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CLASSIFICATION OF 4-DIMENSIONAL HOMOGENEOUS WEAKLY EINSTEIN MANIFOLDS

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Abstract. Y. Euh, J. Park and K. Sekigawa were the first authors who defined the concept of a weakly Einstein Riemannian manifold as a modification of that of an Einstein Riemannian manifold. The defining formula is expressed in terms of the Riemannian scalar invariants of degree two. This concept was inspired by that of a super-Einstein manifold introduced earlier by A. Gray and T. J. Willmore in the context of mean-value theorems in Riemannian geometry. The dimension 4 is the most interesting case, where each Einstein space is weakly Einstein. The original authors gave two examples of homogeneous weakly Einstein manifolds (depending on one, or two parameters, respectively) which are not Einstein. The goal of this paper is to prove that these examples are the only existing examples. We use, for this purpose, the classification of 4-dimensional homogeneous Riemannian manifolds given by L. Bérard Bergery and, also, the basic method and many explicit formulas from our previous article with different topic published in Czechoslovak Math. J. in 2008. We also use Mathematica 7.0 to organize better the tedious routine calculations. The problem of existence of non-homogeneous weakly Einstein spaces in dimension 4 which are not Einstein remains still unsolved.

Keywords: Riemannian homogeneous manifold; Einstein manifold; weakly Einstein manifold

MSC 2010: 53C21, 53C30, 53B21, 53C25

1. Introduction and preliminaries

The first definition of a weakly Einstein manifold in general dimension appeared in [5] and a more detailed study for dimension 4 continued in [6] and [7]. This definition was inspired by that of a super-Einstein manifold as defined in [8]. An n-dimensional

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Einstein manifold $M = (M, g)$ is said to be super-Einstein if the following curvature identity is satisfied:

$$\check{\mathcal{R}} = \frac{1}{n}|\mathcal{R}|^2 g \tag{1.1}$$

where $\check{\mathcal{R}}$ is the tensor given by the coordinate formula $\check{\mathcal{R}}_{ij} = \sum_{a,b,c=1}^{n} \mathcal{R}_{abci}\mathcal{R}_{abcj}$ for $i, j = 1, \ldots, n$. Here, $|\mathcal{R}|^2$ was supposed to be constant. With respect to an orthonormal frame, (1.1) can re-written in the form

$$\sum_{a,b,c=1}^{n} \mathcal{R}_{abci}\mathcal{R}_{abcj} = \frac{1}{n}|\mathcal{R}|^2\delta_{ij}, \quad i, j = 1, \ldots, n. \tag{1.2}$$

Now, a Riemannian manifold $M = (M, g)$ is said to be *weakly Einstein* if it satisfies the formula (1.1) (or (1.2), respectively). It is known from [3] that, for $n \neq 4$, the constancy of $|\mathcal{R}|$ is automatically satisfied. Thus, the dimension 4 is the most interesting case. In [5], the authors prove that, in the 4-dimensional case, each Einstein manifold is weakly Einstein. The converse does not hold. In [6], the authors present two different examples of homogeneous weakly Einstein spaces which are not Einstein.

Example 1.1 ([6], Ex. 4)**.** The Riemannian product manifold of 2-dimensional Riemannian manifolds $M_1(c)$ and $M_2(-c)$ of constant Gaussian curvatures c and $-c$ $(c \neq 0)$, respectively.

Example 1.2 ([6], Ex. 5)**.** A connected and simply connected solvable Lie group $(G, g)_{\alpha,\beta}$ whose associated Lie algebra $\mathfrak{g}_{\alpha,\beta} = \mathrm{span}_{\mathbb{R}}\{e_1, e_2, e_3, e_4\}$ is equipped with the following Lie bracket operation:

$$\begin{aligned} &[e_1, e_2] = \alpha e_2, \quad &&[e_1, e_3] = -\alpha e_3 - \beta e_4, \quad &&[e_1, e_4] = \beta e_3 - \alpha e_4, \\ &[e_2, e_3] = 0, &&[e_2, e_4] = 0, &&[e_3, e_4] = 0, \end{aligned} \tag{1.3}$$

where $\alpha \neq 0$, β are constants. Here, g is the left-invariant Riemannian metric on G determined by the inner product $\langle , \rangle$ on $\mathfrak{g}_{\alpha,\beta}$ defined by $\langle e_i, e_j \rangle = \delta_{ij}$. In the sequel, we will use for the Riemannian group spaces $(G, g)_{\alpha,\beta}$ the name EPS *spaces* giving hereby the credit to the authors of the papers [5], [6], [7].

The main goal of this paper is to find all homogeneous weakly Einstein examples in dimension 4 using the classification given by L. Bérard Bergery. Our final result is formulated in the following

Main theorem. *Every homogeneous weakly Einstein* 4*-manifold which is not Einstein is isometric either to a direct product from Example* 1.1 *or to an* EPS *space from Example* 1.2.

In [2], L. Bérard Bergery published the classification of Riemannian homogeneous 4-spaces. In particular, he obtained the following

Proposition 1.1. *In dimension* 4*, each simply connected Riemannian homogeneous space* M *is either symmetric or isometric to a Lie group with a left-invariant metric. In the second case, either* M *is a solvable group or it is one of the groups* $\mathrm{SU}(2) \times \mathbb{R}$, $\widetilde{\mathrm{Sl}(2,\mathbb{R})} \times \mathbb{R}$. *Moreover, the solvable and simply connected Lie groups are:*

(a) *The non-trivial semi-direct products* $E(2) \rtimes \mathbb{R}$ *and* $E(1,1) \rtimes \mathbb{R}$.
(b) *The non-nilpotent semi-direct products* $H \rtimes \mathbb{R}$*, where* H *is the Heisenberg group.*
(c) *All semi-direct products* $\mathbb{R}^3 \rtimes \mathbb{R}$.

Now, the main part of our computations is to check which of these spaces are weakly Einstein and not Einstein. We shall work at the Lie algebra level and use Mathematica 7.0 for the computations.

Let us start with the symmetric case. Using the de Rham decomposition theorem we can see easily the following:

Theorem 1.2. *The only symmetric weakly Einstein spaces are the following ones:*

(a) *Any irreducible* 4*-dimensional symmetric space (which is known to be Einstein).*
(b) *The direct products* $M_2(c) \times M_2(d)$ *where* $c = \pm d$*. Here, the only non-Einstein weakly Einstein spaces are the direct products* $M_2(c) \times M_2(-c)$.

We have recovered Example 1.1.

Let us continue with the non-solvable group case and later we shall work with the solvable case.

2. Non-solvable cases ($\mathrm{SU}(2) \times \mathbb{R}$ and $\widetilde{\mathrm{Sl}(2,\mathbb{R})} \times \mathbb{R}$)

Let $\mathfrak{g}_3$ be a unimodular Lie algebra with a scalar product $\langle , \rangle_3$. According to [10], page 305, there is an orthonormal basis $\{f_1, f_2, f_3\}$ of $\mathfrak{g}_3$ such that

$$[f_2, f_3] = af_1, \quad [f_3, f_1] = bf_2, \quad [f_1, f_2] = cf_3, \tag{2.1}$$

where a, b, c are real numbers. In the following, we shall study the cases $\mathfrak{g}_3 = \mathfrak{su}(2)$ and $\mathfrak{g}_3 = \mathfrak{sl}(2,\mathbb{R})$, which are characterized by the inequality $abc \neq 0$.

Let now $\mathfrak{g} = \mathfrak{g}_3 \oplus \mathbb{R}$ be the direct sum, and $\langle , \rangle$ a scalar product on $\mathfrak{g}$ defined as follows: we choose a basis $\{f_1, f_2, f_3, f_4\}$ of unit vectors such that $\{f_1, f_2, f_3\}$ is an orthonormal basis of $\mathfrak{g}_3$ satisfying (2.1) and f_4 spans $\mathbb{R}$. Here $\mathbb{R}$ need not be orthogonal to $\mathfrak{g}_3$. In particular, we assume

$$[f_i, f_4] = 0, \quad \langle f_i, f_4 \rangle = k_i, \quad i = 1, 2, 3. \tag{2.2}$$

Here k_1, k_2, k_3 are arbitrary parameters where $\sum\limits_{i=1}^{3} k_i^2 < 1$ due to the Cauchy-Schwarz inequality. Choosing a convenient orientation of f_4, we can always assume that $k_3 \geqslant 0$.

Now we replace the basis $\{f_i\}$ by the new basis $\{e_i\}$ $(i = 1, 2, 3, 4)$ putting

$$e_i = f_i, \quad i = 1, 2, 3, \qquad e_4 = \frac{1}{R}\left(f_4 - \sum_{i=1}^{3} k_i f_i\right) \tag{2.3}$$

where $R = \sqrt{1 - \sum_{i=1}^{3} k_i^2} > 0$. Then we get an orthonormal basis for which

$$\begin{gathered}
[e_2, e_3] = a e_1, \quad [e_3, e_1] = b e_2, \quad [e_1, e_2] = c e_3, \\
[e_1, e_4] = \frac{1}{R}(k_3 b e_2 - k_2 c e_3), \quad [e_2, e_4] = \frac{1}{R}(k_1 c e_3 - k_3 a e_1), \\
[e_3, e_4] = \frac{1}{R}(k_2 a e_1 - k_1 b e_2).
\end{gathered} \tag{2.4}$$

Next, we shall consider the simply connected Lie group G with a left invariant Riemannian metric g corresponding to the Lie algebra $\mathfrak{g}$ and the scalar product $\langle , \rangle$ on it. Here the vectors e_i determine some left-invariant vector fields on G.

According to our construction, the underlying group G is the direct product of the group $\mathrm{SU}(2)$, or $\widetilde{\mathrm{Sl}(2, \mathbb{R})}$, and the multiplicative group $\mathbb{R}^+$.

Theorem 2.1. *The Riemannian manifolds* $(\mathrm{SU}(2) \times \mathbb{R}, g)$ *and* $(\widetilde{\mathrm{Sl}(2, \mathbb{R})} \times \mathbb{R}, g)$ *are not weakly Einstein.*

We shall prove this theorem step by step. First, we calculate the conditions for (G, g) to be a weakly Einstein manifold. From [1] we know the expression for the curvature tensor. We denote by A_{ij} the elementary skew-symmetric operators whose corresponding action is given by the formulas $A_{ij}(e_l) = \delta_{il} e_j - \delta_{jl} e_i$.

Lemma 2.2 ([1]). *The components of the curvature operator are*

$$
\begin{aligned}
\mathcal{R}(e_1,e_2) &= \alpha_{1212}A_{12} + \alpha_{1213}A_{13} + \alpha_{1214}A_{14} + \alpha_{1223}A_{23} + \alpha_{1224}A_{24},\\
\mathcal{R}(e_1,e_3) &= \alpha_{1312}A_{12} + \alpha_{1313}A_{13} + \alpha_{1314}A_{14} + \alpha_{1323}A_{23} + \alpha_{1334}A_{34},\\
\mathcal{R}(e_1,e_4) &= \alpha_{1412}A_{12} + \alpha_{1413}A_{13} + \alpha_{1414}A_{14} + \alpha_{1424}A_{24} + \alpha_{1434}A_{34},\\
\mathcal{R}(e_2,e_3) &= \alpha_{2312}A_{12} + \alpha_{2313}A_{13} + \alpha_{2323}A_{23} + \alpha_{2324}A_{24} + \alpha_{2334}A_{34},\\
\mathcal{R}(e_2,e_4) &= \alpha_{2412}A_{12} + \alpha_{2414}A_{14} + \alpha_{2423}A_{23} + \alpha_{2424}A_{24} + \alpha_{2434}A_{34},\\
\mathcal{R}(e_3,e_4) &= \alpha_{3413}A_{13} + \alpha_{3414}A_{14} + \alpha_{3423}A_{23} + \alpha_{3424}A_{24} + \alpha_{3434}A_{34},
\end{aligned}
\tag{2.5}
$$

where the coefficients $\alpha_{ijlm} = g(\mathcal{R}(e_i,e_j)e_l,e_m)$ satisfy the standard symmetries with respect to their indices and

$$
\begin{aligned}
\alpha_{1212} &= \frac{1}{4R^2}((3c^2-(a-b)^2-2c(a+b))R^2-(a-b)^2k_3^2),\\
\alpha_{1213} &= \frac{1}{4R^2}((a-b)(a-c)k_2k_3),\\
\alpha_{1214} &= \frac{1}{4R}((a-c)(a-b+3c)k_2),\\
\alpha_{1223} &= \frac{1}{4R^2}((a-b)(b-c)k_1k_3),\\
\alpha_{1224} &= \frac{1}{4R}((b-c)(a-b-3c)k_1),\\
\alpha_{1313} &= \frac{1}{4R^2}((3b^2-(a-c)^2-2b(a+c))R^2-(a-c)^2k_2^2),\\
\alpha_{1314} &= \frac{1}{4R}((a-b)(a-c+3b)k_3),\\
\alpha_{1323} &= \frac{1}{4R^2}((a-c)(b-c)k_1k_2),\\
\alpha_{1334} &= \frac{1}{4R}((c-b)(c-a+3b)k_1),\\
\alpha_{1414} &= \frac{1}{4R^2}((4c^2-(a+c)^2)k_2^2+(4b^2-(a+b)^2)k_3^2),\\
\alpha_{1424} &= \frac{1}{4R^2}((c(a+b-3c)+ab)k_1k_2),\\
\alpha_{1434} &= \frac{1}{4R^2}((b(a+c-3b)+ac)k_1k_3),\\
\alpha_{2323} &= \frac{1}{4R^2}((3a^2-(b-c)^2-2a(b+c))R^2-(b-c)^2k_1^2),\\
\alpha_{2324} &= \frac{1}{4R}((b-a)(3a+b-c)k_3),\\
\alpha_{2334} &= \frac{1}{4R}((a-c)(3a-b+c)k_2),
\end{aligned}
\tag{2.6}
$$

$$\alpha_{2424} = \frac{1}{4R^2}((4c^2 - (b+c)^2)k_1^2 + (4a^2 - (a+b)^2)k_3^2),$$
$$\alpha_{2434} = \frac{1}{4R^2}((a(-3a+b+c)+bc)k_2k_3),$$
$$\alpha_{3434} = \frac{1}{4R^2}((4b^2 - (b+c)^2)k_1^2 + (4a^2 - (a+c)^2)k_2^2).$$

