcal{K}$* (or *be class-$\mathcal{K}$ function*) if $\phi(0) = 0$ and ϕ is strictly increasing.

Definition 4.10. If $\phi : [0, \infty) \to [0, \infty)$, $\phi \in \mathcal{K}$, and $\lim_{r\to\infty} \phi(r) = \infty$, then ϕ is said to *belong to class-$\mathcal{KR}$* (or *be class-$\mathcal{KR}$ function*).

Let $\mathcal{U}$ be a neighbourhood of the origin.

Definition 4.11. A real valued function V defined on $\mathbb{N}_0 \times \mathcal{U}$ is said to be *positive definite* if and only if $V(k, 0) = 0$ for all $k \in \mathbb{N}_0$ and there exists $\phi \in \mathcal{K}$ such that $\phi(r) \leq V(k, X)$, $\|X\| = r$, $(k, X) \in \mathbb{N}_0 \times \mathcal{U}$.

Definition 4.12. A real valued function V defined on $\mathbb{N}_0 \times \mathcal{U}$, is said to be *decrescent* if and only if $V(k, 0) = 0$ for all $k \in \mathbb{N}_0$ and there exists $\varphi \in \mathcal{K}$ such that $V(k, X) \leq \varphi(r)$, $\|X\| = r$, $(k, X) \in \mathbb{N}_0 \times \mathcal{U}$.

Now, let us formulate conditions providing stability of solutions of the nonlinear h-fractional system with n orders given by (21), in particular by (14) or (17). Since the solutions of both (14) and (17) are given as sequences $(X(n, X_0))_{n\in\mathbb{N}_0}$ parameterized by h, the sufficient conditions for Lyapunov stability given in [35] for systems with two fractional orders with the Caputo–type operator with $h = 1$ can be generalized for system (21). We present them as facts; for the proofs see [35].

Let $\overline{V} : \mathbb{N}_0 \to \mathbb{R}$ and $\overline{V}(k) := V(k, X(k))$ for $k \in \mathbb{N}_0$.

Fact 4.13. If there exist a neighborhood $\mathcal{U}$ of the origin and a continuous, positive definite and decrescent scalar function $V : \mathbb{N}_0 \times \mathcal{U} \to [0, \infty)$ such that

$$\left(\Delta\overline{V}\right)(k) := \overline{V}(k + 1) - \overline{V}(k) \leq 0$$

for all $k \in \mathbb{N}_0$, then the trivial solution of (21) (or equivalently system (14) or (17)) is uniformly stable.

Fact 4.14. If there exist a neighbourhood $\mathcal{U}$ of the origin and a continuous, positive definite and decrescent scalar function $V : \mathbb{N}_0 \times \mathcal{U} \to [0, \infty)$ such that

$$\left(\Delta\overline{V}\right)(k) \leqslant -\psi\left(\|X(k)\|\right)$$

for all $k \in \mathbb{N}_0$, where $\psi \in \mathcal{K}$, then the trivial solution of (21) (or equivalently system (14) or (17)) is uniformly asymptotically stable.

Fact 4.15. If there exists a continuous function $V : \mathbb{N}_0 \times \mathbb{R}^N \to [0, \infty)$ such that

$$\forall (k, X) \in \mathbb{N}_{n_0} \times \mathbb{R}^n : \quad \phi(\|X(k)\|) \leq \overline{V}(k) \leq \varphi(\|X(k)\|),$$

$$\forall n_0 \in \mathbb{N}_0,\ (k, X) \in \mathbb{N}_{n_0} \times \mathbb{R}^n : \ \left(\Delta\overline{V}\right)(k) \leq -\psi\left(\|X(k)\|\right),$$

where $\phi, \varphi, \psi \in \mathcal{KR}$, then the trivial solution of (21) (or equivalently system (14) or (17)) is globally uniformly asymptotically stable.

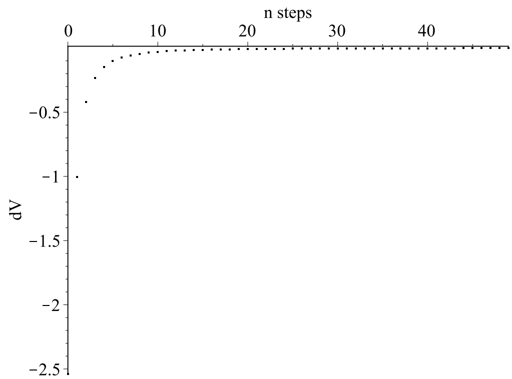


Fig. 3. The graph of $\Delta\overline{V}(k)$ for $0 \leq k \leq 50$.

Example 4.16. (Continuation of Example 4.8) For system (30) one can choose the function $V(x_1, x_2) = x_1^2 + x_2^2$ that is positive definite and decrescent. The simulations made in Maple (see Figure 3) show that

$$\Delta\overline{V}(k) \leq 0\,,$$

where $\overline{V}(k) = V(x_1(kh - 0.5h), x_2(kh - 0.75h))$ and $\Delta\overline{V}(k) = (x_1(kh + 0.5h))^2 + (x_2(kh + 0.25h))^2 - (x_1(kh - 0.5h))^2 - (x_2(kh - 0.75h))^2$. Since $\Delta\overline{V}(k) \leq 0$ holds for all $k \geq 0$, by Fact 4.13 the trivial solution of the considered system is uniformly stable.

5. CONCLUSIONS

The sufficient conditions for stability of the fractional h-difference systems with n-orders are presented. We discuss the asymptotic stability of the considered systems. Additionally, we show that the well known Lyapunov direct method can be used to study the stability of considered systems since these systems can be described in recurrence way as sequences that are parameterized by the orders α_i, $i = 1, \ldots, n$ and h. Therefore the conditions for stability are like in the discrete time case.

Our future work will be devoted to study the Mittag–Leffler stability of fractional difference systems.

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Małgorzata Wyrwas, Faculty of Computer Science, Bialystok University of Technology, Wiejska 45A, 15-351 Białystok. Poland.
e-mail: m.wyrwas@pb.edu.pl

Ewa Pawluszewicz, Faculty of Mechanical Engineering, Bialystok University of Technology, Wiejska 45C, 15-333 Białystok. Poland.
e-mail: e.pawluszewicz@pb.edu.pl

Ewa Girejko, Faculty of Computer Science, Bialystok University of Technology, Wiejska 45A, 15-351 Białystok. Poland.
e-mail: e.girejko@pb.edu.pl