Next, (G, g) is weakly Einstein if and only if the expression

$$\sum_{a,b,c=1}^{4} \alpha_{abci}\alpha_{abcj} - \frac{1}{4}|\mathcal{R}|^2\delta_{ij} \tag{2.7}$$

is equal to zero for every pair of indices (i, j), $i, j = 1, \ldots, 4$. Moreover, note that

$$\sum_{i=1}^{4} \sum_{a,b,c=1}^{4} \alpha_{abci}\alpha_{abci} = \sum_{i,a,b,c=1}^{4} \alpha_{abci}^2 = |\mathcal{R}|^2. \tag{2.8}$$

Here we obtain, by a lengthy but routine calculation

Lemma 2.3.

$$\begin{aligned}
|\mathcal{R}|^2 = \sum_{i,j,k,l=1}^{4} \alpha_{ijkl}^2 &= \frac{1}{4R^4}((-11(a^4+b^4+c^4) + 12abc(a+b+c))R^4 \\
&+ 2(b-c)^2(-a^2 - 6a(b+c))R^2k_1^2 \\
&+ 2(a-c)^2(-b^2 - 6b(a+c))R^2k_2^2 \\
&+ 2(a-b)^2(-c^2 - 6c(a+b))R^2k_3^2 \\
&+ (b-c)^2(11b^2 + 10bc + 11c^2)(R^2 + k_1^2)^2 \\
&+ (a-c)^2(11a^2 + 10ac + 11c^2)(R^2 + k_2^2)^2 \\
&+ (a-b)^2(11a^2 + 10ab + 11b^2)(R^2 + k_3^2)^2 \\
&+ 2(c^2(11c^2 - 6ac - a^2) - 2bc(3c^2 - 2ac + 3a^2) \\
&+ b^2(-c^2 - 6ac + 11a^2))k_1^2k_2^2 \\
&+ 2(b^2(11b^2 - 6bc - c^2) - 2ab(3b^2 - 2bc + 3c^2) \\
&+ a^2(-b^2 - 6bc + 11c^2))k_1^2k_3^2 \\
&+ 2(a^2(11a^2 - 6ab - b^2) - 2ac(3a^2 - 2ab + 3b^2) \\
&+ c^2(-a^2 - 6ab + 11b^2))k_2^2k_3^2).
\end{aligned} \tag{2.9}$$

Lemma 2.4. *The vanishing of the term* (2.7) *for each pair* (i,j) *is equivalent to the system of algebraic equations*

$$
\begin{aligned}
(1,2) &= k_1k_2((a-c)(c-b)(2ab+c(a+b)-5c^2)R^2 \\
&\quad + (b-c)(ab(b+c)+c^2(2a+b-5c))k_1^2 \\
&\quad + (a-c)(ab(a+c)+c^2(a+2b-5c))k_2^2 \\
&\quad + (-2ab(ab+3c^2)+(a^2+b^2)c(a+b+2c))k_3^2) = 0, \\
(1,3) &= k_1k_3((a-b)(b-c)(2ac+b(a+c)-5b^2)R^2 \\
&\quad + (c-b)(ac(b+c)+b^2(2a-5b+c))k_1^2 \\
&\quad + (-2ac(3b^2+ac)+(a^2+c^2)b(a+2b+c))k_2^2 \\
&\quad + (a-b)(ac(b+a)+b^2(2c-5b+a))k_3^2) = 0, \\
(2,3) &= k_2k_3((b-a)(a-c)(2bc+a(b+c)-5a^2)R^2 \\
&\quad + (-2bc(3a^2+bc)+(b^2+c^2)a(2a+b+c))k_1^2 \\
&\quad + (c-a)(bc(a+c)+a^2(2b-5a+c))k_2^2 \\
&\quad + (b-a)(bc(a+b)+a^2(2c-5a+b))k_3^2) = 0, \\
(1,4) &= k_1((b-c)^2(5c^2+(a-b)(a-5b-6c))R^2 \\
&\quad + (b-c)^2(5b^2+2(3bc-a(b+c))+5c^2)k_1^2 \\
&\quad + (a-c)((a-5c)c^2+ab(a-2c)+b^2(2a+3c))k_2^2 \\
&\quad + (a-b)((a-5b)b^2+ac(a-2b)+c^2(2a+3b))k_3^2) = 0, \\
(2,4) &= k_2((c-a)^2(5a^2+(b-c)(b-5c-6a))R^2 \\
&\quad + (b-c)((b-5c)c^2+ab(b-2c)+a^2(2b+3c))k_1^2 \\
&\quad + (a-c)^2(5a^2+2(3ac-b(a+c))+5c^2)k_2^2 \\
&\quad + (b-a)((b-5a)a^2+bc(b-2a)+c^2(2b+3a))k_3^2) = 0, \\
(3,4) &= k_3((a-b)^2(5b^2+(c-a)(c-5a-6b))R^2 \\
&\quad + (c-b)((c-5b)b^2+ac(c-2b)+a^2(2c+3b))k_1^2 \\
&\quad + (c-a)((c-5a)a^2+bc(c-2a)+b^2(2c+3a))k_2^2 \\
&\quad + (a-b)^2(5a^2+2(3ab-c(a+b))+5b^2)k_3^2) = 0, \\
(1,1) &= (7a^4+4abc(5b-7a+5c)-2bc(2b+c)(b+2c))R^4 \\
&\quad - (b-c)^2(11b^2+10bc+11c^2)k_1^4 \\
&\quad + (c-a)^3(7a+9c)(R^2+k_2^2)^2 \\
&\quad + (b-a)^3(7a+9b)(R^2+k_3^2)^2 \\
&\quad + 2(b-c)^2(3a^2-2a(b+c)-(b-c)^2)R^2k_1^2 \\
&\quad + 2(a-c)^2(2b(3a-c)-3b^2)R^2k_2^2
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
&+ 2(a-b)^2(2c(3a-b)-3c^2)R^2k_3^2\\
&+ 2(b-c)(c-a)(3a+c)(c+3b)k_1^2k_2^2\\
&+ 2(a-b)(b-c)(3a+b)(b+3c)k_1^2k_3^2\\
&+ 2(a^2(-7a^2+6ab-3b^2)+2ac(3a^2+6ab-5b^2)\\
&+ c^2(9b^2-10ab-3a^2))k_2^2k_3^2 = 0,\\
(2,2) = &\,(7b^4+4abc(5a-7b+5c)-2ac(2a+c)(a+2c))R^4\\
&+ (c-b)^3(7b+9c)(R^2+k_1^2)^2\\
&- (a-c)^2(11a^2+10ac+11c^2)k_2^4\\
&+ (a-b)^3(9a+7b)(R^2+k_3^2)^2\\
&+ 2(b-c)^2(2a(3b-c)-3a^2)R^2k_1^2\\
&+ 2(a-c)^2(3b^2-2b(a+c)-(a-c)^2)R^2k_2^2\\
&+ 2(a-b)^2(2c(3b-a)-3c^2)R^2k_3^2\\
&+ 2(b-c)(c-a)(3a+c)(c+3b)k_1^2k_2^2\\
&+ 2(b^2(-7b^2+6bc-3c^2)+2ab(3b^2+6bc-5c^2)\\
&+ a^2(9c^2-10bc-3b^2))k_1^2k_3^2\\
&+ 2(b-a)(a-c)(3b+a)(a+3c)k_2^2k_3^2 = 0,\\
(3,3) = &\,(7c^4+4abc(5a-7c+5b)-2ab(2a+b)(a+2b))R^4\\
&+ (b-c)^3(9b+7c)(R^2+k_1^2)^2\\
&+ (a-c)^3(9a+7c)(R^2+k_2^2)^2\\
&- (a-b)^2(11a^2+10ab+11b^2)k_3^4\\
&+ 2(b-c)^2(2a(3c-b)-3a^2)R^2k_1^2\\
&+ 2(a-c)^2(2b(3c-a)-3b^2)R^2k_2^2\\
&+ 2(a-b)^2(3c^2-2c(a+b)-(a-b)^2)R^2k_3^2\\
&+ 2(c^2(-7c^2+6ac-3a^2)+2bc(3c^2+6ac-5a^2)\\
&+ b^2(9a^2-10ac-3c^2))k_1^2k_2^2\\
&+ 2(a-b)(b-c)(3a+b)(b+3c)k_1^2k_3^2\\
&+ 2(b-a)(a-c)(3b+a)(a+3c)k_2^2k_3^2 = 0.
\end{aligned}
$$

Here the symbol "(i,j)" marks the substitution of the corresponding $i \leqslant j$ in (2.7). Note that $\sum_{k=1}^{4}(k,k) = 0$ in this notation.

Now, our goal is to find the values of a, b, c, k_1, k_2 and k_3 which satisfy the system of equations (2.10) and to study each of the particular cases.

Lemma 2.5. *$a = b = c \neq 0$, k_1, k_2, k_3 arbitrary, is not a solution of the system* (2.10).

P r o o f. Substituting a and c by b in equation $(1,1) = 0$ of (2.10) we get that $b^4R^4 = 0$ which is a contradiction with the assumptions $abc \neq 0$ and $R > 0$. □

Proposition 2.6. *The system of algebraic equations* (2.10) *does not have any solution.*

P r o o f. Because we can re-numerate the basis $\{e_1, e_2, e_3\}$ in arbitrary way (which implies the corresponding permutation of the symbols a, b, c and the corresponding re-numeration of the parameters k_1, k_2, k_3), the system (2.10) is *symmetric* with respect to all such permutations and re-numerations. Then, in order to solve this system of equations, we can just consider the following cases:

A. $k_1k_2k_3 \neq 0$.
B. $k_1 = 0$ and $k_2k_3 \neq 0$.
C. $k_1 = k_2 = 0$, k_3 arbitrary.

Case A. $k_1k_2k_3 \neq 0$. We first replace R^2 by its value $1 - \sum_{i=1}^{3} k_i^2$ in the equations $(1,2) = 0$, $(1,3) = 0$, $(2,3) = 0$, $(1,4) = 0$, $(2,4) = 0$ and $(3,4) = 0$ of (2.10). Moreover, we divide them by their nonzero coefficients k_1k_2, k_1k_3, k_2k_3, k_1, k_2 and k_3, respectively. Now, we consider the system formed by the equations $(i,4) = 0$, $i = 1,2,3$ as a system of linear equations with respect to k_1^2, k_2^2 and k_3^2 whose determinant D is the following:

$$D = -9a(a-b)^2b(a-c)^2(b-c)^2c(a+b+c)F$$

where $F = 3(a^2+b^2+c^2) + 2(a+b+c)^2 > 0$ due to $abc \neq 0$.

If $D \neq 0$, we get by solving the system formed by $(i,4) = 0$, $i = 1,2,3$ that

(2.11)

$$\begin{aligned}
k_1^2 = {} & \frac{1}{9a(a-b)(a-c)(a+b+c)F}(45a^6 + 11a^5(b+c) \\
& + 2a^3(b+c)(67b^2 - 35bc + 67c^2) - a^4(186b^2 + 337bc + 186c^2) \\
& + a^2(21b^4 + 214b^3c + 216b^2c^2 + 214bc^3 + 21c^4) \\
& - a(b+c)(25b^4 - 18b^3c + 38b^2c^2 - 18bc^3 + 25c^4) \\
& - bc(25b^4 + 54b^3c + 178b^2c^2 + 54bc^3 + 25c^4)), \\
k_2^2 = {} & \frac{1}{9(a-b)b(b-c)(a+b+c)F}((a-b)^2b(25a^3 + 29a^2b - 101ab^2 - 45b^3) \\
& + (25a^5 + 7a^4b - 214a^3b^2 - 64a^2b^3 + 337ab^4 - 11b^5)c \\
& + 2(27a^4 + 10a^3b - 108a^2b^2 - 32ab^3 + 93b^4)c^2
\end{aligned}$$

$$
\begin{aligned}
&+ 2(89a^3 + 10a^2b - 107ab^2 - 67b^3)c^3 \\
&+ (54a^2 + 7ab - 21b^2)c^4 + 25(a+b)c^5), \\
k_3^2 = &\frac{1}{9(a-c)(b-c)c(a+b+c)F}(-ab(25a^4 + 54a^3b + 178a^2b^2 + 54ab^3 + 25b^4) \\
&- (a+b)(25a^4 - 18a^3b + 38a^2b^2 - 18ab^3 + 25b^4)c \\
&+ (21a^4 + 214a^3b + 216a^2b^2 + 214ab^3 + 21b^4)c^2 \\
&+ 2(a+b)(67a^2 - 35ab + 67b^2)c^3 \\
&- (186a^2 + 337ab + 186b^2)c^4 + 11(a+b)c^5 + 45c^6).
\end{aligned}
$$

Substituting the values of k_1^2, k_2^2 and k_3^2 given in (2.11) in the equations $(1,2)=0$, $(1,3)=0$ and $(2,3)=0$, and multiplying all of them by $3(a+b+c)F$ we get

(2.12)

$$
\begin{aligned}
(1,2)' = &- ab(a+b)(25a^2 - ab + 25b^2) - (25a^4 + 52a^3b - 87a^2b^2 + 52ab^3 + 25b^4)c \\
&- 3(a+b)(13a^2 - 57ab + 13b^2)c^2 + (45a^2 + 86ab + 45b^2)c^3 \\
&- 76(a+b)c^4 - 45c^5 = 0, \\
(1,3)' = &\, 25a^4(b+c) + a^3(39b^2 + 52bc + 24c^2) - 3a^2(15b^3 + 44b^2c + 29bc^2 - 8c^3) \\
&+ b(45b^4 + 76b^3c - 45b^2c^2 + 39bc^3 + 25c^4) \\
&+ a(76b^4 - 86b^3c - 132b^2c^2 + 52bc^3 + 25c^4) = 0, \\
(2,3)' = &- 45a^5 - 76a^4(b+c) - 3a^2(b+c)(13b^2 - 57bc + 13c^2) \\
&- bc(b+c)(25b^2 - bc + 25c^2) + a^3(45b^2 + 86bc + 45c^2) \\
&- a(25b^4 + 52b^3c - 87b^2c^2 + 52bc^3 + 25c^4) = 0.
\end{aligned}
$$

Now, we consider the following system:

$$
\begin{aligned}
(1,2)' - (2,3)' &= 3(a-c)F_{1223} = 0, \\
(1,3)' + (2,3)' &= 3(b-a)F_{1323} = 0,
\end{aligned}
\tag{2.13}
$$

where

$$
\begin{aligned}
F_{1223} = &\, 15a^4 + 17a^3b - 23a^2b^2 + 5ab^3 + 32a^3c - 29a^2bc - 38ab^2c \\
&+ 5b^3c + 4a^2c^2 - 29abc^2 - 23b^2c^2 + 32ac^3 + 17bc^3 + 15c^4, \\
F_{1323} = &\, 15a^4 + 32a^3b + 4a^2b^2 + 32ab^3 + 17a^3c - 29a^2bc - 29ab^2c \\
&+ 17b^3c - 23a^2c^2 - 38abc^2 - 23b^2c^2 + 5ac^3 + 5bc^3 + 15b^4.
\end{aligned}
$$

Due to $D \neq 0$, the previous system is equivalent to $\{F_{1223} = 0, F_{1323} = 0\}$. Here, $F_{1223} - F_{1323} = 0$ gives $-3(b-c)(a+b+c)F = 0$ and hence $D = 0$, a contradiction.

Finally we study the case $D = 0$ which is equivalent to the case $(a-b)(a-c)(b-c) \times (a+b+c) = 0$. Obviously, because the system (2.10) is symmetric with respect to all permutations, if we assume $a - b = 0$, we get also $b - c = 0$ and $c - a = 0$ which cannot occur due to Lemma 2.5. Therefore, we can assume $(a+b+c) = 0$. Moreover, at least one of the products ab, bc, ac is positive. Suppose that $bc > 0$, the other cases are analogous. Substituting a by $-(b+c)$ in the equations $(1,4) = 0$, $(2,4) = 0$ and $(3,4) = 0$ we get

$$
\begin{aligned}
(1,4)' &= (b-c)(4(b-c)(3b^2+5bc+3c^2) - 5(b-c)(b+c)^2k_1^2 \\
&\quad - b(11b^2+10bc+6c^2)k_2^2 + c(6b^2+10bc+11c^2)k_3^2) = 0, \\
(2,4)' &= (b+2c)(4(b+2c)(3b^2+bc+c^2) - (b+c)(11b^2+12bc+7c^2)k_1^2 \\
&\quad - 5b^2(b+2c)k_2^2 - c(6b^2+2bc+7c^2)k_3^2) = 0, \\
(3,4)' &= (2b+c)(4(2b+c)(b^2+bc+3c^2) - (b+c)(7b^2+12bc+11c^2)k_1^2 \\
&\quad - b(7b^2+2bc+6c^2)k_2^2 - 5c^2(2b+c)k_3^2) = 0.
\end{aligned} \tag{2.14}
$$

Now, we will show that the system (2.14) does not have any solution. Adding $(1,4)' = 0$, $(2,4)' = 0$ and $(3,4)' = 0$ we get $10(b^2+bc+c^2)G = 0$ where $G = (b-c)^2 + 3b^2(1-k_1^2-k_2^2) + 3c^2(1-k_1^2-k_3^2) + 6bc(1-k_1^2)$. Here, obviously $b^2+bc+c^2 > 0$ and $G > 0$ due to $bc > 0$, a contradiction.

Case B. $k_1 = 0$ and $k_2k_3 \neq 0$. We first replace R^2 by its value $1 - k_2^2 - k_3^2$ and we put $k_1 = 0$ in the equations $(2,3) = 0$, $(2,4) = 0$, $(3,4) = 0$, $(1,1) = 0$ and $(2,2) = 0$ of (2.10). Moreover, we divide $(2,3) = 0$, $(2,4) = 0$ and $(3,4) = 0$ by their nonzero coefficients k_2k_3, k_2 and k_3, respectively. We get

(2.15)

$$
\begin{aligned}
(2,3) &= (b-a)(a-c)(-5a^2+2bc+a(b+c)) + b(a-c)(4a^2-ab-2bc-c^2)k_2^2 \\
&\quad + (a-b)c(4a^2-b^2-ac-2bc)k_3^2 = 0, \\
(2,4) &= (a-c)^2(5a^2-6ab+b^2+6ac-6bc+5c^2) + b(a-c)^2(4a-b+4c)k_2^2 \\
&\quad + c(4a^3-4a^2b-ab^2+b^3-a^2c-5abc+b^2c+4ac^2+6bc^2-5c^3)k_3^2 = 0, \\
(3,4) &= (a-b)^2(5a^2+6ab+5b^2-6ac-6bc+c^2) \\
&\quad + b(4a^3-a^2b+4ab^2-5b^3-4a^2c-5abc+6b^2c-ac^2+bc^2+c^3)k_2^2 \\
&\quad + (a-b)^2(4a+4b-c)ck_3^2 = 0, \\
(1,1) &= (b-a)^3(7a+9b) + 4(a-b)^2(3a-b)c + 2(3a^2+10ab-5b^2)c^2 - 4(5a+b)c^3 \\
&\quad + 9c^4 + 2b(4b^2c+7bc^2+2c^3+a^2(14c-9b)+2a(10b^2-7bc-5c^2)-6a^3 \\
&\quad - 9b^3)k_2^2 + b^2(12a^2-20ab+9b^2+8ac-4bc-4c^2)k_2^4
\end{aligned}
$$

$$
\begin{aligned}
&+ 2c(2(a-b)^2(b-3a) + (7b^2 - 14ab - 9a^2)c + 4(5a+b)c^2 - 9c^3)k_3^2 \\
&+ c^2(12a^2 + 8ab - 4b^2 - 20ac - 4bc + 9c^2)k_3^4 \\
&+ 2bc(12a^2 + (2c-b)(2b-c) - 6a(b+c))k_2^2k_3^2 = 0,
\end{aligned}
$$

$$
\begin{aligned}
(2,2) = {}& (a-b)^3(7b+9a) + 4(a-b)^2(3b-a)c + 2(3b^2 + 10ab - 5a^2)c^2 - 4(5b+a)c^3 \\
&+ 9c^4 - 2(10a^4 + a^2(3b^2 + 18bc - 4c^2) + 2a(b-c)(6b^2 - 5bc + 4c^2) \\
&- 2a^3(9b+4c) - (b-c)(7b^3 - 5b^2c - 8bc^2 + 10c^3))k_2^2 \\
&+ b(-16a^3 - 7b^3 + 16a^2c + 12b^2c - 16c^3 + 4a(3b^2 - 4bc + 4c^2))k_2^4 \\
&+ 2c(2(a-b)^2(a-3b) + (7a^2 - 14ab - 9b^2)c + 4(5b+a)c^2 - 9c^3)k_3^2 \\
&+ c^2(12b^2 + 8ab - 4a^2 - 20bc - 4ac + 9c^2)k_3^4 + 2c(-8a^3 + 6b^3 - 3b^2c \\
&- 18bc^2 + 10c^3 + 2a^2(2b+c) - 2a(b^2 - 9bc + 4c^2))k_2^2k_3^2 = 0.
\end{aligned}
$$

Now, we consider the system formed by the equations $(2,3)=0$ and $(2,4)=0$ of (2.15) as a system of linear equations with respect to k_2^2 and k_3^2 whose determinant is

$$D_1 = b(c-a)(b-c)c^2F_1$$

where $F_1 = (28a^3 - 12a^2b - 4ab^2 + 3b^3 - 23a^2c - 6abc + 4b^2c - 4ac^2 + 9bc^2 + 5c^3)$. Moreover, if $D_1 \neq 0$, we get

(2.16)

$$
\begin{aligned}
k_2^2 &= \frac{a-b}{b(c-b)F_1}(8a^4 + 24a^3b - 13a^2b^2 - 2ab^3 + 3b^4 - (26a^3 + 11a^2b + ab^2 + 2b^3)c \\
&\quad + 5(2a^2 + ab + b^2)c^2), \\
k_3^2 &= \frac{(a-c)^2}{c(b-c)F_1}(8a^3 + a(b-2c)(5b+3c) + a^2(-13b+19c) + c(3b^2 - 4bc - 5c^2)).
\end{aligned}
$$

In addition, $a-b \neq 0$ due to $R^2 > 0$, and using (2.16) we get

$$R^2 = 1 - k_2^2 - k_3^2 = \frac{b-a}{bcF_1}(8a^4 + 3b^2c^2 + abc(b+2c) + a^2c(7b+10c) - a^3(5b+26c)).$$

Now, substituting (2.16) in $(3,4)=0$ and $(2,2)=0$ of (2.15), these equations became equivalent to

(2.17)

$$
\begin{aligned}
(3,4)_1 = {}& 69a^4 - 21a^3(b+c) + 5a(b-c)^2(b+c) + 3bc(5b^2 + 6bc + 5c^2) \\
&- a^2(13b^2 + 49bc + 13c^2) = 0, \\
(2,2)_1 = {}& 5508a^8 - 8a^7(538b + 2305c) + 8a^6(-27b^2 + 1067bc + 3159c^2) \\
&+ 4a^5(300b^3 + 443b^2c - 1413bc^2 - 3152c^3) \\
&+ 3b^2c^2(21b^4 - 12b^3c + 46b^2c^2 - 12bc^3 - 7c^4)
\end{aligned}
$$

$$
\begin{aligned}
&- a^4(309b^4 + 2198b^3c + 2385b^2c^2 + 5944bc^3 + 188c^4)\\
&+ 2abc(21b^5 - 3b^4c - 127b^3c^2 - 29b^2c^3 - 140bc^4 - 94c^5)\\
&+ a^3(-30b^5 + 440b^4c + 3194b^3c^2 + 2396b^2c^3 + 5792bc^4 + 1896c^5)\\
&+ a^2(7b^6 - 88b^5c - 863b^4c^2 - 886b^3c^3 - 50b^2c^4 - 800bc^5 - 416c^6) = 0.
\end{aligned}
$$

On the other hand, we consider the system formed by the equations $(2,3) = 0$ and $(3,4) = 0$ of (2.15) as a system of linear equations with respect to k_2^2 and k_3^2 whose determinant is

$$D_2 = c(b-a)(b-c)b^2F_2$$

where $F_2 = F_1 + (c-b)(11a^2 - 2b^2 - 7bc - 2c^2)$. Moreover, if $D_2 \neq 0$, we get

(2.18)

$$
\begin{aligned}
k_2^2 &= \frac{(a-b)^2}{b(b-c)F_2}(-8a^3 + a(2b-c)(3b+5c) + a^2(-19b+13c) + b(5b^2+4bc-3c^2)),\\
k_3^2 &= \frac{a-c}{c(b-c)F_2}(2a^2(4a^2 - 13ab + 5b^2) + ac(24a^2 - 11ab + 5b^2) - 2(a+b)c^3 + 3c^4\\
&\quad - (13a^2 + ab - 5b^2)c^2).
\end{aligned}
$$

In addition, $a - c \neq 0$ due to $R^2 > 0$ and we have using (2.18) that

$$R^2 = 1 - k_2^2 - k_3^2 = \frac{c-a}{bcF_2}(8a^4 + 3b^2c^2 + abc(2b+c) - a^3(26b+5c) + a^2b(10b+7c)).$$

Now, substituting (2.18) in $(2,4) = 0$ and $(1,1) = 0$ of (2.15), these equations became equivalent to

(2.19)

$$
\begin{aligned}
(2,4)_2 &= 69a^4 - 21a^3(b+c) + 5a(b-c)^2(b+c) + 3bc(5b^2 + 6bc + 5c^2)\\
&\quad - a^2(13b^2 + 49bc + 13c^2) = 0,\\
(1,1)_2 &= 4a^3(4683a^6 - 15468a^5b + 19468a^4b^2 - 8924a^3b^3 - 1137a^2b^4 + 2080ab^5\\
&\quad - 446b^6) + 4a^2c(-7071a^6 + 18895a^5b - 17704a^4b^2 + 1740a^3b^3 + 5223a^2b^4\\
&\quad - 2147ab^5 + 168b^6) + ac^2(11884a^6 - 21352a^5b + 6453a^4b^2 + 17292a^3b^3\\
&\quad - 12082a^2b^4 + 1676ab^5 + 33b^6) + c^3(2560a^6 - 7758a^5b + 9829a^4b^2\\
&\quad - 4634a^3b^3 - 364a^2b^4 - 168ab^5 + 279b^6) + c^4(-3123a^5 + 4898a^4b\\
&\quad - 4131a^3b^2 - 228a^2b^3 + 192ab^4 + 360b^5) + c^5(593a^4 - 820a^3b + 787a^2b^2\\
&\quad + 378ab^3 - 90b^4) + (a+3b)c^6(47a^2 + 37ab - 24b^2) - 9c^7(a+3b)^2 = 0.
\end{aligned}
$$

Note that $(2,4)_2 = (3,4)_1$. Moreover, if $D_1D_2 \neq 0$, Mathematica 7.0 shows that the only possible solution of the system formed by the equations $(3,4)_1 = 0$, $(2,2)_1 = 0$

of (2.17) and $(1,1)_2 = 0$ of (2.19) is $a = b = c \neq 0$ which cannot happen due to Lemma 2.5.

Now, we will study the case $D_1 D_2 = 0$. First, we shall show that we can assume $(a-b)b^3(a-c)(b-c)^2c^3 \neq 0$. If $a = c$, the equations $(2,3) = 0$ and $(2,4) = 0$ of the system (2.15) become

$$\begin{aligned}(2,3)' &= c(c-b)^2(b+3c)k_3^2 = 0,\\ (2,4)' &= c(c-b)^2(b+2c)k_3^2 = 0.\end{aligned}$$

Therefore, the only possible solution of the previous system is $a = b = c$, which cannot occur due to Lemma 2.5. If $a = b$, the equations $(2,3) = 0$ and $(2,4) = 0$ of the system (2.15) become

$$\begin{aligned}(2,3)' &= b(b-c)^2(3b+c)k_2^2 = 0,\\ (2,4)' &= (b-c)^2(b(3b+4c)k_2^2 + 5c^2(1-k_3^2)) = 0,\end{aligned}$$

and the only possible solutions of the previous system are: $a = b = c$, which cannot occur, and $c = -3b$, $k_3^2 = (1/5)(5-k_2^2)$, which does not satisfy the condition $R^2 > 0$. If $b = c$, the equations $(2,3) = 0$ and $(2,4) = 0$ of the system (2.15) become

$$\begin{aligned}(2,3)' &= (a-c)^2(5a^2 - 2c(a+c) + (4ac+3c^2)(k_2^2+k_3^2)) = 0,\\ (2,4)' &= (a-c)^2(5a^2 + (4ac+3c^2)(k_2^2+k_3^2)) = 0.\end{aligned}$$

The only possible solutions of the previous system are: $a = b = c$, which cannot occur, and $a = -c$, $k_2^2 = 5 - k_3^2$, which does not satisfy the condition $R^2 > 0$.

Therefore, $D_1 D_2 = 0$ is equivalent to $F_1 F_2 = 0$. Finally, we will study the only three different possibilities that can occur when $F_1 F_2 = 0$. If $F_1 = 0$ and $F_2 = 0$, we get that $F_2 = 0$ becomes equivalent to $(11a^2 - 2b^2 - 7bc - 2c^2) = 0$. Mathematica 7.0 shows that the only possible solution of the system formed by $F_1 = 0$, $F_2 = 0$ and by equations $(2,3) = 0$ and $(2,4) = 0$ of (2.15) is $a = b = c$ which cannot occur by Lemma 2.5.

If $F_1 = 0$ and $F_2 \neq 0$, Mathematica 7.0 shows that the possible solutions of the system formed by $F_1 = 0$ and $(2,4)_2 = 0$ of (2.19) imply that k_2^2 and k_3^2 of (2.18) do not satisfy $R^2 > 0$.

If $F_2 = 0$ and $F_1 \neq 0$, Mathematica 7.0 shows that the possible solutions of the system formed by $F_2 = 0$ and $(3,4)_1 = 0$ of (2.17) imply that k_2^2 and k_3^2 of (2.16) do not satisfy $R^2 > 0$.

Case C. $k_1 = k_2 = 0$, k_3 arbitrary. We first put $k_1 = k_2 = 0$ and replace R^2 by $1 - k_3^2$ in (2.10). We get

(2.20)
$$\begin{aligned}
(3,4) &= k_3(a-b)^2(5a^2+5b^2+6a(b-c)-6bc+c^2+(4a+4b-c)ck_3^2) = 0,\\
(1,1) &= (b-a)^3(7a+9b)+4(a-b)^2(3a-b)c+2(3a^2+10ab-5b^2)c^2\\
&\quad -4(5a+b)c^3+9c^4+c^2(12a^2+8ab-4b^2-20ac-4bc+9c^2)k_3^4\\
&\quad +2c(2(a-b)^2(b-3a)+(-9a^2-14ab+7b^2)c+4(5a+b)c^2-9c^3)k_3^2 = 0,\\
(2,2) &= (a-b)^3(9a+7b)-4(a-3b)(a-b)^2c+2(-5a^2+10ab+3b^2)c^2\\
&\quad -4(a+5b)c^3+9c^4+c^2(-4a^2+8ab+12b^2-4ac-20bc+9c^2)k_3^4\\
&\quad +2c(2(a-3b)(a-b)^2+(7a^2-14ab-9b^2)c+4(a+5b)c^2-9c^3)k_3^2 = 0,\\
(3,3) &= 4a(3c-b)(b-c)^2-4a^3(b+5c)+(b-c)^3(9b+7c)\\
&\quad +a^2(-10b^2+20bc+6c^2)+9a^4+c(-16(a-b)^2(a+b)\\
&\quad -16abc+12(a+b)c^2-7c^3)k_3^4+2(18(a-b)^2(a+b)c\\
&\quad -2(a-b)^2(5a^2+6ab+5b^2)+(-3a^2+22ab-3b^2)c^2\\
&\quad -12(a+b)c^3+7c^4)k_3^2 = 0.
\end{aligned}$$

Here we can assume that $(a-b)k_3 \neq 0$ because if we put $a = b$ in the previous system, we get that equations $(1,1) = 0$ and $(3,3) = 0$ of (2.20) become

$$\begin{aligned}
(1,1)' &= c^2R^4(4b-3c)^2 = 0,\\
(3,3)' &= c^2R^4(16b^2-24bc+7c^2) = 0.
\end{aligned}$$

Now, $(1,1)' - (3,3)' = 0$ gives $c = 0$, a contradiction. If $k_3 = 0$ the equations $(1,1) = 0$ and $(2,2) = 0$ of (2.20) reduce to the following:

(2.21)
$$\begin{aligned}
(1,1)^* &= (b-a)^3(7a+9b)+4(a-b)^2(3a-b)c\\
&\quad +2(3a^2+10ab-5b^2)c^2-4(5a+b)c^3+9c^4 = 0,\\
(2,2)^* &= (a-b)^3(9a+7b)-4(a-3b)(a-b)^2c\\
&\quad +2(-5a^2+10ab+3b^2)c^2-4(a+5b)c^3+9c^4 = 0.
\end{aligned}$$

Adding equations $(1,1)^* = 0$ and $(2,2)^* = 0$ of (2.21) we get

$$16(b-a)(a-b-c)(a+b-c)(a-b+c) = 0.$$

Then, we only have to study three different possibilities: $a = b + c$, $a = c - b$ and $a = b - c$. If we replace $a = b + c$ in $(1,1)^* = 0$ of (2.21) we get $16b^2c^2 = 0$ which is a contradiction. If we put $a = c - b$ or $a = b - c$ in $(1,1)^* = 0$ of (2.21) we get $b = c$ which is a contradiction with $a \neq 0$.

Therefore, we get that equation $(3,4) = 0$ of (2.20) is equivalent to

$$(3,4)^\sharp = (5a^2 + 5b^2 + 6a(b-c) - 6bc + c^2 + (4(a+b) - c)ck_3^2) = 0.$$

Now, we divide the study into two different cases: $4(a+b)-c = 0$ and $4(a+b)-c \neq 0$. If $4(a+b) - c = 0$, we reduce the equations $(3,4)^\sharp = 0$ and $(3,3) = 0$ of (2.20) to

$$\begin{aligned}
(3,4)^\flat &= (a-b)^2(3a+b)(a+3b)k_3 = 0,\\
(3,3)^\flat &= -999a^4 - 4356a^3b - 6698a^2b^2 - 4356ab^3 - 999b^4 - 1088(a+b)^4k_3^4\\
&\quad + 4(519a^4 + 2180a^3b + 3306a^2b^2 + 2180ab^3 + 519b^4)k_3^2 = 0.
\end{aligned}$$

Now, if we put $a = -3b$ or $b = -3a$ in $(3,3)^\flat = 0$ we get $(45 - 108k_3^2 + 68k_3^4) = 0$ or equivalently that $k_3^2 = \frac{27}{34} \pm \frac{3}{17}\mathrm{i}$, a contradiction.

If $4(a+b) - c \neq 0$, we get that $(3,4)^\sharp = 0$ is satisfied if and only if

$$k_3^2 = \frac{5a^2 + 6ab + 5b^2 - 6(a+b)c + c^2}{(c - 4(a+b))c}. \tag{2.22}$$

Thus, the equations $(1,1) = 0$ and $(2,2) = 0$ of (2.20) become equivalent to the following:

$$\begin{aligned}
(1,1)^\dagger &= 107a^6 + 214a^5b + 369a^4b^2 + 260a^3b^3 + 45a^2b^4 + 38ab^5 - 9b^6\\
&\quad - 8c(3a+b)(2a^2 + ab + b^2)(5a^2 + 6ab + 5b^2)\\
&\quad + 4c^2(48a^4 + 99a^3b + 113a^2b^2 + 69ab^3 + 23b^4)\\
&\quad - 4c^3(a+b)(17a^2 + 18ab + 13b^2) + 9c^4(a+b)^2 = 0,\\
(2,2)^\dagger &= 9a^6 - 38a^5b - 45a^4b^2 - 260a^3b^3 - 369a^2b^4 - 214ab^5 - 107b^6\\
&\quad + 8c(a+3b)(a^2 + ab + 2b^2)(5a^2 + 6ab + 5b^2)\\
&\quad - 4c^2(23a^4 + 69a^3b + 113a^2b^2 + 99ab^3 + 48b^4)\\
&\quad + 4c^3(a+b)(13a^2 + 18ab + 17b^2) - 9c^4(a+b)^2 = 0.
\end{aligned} \tag{2.23}$$

Moreover, the previous system (2.23) is equivalent to

(2.24)

$$\begin{aligned}
(1,1)^\dagger + (2,2)^\dagger &= 29a^5 + 73a^4b + 154a^3b^2 + 154a^2b^3 + 73ab^4 + 29b^5 - 4c^3(a+b)^2\\
&\quad - 2c(5a^2 + 6ab + 5b^2)^2 + 5c^2(a+b)(5a^2 + 6ab + 5b^2) = 0,\\
(1,1)^\dagger - (2,2)^\dagger &= 7a^3 + 9a^2b + 9ab^2 + 7b^3 - 2(5a^2 + 6ab + 5b^2)c + 3(a+b)c^2 = 0.
\end{aligned}$$

Here, we have $a+b \neq 0$ because if we replace a by $-b$ in (2.22) we get $k_3^2 = 1+4b^2/c^2$ and hence $R^2 = 1-k_3^2 < 0$, a contradiction. Therefore from the equation $(1,1)^\dagger - (2,2)^\dagger = 0$ we get

$$c = \frac{5a^2+6ab+5b^2 \pm 2\sqrt{E}}{3(a+b)}, \tag{2.25}$$

where

$$E = a^4+3a^3b+8a^2b^2+3ab^3+b^4 = \frac{1}{2}(a^4+5a^2b^2+b^4+(a^2+3ab+b^2)^2) > 0.$$

Now, substituting (2.25) in $(1,1)^\dagger + (2,2)^\dagger = 0$ and multiplying by $-\frac{27}{16}(a+b)$ we get:

$$(a^2-3ab-2b^2)(2a^2+3ab-b^2)(a^2+6ab+b^2) = \mp 2E^{3/2}. \tag{2.26}$$

Taking squares on both sides of the equality (2.26), it becomes equivalent to

$$-27a^2(a-b)^2b^2(a+b)^2(3a+b)^2(a+3b)^2 = 0. \tag{2.27}$$

Finally, equations (2.25) and (2.27) show that the only solutions of (2.24) such that $abc \neq 0$ are:

- ▷ $b=-3a$, $c_\pm = \frac{8}{3}a(-2\pm 1)$, which for c_+ gives a contradiction with $k_3^2 > 0$. That is, substituting this possible solution in (2.22) we get $k_3^2 = -\frac{1}{2}$. For c_-, it gives a contradiction with $4(a+b)-c \neq 0$.
- ▷ $b = -\frac{1}{3}a$, $c_\pm = \frac{8}{9}a(2\pm 1)$, which also gives a contradiction. For c_+ we get $4(a+b)-c=0$. For c_-, we obtain as before that $k_3^2 = -\frac{1}{2}$.

Proposition 2.6 is proved. □

This completes the proof of the Theorem 2.1. □

3. Solvable case

As concerns the semidirect products of the form $G = G_3 \rtimes \mathbb{R}$ in Proposition 1.1 and all possible left-invariant metrics on them, we can construct all of them on the level of Lie algebras as follows: we consider the Lie algebra $\mathfrak{g}_3$ and the vector space $\mathfrak{g} = \mathfrak{g}_3 + \mathbb{R}$. Let $\{f_1,\dots,f_4\}$ be any basis of $\mathfrak{g}$ such that $\mathfrak{g}_3 = \operatorname{span}\{f_1,f_2,f_3\}$, $\mathbb{R} = \operatorname{span}\{f_4\}$. Let D be an arbitrary derivation of the algebra $\mathfrak{g}_3$ and let us define

$$[f_4, f_i] = Df_i \quad \text{for } i=1,2,3. \tag{3.1}$$

(This completes the multiplication table of the algebra $\mathfrak{g}_3$ to the multiplication table of $\mathfrak{g}$). Then we choose any scalar product $\langle , \rangle$ on $\mathfrak{g}$ for which $\{f_1, f_2, f_3\}$ forms an orthonormal triplet but f_4 is just a unit vector which need not be orthonormal to $\mathfrak{g}_3$. Thus we have, as in the formula (2.2), $\langle f_i, f_4 \rangle = k_i$, $i = 1, 2, 3$. Now, all semi-direct products $G_3 \rtimes \mathbb{R}$ with left-invariant metrics correspond to various choices of the derivations D of $\mathfrak{g}_3$ and to all scalar products given by the above rule. The algebra of all derivations D of $\mathfrak{g}_3$ will be usually represented in the corresponding matrix form.

Now, we shall study each of the solvable cases from Proposition 1.1 separately following the construction indicated above and preserving the style of Section 2.

4. Non-trivial semi-direct products $E(2) \rtimes \mathbb{R}$

Let $\mathfrak{e}(2)$ be the Lie algebra of $E(2)$ with a scalar product $\langle , \rangle_3$. Then, there is an orthonormal basis $\{f_1, f_2, f_3\}$ of $\mathfrak{e}(2)$ such that

$$(4.1) \qquad [f_2, f_3] = -\gamma f_1, \quad [f_3, f_1] = -\gamma f_2, \quad [f_1, f_2] = 0$$

where $\gamma \neq 0$ is a real number. The algebra of all derivations D of $\mathfrak{e}(2)$ is

$$\left\{ \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ c & d & 0 \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\},$$

when represented in the matrix form.

According to the general scheme, we consider the algebra $\mathfrak{g} = \mathfrak{e}(2) + \mathbb{R}$, where the multiplication table is given by (4.1) and, according to the general formula (3.1), also by

$$(4.2) \qquad [f_4, f_1] = af_1 + bf_2, \quad [f_4, f_2] = -bf_1 + af_2, \quad [f_4, f_3] = cf_1 + df_2,$$
$$\langle f_i, f_4 \rangle = k_i, \quad i = 1, 2, 3.$$

Here $\gamma \neq 0$, a, b, c, d, k_1, k_2, k_3 are arbitrary parameters where $\sum_{i=1}^{3} k_i^2 < 1$ due to the positivity of the scalar product. We exclude the case $a = b = c = d = 0$, i.e., the direct product $E(2) \times \mathbb{R}$.

This gives rise to a simply connected group space $(G = E(2) \rtimes \mathbb{R}, g)$.

Theorem 4.1. *The only metric which makes $(E(2) \rtimes \mathbb{R}, g)$ a weakly Einstein manifold is the flat one. Moreover, the corresponding Lie algebra is determined by* (4.2) *where $d = \gamma k_1$, $c = -\gamma k_2$, $a = 0$, $\gamma \neq 0$, and b, k_1, k_2, k_3 are arbitrary.*

In the remainder of the section we will prove the announced theorem. We replace the basis $\{f_i\}$ by the new basis $\{e_i\}$, as in the formula (2.3). Then we get an orthonormal basis for which

$$
\begin{gathered}
[e_2, e_3] = -\gamma e_1, \quad [e_3, e_1] = -\gamma e_2, \quad [e_1, e_2] = 0, \\
[e_4, e_1] = \frac{1}{R}(ae_1 + (b + k_3\gamma)e_2), \quad [e_4, e_2] = \frac{1}{R}(-(b + k_3\gamma)e_1 + ae_2), \\
[e_4, e_3] = \frac{1}{R}((c + k_2\gamma)e_1 + (d - k_1\gamma)e_2).
\end{gathered}
\tag{4.3}
$$

Now we are going to calculate the expression for the condition for (G, g) to be a weakly Einstein manifold. From [1] we know

Lemma 4.2. *The components of the curvature operator are*

$$
\begin{aligned}
\mathcal{R}(e_1, e_2) &= \alpha_{1212}A_{12} + \alpha_{1213}A_{13} + \alpha_{1223}A_{23}, \\
\mathcal{R}(e_1, e_3) &= \alpha_{1312}A_{12} + \alpha_{1313}A_{13} + \alpha_{1323}A_{23} + \alpha_{1334}A_{34}, \\
\mathcal{R}(e_1, e_4) &= \alpha_{1414}A_{14} + \alpha_{1424}A_{24} + \alpha_{1434}A_{34}, \\
\mathcal{R}(e_2, e_3) &= \alpha_{2312}A_{23} + \alpha_{2313}A_{13} + \alpha_{2323}A_{23} + \alpha_{2334}A_{34}, \\
\mathcal{R}(e_2, e_4) &= \alpha_{2414}A_{14} + \alpha_{2424}A_{24} + \alpha_{2434}A_{34}, \\
\mathcal{R}(e_3, e_4) &= \alpha_{3413}A_{13} + \alpha_{3414}A_{14} + \alpha_{3423}A_{23} + \alpha_{3424}A_{24} + \alpha_{3434}A_{34},
\end{aligned}
\tag{4.4}
$$

where the coefficients $\alpha_{ijlm} = g(\mathcal{R}(e_i, e_j)e_l, e_m)$ satisfy the standard symmetries with respect to their indices and

(4.5)

$$
\begin{gathered}
\alpha_{1212} = \frac{a^2}{R^2}, \quad \alpha_{1213} = \frac{a(d - \gamma k_1)}{2R^2}, \quad \alpha_{1223} = -\frac{a(c + \gamma k_2)}{2R^2}, \quad \alpha_{1313} = -\frac{(c + \gamma k_2)^2}{4R^2}, \\
\alpha_{1323} = -\frac{(d - \gamma k_1)(c + \gamma k_2)}{4R^2}, \quad \alpha_{1334} = \frac{\gamma(-d + \gamma k_1)}{2R}, \quad \alpha_{1414} = \frac{4a^2 - (c + \gamma k_2)^2}{4R^2}, \\
\alpha_{1424} = -\frac{(d - \gamma k_1)(c + \gamma k_2)}{4R^2}, \quad \alpha_{1434} = \frac{2a(c + \gamma k_2) + (d - \gamma k_1)(b + \gamma k_3)}{2R^2}, \\
\alpha_{2323} = -\frac{(d - \gamma k_1)^2}{4R^2}, \quad \alpha_{2334} = \frac{\gamma(c + \gamma k_2)}{2R}, \quad \alpha_{2424} = \frac{4a^2 - (d - \gamma k_1)^2}{4R^2}, \\
\alpha_{2434} = \frac{2a(d - \gamma k_1) - (c + \gamma k_2)(b + \gamma k_3)}{2R^2}, \quad \alpha_{3434} = \frac{3((d - \gamma k_1)^2 + (c + \gamma k_2)^2)}{4R^2}.
\end{gathered}
$$

Now we obtain the following analogue of Lemmas 2.3 and 2.4:

Lemma 4.3.

$$(4.6)\ \ |\mathcal{R}|^2 = \sum_{i,j,k,l=1}^{4} \alpha_{ijkl}^2 = \frac{1}{4R^4}(48a^4 + (c+\gamma k_2)^2(8(4a^2+\gamma^2R^2) + 11(c+\gamma k_2)^2) + 8(c+\gamma k_2)^2(b+\gamma k_3)^2 + (d-\gamma k_1)^2(8(4a^2+\gamma^2R^2) + 11(d-\gamma k_1)^2 + 22(c+\gamma k_2)^2 + 8(b+\gamma k_3)^2)).$$

Lemma 4.4. *The condition* (2.7) *is equivalent to the system of algebraic equations*

$$(4.7)\quad \begin{aligned}
(1,2) = & -4a(c+\gamma k_2)^2(b+\gamma k_3) \\
& + (d-\gamma k_1)((c+\gamma k_2)((d-\gamma k_1)^2 + (c+\gamma k_2)^2 + 2(a^2+\gamma^2R^2)) \\
& - 2(b+\gamma k_3)((c+\gamma k_2)(b+\gamma k_3) - 2a(d-\gamma k_1))) = 0, \\
(1,3) = & \, 4a(c+\gamma k_2)(3a^2 + (c+\gamma k_2)^2) + (d-\gamma k_1)(4a(d-\gamma k_1)(c+\gamma k_2) \\
& + (4a^2 + 3(c+\gamma k_2)^2 + 3(d-\gamma k_1)^2)(b+\gamma k_3)) = 0, \\
(2,3) = & -(c+\gamma k_2)(4a^2 + 3(c+\gamma k_2)^2)(b+\gamma k_3) \\
& + (d-\gamma k_1)(4a(3a^2 + (d-\gamma k_1)^2 + (c+\gamma k_2)^2) \\
& - 3(d-\gamma k_1)(c+\gamma k_2)(b+\gamma k_3)) = 0, \\
(1,4) = & \, 3\gamma(d-\gamma k_1)((d-\gamma k_1)^2 + (c+\gamma k_2)^2) = 0, \\
(2,4) = & -3\gamma(c+\gamma k_2)((d-\gamma k_1)^2 + (c+\gamma k_2)^2) = 0, \\
(3,4) = & -\gamma(b+\gamma k_3)((d-\gamma k_1)^2 + (c+\gamma k_2)^2) = 0, \\
(1,1) = & \, 16a^4 - (d-\gamma k_1)^2(16a^2 + 11(d-\gamma k_1)^2) \\
& - (c+\gamma k_2)((c+\gamma k_2)(8(a^2+\gamma^2R^2) + 7(c+\gamma k_2)^2 + 18(d-\gamma k_1)^2) \\
& + 8(b+\gamma k_3)(c(b+\gamma k_3) - 4a(d-\gamma k_1) + \gamma k_2(b+\gamma k_3))) = 0, \\
(2,2) = & \, 16a^4 - (c+\gamma k_2)^2(16a^2 + 11(c+\gamma k_2)^2) \\
& - (d-\gamma k_1)((d-\gamma k_1)(8(a^2+\gamma^2R^2) + 7(d-\gamma k_1)^2 + 18(c+\gamma k_2)^2) \\
& + 8(b+\gamma k_3)(4a(c+\gamma k_2) + d(b+\gamma k_3) - \gamma k_1(b+\gamma k_3))) = 0, \\
(4,4) = & \, 16a^4 + (c+\gamma k_2)^2(16a^2 + 9(c+\gamma k_2)^2) + 8(c+\gamma k_2)^2(b+\gamma k_3)^2 \\
& + (d-\gamma k_1)^2(16a^2 + 9(d-\gamma k_1)^2 + 18(c+\gamma k_2)^2 + 8(b+\gamma k_3)^2) = 0.
\end{aligned}$$

Here the symbol "(i,j)" marks again the substitution of the corresponding $i \leqslant j$ in (2.7). *Moreover,* $\sum_{k=1}^{4}(k,k) = 0$.

Now, the goal is to find the values of a, b, c, d, k_1, k_2, k_3 and $\gamma \neq 0$ which satisfy the system of equations (4.7).

Proposition 4.5. *The set of all solutions of the system of algebraic equations* (4.7) *is given by the formulas*

$$d = \gamma k_1, \quad c = -\gamma k_2, \quad a = 0, \quad \gamma \neq 0,\ b,\ k_1,\ k_2,\ k_3,\ \textit{arbitrary}. \tag{4.8}$$

The corresponding spaces are flat.

Proof. From the subsystem of (4.7) formed by the equations $(1,4) = 0$ and $(2,4) = 0$ we obtain $(d - \gamma k_1) = (c + \gamma k_2) = 0$ due to $\gamma \neq 0$. Then, the remaining equations (4.7) are automatically satisfied except $(1,1) = 0$, $(2,2) = 0$ and $(4,4) = 0$ which became equivalent to the equation $16a^4 = 0$. Thus $a = 0$.

Substituting the equalities $d = \gamma k_1$, $c = -\gamma k_2$ and $a = 0$ into the right-hand side of (4.6), we see that $|\mathcal{R}|^2 = 0$ and hence $\mathcal{R} = 0$. □

This completes the proof of Theorem 4.1. □

5. Non-trivial semi-direct products $E(1,1) \rtimes \mathbb{R}$

Let $\mathfrak{e}(1,1)$ be the Lie algebra of $E(1,1)$ with a scalar product $\langle , \rangle_3$. Then, there is an orthonormal basis $\{f_1, f_2, f_3\}$ of $\mathfrak{e}(1,1)$ such that

$$[f_2, f_3] = \gamma f_2, \quad [f_3, f_1] = \gamma f_1, \quad [f_1, f_2] = 0 \tag{5.1}$$

where $\gamma \neq 0$ is a real number. The algebra of all derivations D of $\mathfrak{e}(1,1)$ is

$$\left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & c & 0 \end{pmatrix} : a, b, c \in \mathbb{R} \right\},$$

when represented in the matrix form.

According to the general scheme, we consider the algebra $\mathfrak{g} = \mathfrak{e}(1,1) + \mathbb{R}$, where the multiplication table is given by (5.1) and, according to the general formula (3.1), also by

$$[f_4, f_1] = af_1,\ [f_4, f_2] = af_2,\ [f_4, f_3] = bf_1 + cf_2,\ \langle f_i, f_4 \rangle = k_i,\ i = 1,2,3. \tag{5.2}$$

Here $\gamma \neq 0$, a, b, c, k_1, k_2, k_3 are arbitrary parameters where $\sum_{i=1}^{3} k_i^2 < 1$, and we exclude the case $a = b = c = 0$.

This gives rise to a simply connected group space $(G = E(1,1) \rtimes \mathbb{R}, g)$.

Theorem 5.1. *The only family of metrics which makes $(E(1,1) \rtimes \mathbb{R}, g)$ weakly Einstein manifold is Einstein and locally symmetric. Moreover, the corresponding Lie algebra it is determined by (5.2) where $a = \gamma\sqrt{1 - k_1^2 - k_2^2}$, $b = -\gamma k_1$, $c = \gamma k_2$, $k_3 = 0$, $\gamma \neq 0$, and k_1, k_2 are arbitrary.*

In what follows, we will prove the announced theorem. We replace the basis $\{f_i\}$ by the new basis $\{e_i\}$, as in the formula (2.3). Then we get an orthonormal basis for which

$$
\begin{gathered}
[e_2, e_3] = \gamma e_2, \quad [e_3, e_1] = \gamma e_1, \quad [e_1, e_2] = 0, \\
[e_4, e_1] = \frac{1}{R}((a - k_3\gamma)e_1), \quad [e_4, e_2] = \frac{1}{R}((a + k_3\gamma)e_2), \\
[e_4, e_3] = \frac{1}{R}((b + k_1\gamma)e_1 + (c - k_2\gamma)e_2).
\end{gathered} \tag{5.3}
$$

Now we are going to calculate the expression of the condition for (G, g) to be a weakly Einstein manifold. From [1] we know

Lemma 5.2. *The components of the curvature operator are*

$$
\begin{aligned}
\mathcal{R}(e_1, e_2) &= \alpha_{1212}A_{12} + \alpha_{1213}A_{13} + \alpha_{1214}A_{14} + \alpha_{1223}A_{23} + \alpha_{1224}A_{24}, \\
\mathcal{R}(e_1, e_3) &= \alpha_{1312}A_{12} + \alpha_{1313}A_{13} + \alpha_{1314}A_{14} + \alpha_{1323}A_{23} + \alpha_{1334}A_{34}, \\
\mathcal{R}(e_1, e_4) &= \alpha_{1412}A_{12} + \alpha_{1413}A_{13} + \alpha_{1414}A_{14} + \alpha_{1424}A_{24} + \alpha_{1434}A_{34}, \\
\mathcal{R}(e_2, e_3) &= \alpha_{2312}A_{23} + \alpha_{2313}A_{13} + \alpha_{2323}A_{23} + \alpha_{2324}A_{24} + \alpha_{2334}A_{34}, \\
\mathcal{R}(e_2, e_4) &= \alpha_{2412}A_{12} + \alpha_{2414}A_{14} + \alpha_{2423}A_{23} + \alpha_{2424}A_{24} + \alpha_{2434}A_{34}, \\
\mathcal{R}(e_3, e_4) &= \alpha_{3413}A_{13} + \alpha_{3414}A_{14} + \alpha_{3423}A_{23} + \alpha_{3424}A_{24} + \alpha_{3434}A_{34},
\end{aligned} \tag{5.4}
$$

where the coefficients $\alpha_{ijlm} = g(\mathcal{R}(e_i, e_j)e_l, e_m)$ satisfy the standard symmetries with respect to their indices and

$$
\begin{aligned}
\alpha_{1212} &= \frac{a^2 + \gamma^2(-1 + k_1^2 + k_2^2)}{R^2}, & \alpha_{1213} &= \frac{(c - \gamma k_2)(a - \gamma k_3)}{2R^2}, \\
\alpha_{1214} &= \frac{-\gamma(c - \gamma k_2)}{2R}, & \alpha_{1223} &= \frac{-(b + \gamma k_1)(a + \gamma k_3)}{2R^2}, \\
\alpha_{1224} &= \frac{-\gamma(b + \gamma k_1)}{2R}, & \alpha_{1313} &= \frac{4R^2\gamma^2 - (b + \gamma k_1)^2}{4R^2}, \\
\alpha_{1314} &= \frac{\gamma(a - \gamma k_3)}{R}, & \alpha_{1323} &= \frac{(b + \gamma k_1)(-c + \gamma k_2)}{4R^2}, \\
\alpha_{1334} &= \frac{\gamma(b + \gamma k_1)}{R}, & \alpha_{1414} &= \frac{4(a - \gamma k_3)^2 - (b + \gamma k_1)^2}{4R^2},
\end{aligned} \tag{5.5}
$$

$$\alpha_{1424} = \frac{(b+\gamma k_1)(-c+\gamma k_2)}{4R^2}, \qquad \alpha_{1434} = \frac{(b+\gamma k_1)(a-\gamma k_3)}{R^2},$$
$$\alpha_{2323} = \frac{4R^2\gamma^2 - (c-\gamma k_2)^2}{4R^2}, \qquad \alpha_{2324} = \frac{-\gamma(a+\gamma k_3)}{R},$$
$$\alpha_{2334} = \frac{\gamma(-c+\gamma k_2)}{R}, \qquad \alpha_{2424} = \frac{4(a+\gamma k_3)^2 - (c-\gamma k_2)^2}{4R^2},$$
$$\alpha_{2434} = \frac{(c-\gamma k_2)(a+\gamma k_3)}{R^2}, \qquad \alpha_{3434} = \frac{3((b+\gamma k_1)^2 + (c-\gamma k_2)^2)}{4R^2}.$$

Further, we obtain easily

Lemma 5.3 ([1]). *The matrix of the Ricci tensor of type* $(1,1)$ *expressed with respect to the basis* $\{e_1, e_2, e_3, e_4\}$ *is of the form*

$$\begin{pmatrix} \beta_{11} & \frac{(b+\gamma k_1)(c-\gamma k_2)}{2R^2} & \frac{(b+\gamma k_1)(-3a+\gamma k_3)}{2R^2} & \frac{\gamma(b+\gamma k_1)}{2R} \\ \frac{(b+\gamma k_1)(c-\gamma k_2)}{2R^2} & \beta_{22} & \frac{-(c-\gamma k_2)(3a+\gamma k_3)}{2R^2} & \frac{\gamma(-c+\gamma k_2)}{2R} \\ \frac{(b+\gamma k_1)(-3a+\gamma k_3)}{2R^2} & \frac{-(c-\gamma k_2)(3a+\gamma k_3)}{2R^2} & \beta_{33} & \frac{2\gamma^2 k_3}{R} \\ \frac{\gamma(b+\gamma k_1)}{2R} & \frac{\gamma(-c+\gamma k_2)}{2R} & \frac{2\gamma^2 k_3}{R} & \beta_{44} \end{pmatrix} \tag{5.6}$$

where

$$\beta_{11} = \frac{(b+\gamma k_1)^2 - 4a(a-\gamma k_3)}{2R^2}, \quad \beta_{22} = \frac{(c-\gamma k_2)^2 - 4a(a+\gamma k_3)}{2R^2},$$
$$\beta_{33} = -\frac{4R^2\gamma^2 + (b+\gamma k_1)^2 + (c-\gamma k_2)^2}{2R^2},$$
$$\beta_{44} = -\frac{4(a^2+\gamma^2 k_3^2) + (b+\gamma k_1)^2 + (c-\gamma k_2)^2}{2R^2}.$$

Now we obtain the following analogue of Lemmas 2.3 and 2.4:

Lemma 5.4.

$$\begin{aligned} |\mathcal{R}|^2 &= \sum_{i,j,k,l=1}^{4} \alpha_{ijkl}^2 \\ &= \frac{1}{4R^4}((b+\gamma k_1)^2(2(11(c-\gamma k_2)^2 + 16(a(a-\gamma k_3) - \gamma^2(-1+k_1^2+k_2^2)))) \\ &\quad + (c-\gamma k_2)^2(11(c-\gamma k_2)^2 + 32(a(a+\gamma k_3) - \gamma^2(-1+k_1^2+k_2^2))) \\ &\quad + 11(b+\gamma k_1)^4 + 16a^2(3a^2 + 8\gamma^2 k_3^2 - 2\gamma^2(-1+k_1^2+k_2^2)) \\ &\quad + 48\gamma^4(-1+k_1^2+k_2^2)^2). \end{aligned} \tag{5.7}$$

Lemma 5.5. *The condition* (2.7) *is equivalent to the system of algebraic equations*

(5.8)

$$
\begin{aligned}
(1,2) = {} & (b+\gamma k_1)(c-\gamma k_2)((b+\gamma k_1)^2+(c-\gamma k_2)^2 \\
& +2(a^2+5\gamma^2(-1+k_1^2+k_2^2)))=0,\\
(1,3) = {} & (b+\gamma k_1)((c-\gamma k_2)^2(2a-5\gamma k_3)+2(b+\gamma k_1)^2(a-\gamma k_3) \\
& +2a^2(2a-\gamma k_3)+2a(a-2\gamma k_3)^2+2\gamma^2(3\gamma k_3-a)(-1+k_1^2+k_2^2))=0,\\
(2,3) = {} & (c-\gamma k_2)((b+\gamma k_1)^2(2a+5\gamma k_3)+2(c-\gamma k_2)^2(a+\gamma k_3) \\
& +2a^2(2a+\gamma k_3)+2a(a+2\gamma k_3)^2-2\gamma^2(a+3\gamma k_3)(-1+k_1^2+k_2^2))=0,\\
(1,4) = {} & -\gamma(b+\gamma k_1)(5(c-\gamma k_2)^2+2(b+\gamma k_1)^2 \\
& +2(3\gamma^2(1-k_1^2-k_2^2)+a^2-4a\gamma k_3))=0,\\
(2,4) = {} & \gamma(c-\gamma k_2)(5(b+\gamma k_1)^2+2(c-\gamma k_2)^2 \\
& +2(3\gamma^2(1-k_1^2-k_2^2)+a^2+4a\gamma k_3))=0,\\
(3,4) = {} & \gamma((b+\gamma k_1)^2(3a-\gamma k_3)-(c-\gamma k_2)^2(3a+\gamma k_3) \\
& -8\gamma k_3(3a^2-\gamma^2(-1+k_1^2+k_2^2)))=0,\\
(1,1) = {} & -(b+\gamma k_1)^2(18(c-\gamma k_2)^2+8(a(a-2\gamma k_3)-\gamma^2(-1+k_1^2+k_2^2))) \\
& -(c-\gamma k_2)^2(11(c-\gamma k_2)^2+16(a(a+4\gamma k_3)-\gamma^2(-1+k_1^2+k_2^2))) \\
& -7(b+\gamma k_1)^4+16a(a^2(a-8\gamma k_3)+2\gamma^2(a+4\gamma k_3)(-1+k_1^2+k_2^2)) \\
& +16\gamma^4(-1+k_1^2+k_2^2)^2=0,\\
(2,2) = {} & -(b+\gamma k_1)^2(18(c-\gamma k_2)^2+16(a(a-4\gamma k_3)-\gamma^2(-1+k_1^2+k_2^2))) \\
& -(c-\gamma k_2)^2(7(c-\gamma k_2)^2+8(a(a+2\gamma k_3)-\gamma^2(-1+k_1^2+k_2^2))) \\
& -11(b+\gamma k_1)^4+16a((a^2(a+8\gamma k_3)+2\gamma^2(a-4\gamma k_3)(-1+k_1^2+k_2^2))) \\
& +16\gamma^4(-1+k_1^2+k_2^2)^2=0,\\
(3,3) = {} & 2(b+\gamma k_1)^2(9(c-\gamma k_2)^2+4(a^2-\gamma k_3(2a+\gamma k_3)-2\gamma^2(-1+k_1^2+k_2^2))) \\
& +(c-\gamma k_2)^2(9(c-\gamma k_2)^2+8(a^2+\gamma k_3(2a-\gamma k_3)-2\gamma^2(-1+k_1^2+k_2^2))) \\
& -16(3a^2-\gamma^2(-1+k_1^2+k_2^2))(a^2+4\gamma^2k_3^2+\gamma^2(-1+k_1^2+k_2^2)) \\
& +9(b+\gamma k_1)^4=0.
\end{aligned}
$$

Now, we have

Proposition 5.6. *The set of all solutions of the system of algebraic equations* (5.8) *is, up to a re-numeration of the triplet* $\{e_1, e_2, e_3\}$,

(5.9) $a=\gamma\sqrt{1-k_1^2-k_2^2},\ \ b=-\gamma k_1,\ \ c=\gamma k_2,\ \ k_3=0,\ \ \gamma\neq 0,\ k_1,\ k_2$ *arbitrary.*

The corresponding spaces are Einstein and locally symmetric.

P r o o f. Suppose first $b+\gamma k_1 \neq 0$. Thus, due to $\gamma \neq 0$ we obtain from (5.8) that

$$(5.10)\quad \begin{aligned}(1,3)' &= (c-\gamma k_2)^2(2a-5\gamma k_3)+2(b+\gamma k_1)^2(a-\gamma k_3)+2a^2(2a-\gamma k_3)\\ &\quad +2a(a-2\gamma k_3)2+2\gamma^2(3\gamma k_3-a)(-1+k_1^2+k_2^2)=0,\\ (1,4)' &= 5(c-\gamma k_2)^2+2(b+\gamma k_1)^2+2(3\gamma^2(1-k_1^2-k_2^2)+a^2-4a\gamma k_3)=0,\\ (2,4)' &= (c-\gamma k_2)(5(b+\gamma k_1)^2+2(3\gamma^2(1-k_1^2-k_2^2)+a^2+4a\gamma k_3))\\ &\quad +2(c-\gamma k_2)^4=0,\\ (3,4)' &= (b+\gamma k_1)^2(3a-\gamma k_3)-(c-\gamma k_2)^2(3a+\gamma k_3)\\ &\quad -8\gamma k_3(3a^2-\gamma^2(-1+k_1^2+k_2^2))=0.\end{aligned}$$

Moreover, if $c-\gamma k_2 \neq 0$ we can reduce the equation $(2,4)'=0$ to

$$(2,4)'' = 2(c-\gamma k_2)^2+5(b+\gamma k_1)^2+2(3\gamma^2(1-k_1^2-k_2^2)+a^2+4a\gamma k_3)=0.$$

Adding the equations $(1,4)'=0$ and $(2,4)''=0$ we get

$$7(c-\gamma k_2)^2+7(b+\gamma k_1)^2+12\gamma^2(1-k_1^2-k_2^2)+4a^2=0,$$

a contradictions due to $1-k_1^2-k_2^2>0$. Thus, $c-\gamma k_2=0$. In this case, (5.10) reduce to

$$(5.11)\quad \begin{aligned}(1,3)^* &= 2(b+\gamma k_1)^2(a-\gamma k_3)+2a^2(2a-\gamma k_3)+2a(a-2\gamma k_3)^2\\ &\quad +2\gamma^2(3\gamma k_3-a)(-1+k_1^2+k_2^2)=0,\\ (1,4)^* &= 2(b+\gamma k_1)^2+2(3\gamma^2(1-k_1^2-k_2^2)+a^2-4a\gamma k_3)=0,\\ (3,4)^* &= (b+\gamma k_1)^2(3a-\gamma k_3)-8\gamma k_3(3a^2-\gamma^2(-1+k_1^2+k_2^2))=0.\end{aligned}$$

Now, we have $a \neq 0$ because if $a=0$ from $(1,4)^*=0$ we get a contradiction with $1-k_1^2-k_2^2>0$. Thus, from $(1,4)^*=0$ we get

$$(5.12)\qquad k_3=\frac{a^2+(b+\gamma k_1)^2+3\gamma^2(1-k_1^2-k_2^2)}{4a\gamma}.$$

Now, substituting (5.12) in $(1,3)^*=0$ we obtain $4a(a^2-\gamma^2(1-k_1^2-k_2^2))=0$. Thus

$$(5.13)\qquad a^2=\gamma^2(1-k_1^2-k_2^2).$$

Finally, substituting (5.12) and (5.13) in $(3,4)^*=0$ we get

$$\frac{1}{4a}((b+\gamma k_1)^2+16\gamma^2(1-k_1^2-k_2^2))((b+\gamma k_1)^2+8\gamma^2(1-k_1^2-k_2^2))=0,$$

a contradiction with $1-k_1^2-k_2^2>0$.

Therefore, we must assume $b+\gamma k_1=0$. In this case, due to $\gamma\neq 0$ we obtain from (5.8) that

$$(5.14)\quad \begin{aligned}(2,3)' &= (c-\gamma k_2)(2(c-\gamma k_2)^2(a+\gamma k_3)+2a^2(2a+\gamma k_3)\\ &\quad +2a(a+2\gamma k_3)^2-2\gamma^2(a+3\gamma k_3)(-1+k_1^2+k_2^2))=0,\\ (2,4)' &= (c-\gamma k_2)(2(c-\gamma k_2)^2+2(3\gamma^2(1-k_1^2-k_2^2)+a^2+4a\gamma k_3))=0,\\ (3,4)' &= -(c-\gamma k_2)^2(3a+\gamma k_3)-8\gamma k_3(3a^2-\gamma^2(-1+k_1^2+k_2^2))=0,\\ (1,1)' &= -(c-\gamma k_2)^2(11(c-\gamma k_2)^2+16(a(a+4\gamma k_3)-\gamma^2(-1+k_1^2+k_2^2)))\\ &\quad +16a(a^2(a-8\gamma k_3)+2\gamma^2(a+4\gamma k_3)(-1+k_1^2+k_2^2))\\ &\quad +16\gamma^4(-1+k_1^2+k_2^2)^2=0.\end{aligned}$$

Now, if we assume $c-\gamma k_2\neq 0$ we can reduce the equations $(2,3)'=0$ and $(2,4)'=0$ of (5.14) to

$$\begin{aligned}(2,3)'' &= 2(c-\gamma k_2)^2(a+\gamma k_3)+2a^2(2a+\gamma k_3)+2a(a+2\gamma k_3)^2\\ &\quad -2\gamma^2(a+3\gamma k_3)(-1+k_1^2+k_2^2)=0,\\ (2,4)'' &= 2(c-\gamma k_2)^2+2(3\gamma^2(1-k_1^2-k_2^2)+a^2+4a\gamma k_3)=0.\end{aligned}$$

Now, we have $a\neq 0$ because if $a=0$ from $(2,4)''=0$ we get a contradiction with $1-k_1^2-k_2^2>0$. Thus, from $(2,4)''=0$ we get

$$(5.15)\qquad k_3=-\frac{a^2+(c-\gamma k_2)^2+3\gamma^2(1-k_1^2-k_2^2)}{4a\gamma}.$$

Now, substituting (5.15) in $(2,3)''=0$ we obtain $4a(a^2-\gamma^2(1-k_1^2-k_2^2))=0$. Thus

$$(5.16)\qquad a^2=\gamma^2(1-k_1^2-k_2^2).$$

Finally, substituting (5.15) and (5.16) in $(3,4)'=0$ of (5.14) we get

$$\frac{32\gamma^4(-1+k_1^2+k_2^2)^2}{a}+\frac{(-c+\gamma k_2)^2((c-\gamma k_2)^2+24\gamma^2(1-k_1^2-k_2^2))}{4a}=0,$$

a contradiction with $1-k_1^2-k_2^2>0$. Therefore, we must assume $c-\gamma k_2=0$. In this case, we get that (5.14) reduces to

$$(5.17)\qquad \begin{aligned}(3,4)^* &= k_3(3a^2+\gamma^2(1-k_1^2-k_2^2))=0,\\ (1,1)^* &= 16a(a^2(a-8\gamma k_3)+2\gamma^2(a+4\gamma k_3)(-1+k_1^2+k_2^2))\\ &\quad +16\gamma^4(-1+k_1^2+k_2^2)^2=0.\end{aligned}$$

Now, from $(3,4)^* = 0$ we get $k_3 = 0$ due to $1 - k_1^2 - k_2^2 > 0$. Finally, putting $k_3 = 0$ in $(1,1)^* = 0$ we obtain $16(a^2 - \gamma^2(1 - k_1^2 - k_2^2))^2 = 0$. That is, the formulas (5.9).

On the other hand, (5.8) is automatically satisfied by the solution (5.9). Moreover, we get substituting (5.9) in (5.6) that the corresponding spaces have the Ricci eigenvalue $\varrho = -2\gamma^2$ with multiplicity four. Then the corresponding spaces are Einstein and by [9] they are locally symmetric. Proposition 5.6 is proved. □

This completes the proof of the Theorem 5.1. □

6. Non-nilpotent semi-direct products $H \rtimes \mathbb{R}$

Let $\mathfrak{h}$ be the Lie algebra of H (the Heisenberg group) with a scalar product $\langle , \rangle_3$. Then, there is an orthonormal basis $\{f_1, f_2, f_3\}$ of $\mathfrak{h}$ such that

$$[f_3, f_2] = 0, \quad [f_3, f_1] = 0, \quad [f_1, f_2] = \gamma f_3 \tag{6.1}$$

where $\gamma \neq 0$ is a real number. The algebra of all derivations D of $\mathfrak{h}$ is

$$\left\{ \begin{pmatrix} a & b & h \\ c & d & f \\ 0 & 0 & a+d \end{pmatrix} : a, b, c, d, h, f \in \mathbb{R} \right\},$$

when represented in the matrix form.

According to the general scheme, we consider the algebra $\mathfrak{g} = \mathfrak{h} + \mathbb{R}$, where the multiplication table is given by (6.1) and, according to the general formula (3.1), also by

$$\begin{gathered} [f_4, f_1] = af_1 + bf_2 + hf_3, \quad [f_4, f_2] = cf_1 + df_2 + ff_3, \\ [f_4, f_3] = (a+d)f_3, \quad \langle f_i, f_4 \rangle = k_i, \quad i = 1, 2, 3. \end{gathered} \tag{6.2}$$

Here $\gamma \neq 0, a, b, c, d, f, h, k_1, k_2, k_3$ are arbitrary parameters where $\sum_{i=1}^{3} k_i^2 < 1$. We exclude the nilpotent case $a = b = c = d = h = 0$. (See [2]).

This give rise to a simply connected group space $(G = H \rtimes \mathbb{R}, g)$.

Theorem 6.1. *The only family of metrics such that $(H \rtimes \mathbb{R}, g)$ is a weakly Einstein manifold is Einstein and locally symmetric. Moreover, the corresponding Lie algebra is determined by (6.2) where $a = d = \pm\gamma R/2$, $b = -c$, $h = -\gamma k_2$, $f = \gamma k_1$, $\gamma \neq 0$, and k_1, k_2, k_3 arbitrary.*

In the remainder of the section, we will prove the announced theorem. We replace the basis $\{f_i\}$ by the new basis $\{e_i\}$, as in the formula (2.3). Then we get an orthonormal basis for which

$$(6.3)\ [e_1, e_2] = \gamma e_3, \quad [e_3, e_2] = [e_3, e_1] = 0, \quad [e_4, e_1] = \frac{1}{R}(ae_1 + be_2 + (h + k_2\gamma)e_3),$$
$$[e_4, e_2] = \frac{1}{R}(ce_1 + de_2 + (f - k_1\gamma)e_3), \quad [e_4, e_3] = \frac{1}{R}((a + d)e_3).$$

Now we are going to calculate, in the new basis, the expression for the condition for (G, g) to be a weakly Einstein manifold. From [1] we know

Lemma 6.2. *The components of the curvature operator are*

(6.4)
$$\begin{aligned}
\mathcal{R}(e_1, e_2) &= \alpha_{1212}A_{12} + \alpha_{1213}A_{13} + \alpha_{1214}A_{14} + \alpha_{1223}A_{23} + \alpha_{1224}A_{24} + \alpha_{1234}A_{34},\\
\mathcal{R}(e_1, e_3) &= \alpha_{1312}A_{12} + \alpha_{1313}A_{13} + \alpha_{1314}A_{14} + \alpha_{1323}A_{23} + \alpha_{1324}A_{24} + \alpha_{1334}A_{34},\\
\mathcal{R}(e_1, e_4) &= \alpha_{1412}A_{12} + \alpha_{1413}A_{13} + \alpha_{1414}A_{14} + \alpha_{1423}A_{23} + \alpha_{1424}A_{24} + \alpha_{1434}A_{34},\\
\mathcal{R}(e_2, e_3) &= \alpha_{2312}A_{23} + \alpha_{2313}A_{13} + \alpha_{2314}A_{14} + \alpha_{2323}A_{23} + \alpha_{2324}A_{24} + \alpha_{2334}A_{34},\\
\mathcal{R}(e_2, e_4) &= \alpha_{2412}A_{12} + \alpha_{2413}A_{13} + \alpha_{2414}A_{14} + \alpha_{2423}A_{23} + \alpha_{2424}A_{24} + \alpha_{2434}A_{34},\\
\mathcal{R}(e_3, e_4) &= \alpha_{3412}A_{12} + \alpha_{3413}A_{13} + \alpha_{3414}A_{14} + \alpha_{3423}A_{23} + \alpha_{3424}A_{24} + \alpha_{3434}A_{34},
\end{aligned}$$

where the coefficients $\alpha_{ijlm} = g(\mathcal{R}(e_i, e_j)e_l, e_m)$ satisfy the standard symmetries with respect to their indices and

$$(6.5)\quad \alpha_{1212} = \frac{4ad + 3\gamma^2R^2 - (b + c)^2}{4R^2}, \quad \alpha_{1213} = \frac{2a(f - \gamma k_1) - (b + c)(h + \gamma k_2)}{4R^2},$$
$$\alpha_{1214} = \frac{-3\gamma(h + \gamma k_2)}{4R}, \quad \alpha_{1223} = \frac{(b + c)(f - \gamma k_1) - 2d(h + \gamma k_2)}{4R^2},$$
$$\alpha_{1224} = \frac{3\gamma(-f + \gamma k_1)}{4R}, \quad \alpha_{1234} = \frac{-(a + d)\gamma}{2R},$$
$$\alpha_{1313} = \frac{4a(a + d) - R^2\gamma^2 - (h + \gamma k_2)^2}{4R^2},$$
$$\alpha_{1323} = \frac{2(a + d)(b + c) + (-f + \gamma k_1)(h + \gamma k_2)}{4R^2}, \quad \alpha_{1314} = \frac{-(b + c)\gamma}{4R},$$
$$\alpha_{1324} = \frac{-d\gamma}{2R}, \quad \alpha_{1334} = \frac{\gamma(f - \gamma k_1)}{4R}, \quad \alpha_{1423} = \frac{a\gamma}{2R},$$
$$\alpha_{1414} = \frac{4a^2 + (3b - c)(b + c) + 3(h + \gamma k_2)^2}{4R^2},$$
$$\alpha_{1424} = \frac{4(ac + bd) + 3(f - \gamma k_1)(h + \gamma k_2)}{4R^2},$$

$$\alpha_{1434} = \frac{(b-c)(f-\gamma k_1) + 4(a+d)(h+\gamma k_2)}{4R^2},$$
$$\alpha_{2323} = \frac{4d(a+d) - R^2\gamma^2 - (f-\gamma k_1)^2}{4R^2},$$
$$\alpha_{2324} = \frac{(b+c)\gamma}{4R}, \quad \alpha_{2334} = \frac{-\gamma(h+\gamma k_2)}{4R},$$
$$\alpha_{2424} = \frac{-(b-3c)(b+c) + 4d^2 + 3(f-\gamma k_1)^2}{4R^2},$$
$$\alpha_{2434} = \frac{4(a+d)(f-\gamma k_1) + (c-b)(h+\gamma k_2)}{4R^2},$$
$$\alpha_{3434} = \frac{4(a+d)^2 - (f-\gamma k_1)^2 - (h+\gamma k_2)^2}{4R^2}.$$

Further, we obtain easily

Lemma 6.3 ([1]). *The matrix of the Ricci tensor of type* $(1,1)$ *expressed with respect to the basis* $\{e_1, e_2, e_3, e_4\}$ *is of the form*

$$\begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \frac{\gamma(-f+\gamma k_1)}{2R} \\ \beta_{12} & \beta_{22} & \beta_{23} & \frac{\gamma(h+\gamma k_2)}{2R} \\ \beta_{13} & \beta_{23} & \beta_{33} & 0 \\ \frac{\gamma(-f+\gamma k_1)}{2R} & \frac{\gamma(h+\gamma k_2)}{2R} & 0 & \beta_{44} \end{pmatrix} \tag{6.6}$$

where

$$\beta_{11} = \frac{-4a(a+d) - b^2 + c^2 - R^2\gamma^2 - (h+\gamma k_2)^2}{2R^2},$$
$$\beta_{12} = \frac{(-f+\gamma k_1)(h+\gamma k_2) - a(b+3c) - d(3b+c)}{2R^2},$$
$$\beta_{13} = \frac{c(f-\gamma k_1) - (2a+3d)(h+\gamma k_2)}{2R^2},$$
$$\beta_{22} = \frac{b^2 - c^2 - 4d(a+d) - R^2\gamma^2 - (f-\gamma k_1)^2}{2R^2},$$
$$\beta_{23} = \frac{(3a+2d)(-f+\gamma k_1) + b(h+\gamma k_2)}{2R^2},$$
$$\beta_{33} = \frac{-4(a+d)^2 + R^2\gamma^2 + (f-\gamma k_1)^2 + (h+\gamma k_2)^2}{2R^2},$$
$$\beta_{44} = \frac{-(b+c)^2 - 4((a+d)^2 - ad) - (f-\gamma k_1)^2 - (h+\gamma k_2)^2}{2R^2}.$$

Now we obtain the following analogue of Lemmas 2.3 and 2.4:

Lemma 6.4.

$$(6.7)\quad |\mathcal{R}|^2 = \sum_{i,j,k,l=1}^{4} \alpha_{ijkl}^2$$
$$= \frac{1}{4R^4}((f-\gamma k_1)(-16(a(b-2c)-(2b-c)d)(h+\gamma k_2)$$
$$+(f-\gamma k_1)(32(c^2+d^2)+8(a+d)(4a+d)-2(b-5c)(b-c)$$
$$+11(f-\gamma k_1)^2+22((h+\gamma k_2)^2+\gamma^2R^2)))+(h+\gamma k_2)^2(40a(a+d)$$
$$+2(11b^2+6bc-c^2+16d^2)+11(h+\gamma k_2)^2+22\gamma^2R^2)$$
$$+(b+c)(-32c(a-d)^2+(b+c)(32a^2+11b^2-10bc+11c^2+8ad+32d^2)$$
$$-2(b+c)\gamma^2R^2)+8d^2(16c^2+6d^2-\gamma^2R^2)+11\gamma^4R^4$$
$$+8(a+d)(2(a+d)(3a^2+2c^2+6d^2)-4d(4c^2+3d^2)+(a+2d)\gamma^2R^2)).$$

Lemma 6.5. *The condition (2.7) is equivalent to the following system of algebraic equations where* $F=f-\gamma k_1$, $H=h+\gamma k_2$ and $S=\gamma^2R^2>0$.

(6.8)
$$(1,2) = 4c(a+d)(4a^2+4c^2+4ad+4d^2)+2(b-c)(2d((b+c)^2+2d^2+2a^2)$$
$$+2(a+d)(bc+3c^2)+2(a+d)^3)+(2ab+4ac+7bd-3cd)F^2$$
$$-2(b+c)(a+d)S-(3ab-7ac-4bd-2cd)H^2+5FH^3+5F^3H$$
$$+FH(2(b^2+3bc+c^2+7d^2)+2(6a-d)(a+d))+5FHS=0,$$
$$(1,3) = F(4bd(2a+3d)+(b+c)(2(a-2d)(a+d)+(b-c)^2)+b(b+c)^2)$$
$$+H((a+d)(2b(3b+c)+(b+c)^2-16ad)+16(a+d)^3$$
$$-d(2b^2+2c^2+(b-c)^2+4d^2))+4(a+d)HS+cF^3$$
$$+(5a+3d)F^2H-bFH^2+4(a+d)H^3=0,$$
$$(2,3) = H(4ac(3a+2d)+(b+c)((b-c)^2+c(b+c)+2(d-2a)(a+d)))$$
$$+F(2(a+d)(-b(3b+c)+2(b+c)^2-2ad)+12(a+d)^3$$
$$+d(2b^2+2c^2+(b-c)^2+4d^2))+4(a+d)FS+bH^3-cF^2H$$
$$+(3a+5d)FH^2+4(a+d)F^3=0,$$
$$(1,4) = H((b-c)(a+6d)+10c(a+d))+5FS+5F^3+5FH^2$$
$$+F(2(a+d)(a+7d)-(3b-2c)(b+c)-2d^2)=0,$$
$$(2,4) = F((b-c)(4a+9d)+10c(a+d))+5HS+5F^2H+5H^3$$
$$+H((2b-3c)(b+c)+4(a+d)(3a+d)-2d^2)=0,$$
$$(3,4) = 2(b-c)((a-d)^2+(b+c)^2)-bH^2+cF^2+(a-d)FH=0,$$

$$
\begin{aligned}
(2,2) = &- H^2(40a^2+18b^2+12bc-6c^2+40ad+16d^2)-2H^2S-11H^4\\
&-(b+c)(24a^2b+7b^3-8a^2c+5b^2c-11bc^2-9c^3+8abd+8acd\\
&+8bd^2-24cd^2)-6(b+c)^2S-8(a+d)(6a^3+6a^2d+4ad^2)\\
&+F^2(8a^2-6b^2+20bc+18c^2+8ad+24d^2)+18F^2S+9F^4-2F^2H^2\\
&+8(a+d)(a+2d)S-2FH(8ab-8ac+12bd-4cd)+(4d^2-3S)^2=0,\\
(3,3) = &-7S^2-H^2(40a^2+18b^2+12bc-6c^2+24ad+8d^2)-14H^2S-7H^4\\
&-(b+c)(16a^2b+11b^3-16a^2c+b^2c+bc^2+11c^3-24abd+40acd\\
&+16bd^2-16cd^2)+8(a+d)(2a^3-4ac^2+10a^2d+12c^2d+4ad^2+8d^3)\\
&+6(b+c)^2S-8(a+d)(a+4d)S-2FH(4ab+28ac+28bd+4cd)\\
&-F^2(8a^2-6b^2+12bc+18c^2+24ad+40d^2)-8d^2(16c^2+6d^2-3S)\\
&-14F^2S-14F^2H^2-7F^4=0,\\
(4,4) = &\,16d^2(8c^2+d^2)+24d^2S+H^2(56a^2+18b^2+4bc-6c^2+56ad+16d^2)\\
&-11S^2-2H^2S+(b+c)(16a^2b+9b^3-16a^2c-5b^2c-5bc^2+9c^3\\
&-8abd+56acd+16bd^2-16cd^2)+2FH(8ab+32ac+32bd+8cd)\\
&+8(a+d)(2a^3+4ac^2+2a^2d-12c^2d+4ad^2)+18F^2H^2-2F^2S\\
&+F^2(16a^2-6b^2+4bc+18c^2+56ad+56d^2)-8(a+d)(2d-a)S\\
&+6(b+c)^2S+9F^4+9H^4=0.
\end{aligned}
$$

Proposition 6.6. *The solutions of the system of algebraic equations* (6.8) *are, up to a re-numeration of the triplet* $\{e_1, e_2, e_3\}$,

$$
a=d=\pm\frac{1}{2}\sqrt{S}, \quad b=-c, \quad H=0, \quad F=0. \tag{6.9}
$$

In this situation, the corresponding spaces are Einstein and locally symmetric.

P r o o f. We obtain solving the system (6.8) numerically with Mathematica 7.0 that its solution set depends exactly on two parameters. Moreover, we get that the solutions are real if and only if $F=H=0$. Thus, we will just study this subcase. If we put $F=H=0$ in (6.8), the system reduces to

(6.10)

$$
\begin{aligned}
(1,2) &= 16c(a+d)(a^2+c^2+ad+d^2)+4(b-c)(c(b+3c)(a+d)+(a+d)^3\\
&\quad+d(2a^2+(b+c)^2+2d^2))-2(b+c)(a+d)S=0\\
(3,4) &= (b-c)((a-d)^2+(b+c)^2)=0,\\
(2,2) &= -(b+c)(24a^2b+7b^3-8a^2c+5b^2c-11bc^2-9c^3+8abd+8acd\\
&\quad+8bd^2-24cd^2)-6(b+c)^2S-8(a+d)(6a^3+6a^2d+4ad^2)
\end{aligned}
$$

$$
\begin{aligned}
&+8(a+d)(a+2d)S+(4d^2-3S)^2=0,\\
(3,3)=&-(b+c)(16a^2b+11b^3-16a^2c+b^2c+bc^2+11c^3-24abd+40acd\\
&+16bd^2-16cd^2)+8(a+d)(2a^3-4ac^2+10a^2d+12c^2d+4ad^2+8d^3)\\
&+6(b+c)^2S-8(a+d)(a+4d)S-8d^2(16c^2+6d^2-3S)=0,\\
(4,4)=&\,16d^2(8c^2+d^2)+24d^2S-11S^2+(b+c)(16a^2b+9b^3-16a^2c-5b^2c\\
&-5bc^2+9c^3-8abd+56acd+16bd^2-16cd^2)-8(a+d)(2d-a)S\\
&+6(b+c)^2S+8(a+d)(2a^3+4ac^2+2a^2d-12c^2d+4ad^2)=0.
\end{aligned}
$$

Now, in order to solve this system of equations, we focus the attention on the equation $(3,4)=0$ and we divide the study into two cases:

Case 1. $a=d$, $b=-c$. Replacing the previous hypothesis in the system (6.10), this becomes equivalent to

$$
\begin{aligned}
(2,2)'&=-3(4d^2-S)(20d^2+3S)=0,\\
(3,3)'&=7(4d^2-S)(12d^2+S)=0,\\
(4,4)'&=(4d^2-S)(36d^2+11S)=0.
\end{aligned}
\tag{6.11}
$$

Then, due to $S>0$, necessarily $d^2=\frac{1}{4}S=\frac{1}{4}\gamma^2R^2$ and we get the desired solution (6.9). Moreover, substituting (6.9) into (6.6) (remember here that $F=f-\gamma k_1$, $H=h+\gamma k_2$ and $S=\gamma^2R^2>0$) we get that the corresponding spaces are irreducible Riemannian manifolds with all Ricci eigenvalues equal to $-\frac{3}{2}\gamma^2$. Then, they are Einstein and by a well-known theorem of G. R. Jensen [9] they are also locally symmetric.

Case 2. $b=c$. After substitution of the previous condition, we can rewrite the system (6.10) as follows:

$$
\begin{aligned}
(1,2)'&=4c(a+d)(4(a^2+c^2+ad+d^2)-S)=0\\
(2,2)'&=-16(3a^2-c^2+3ad-d^2)(a^2+c^2+ad+d^2)\\
&\quad+8(a^2-3c^2+3ad-d^2)S+9S^2=0,\\
(3,3)'&=16(a^2+c^2+ad+d^2)(a^2-3c^2+5ad+d^2)\\
&\quad-8(a^2-3c^2+5ad+d^2)S-7S^2=0,\\
(4,4)'&=16(a^2+c^2+ad+d^2)^2+8(a^2+3c^2-ad+d^2)S-11S^2=0.
\end{aligned}
\tag{6.12}
$$

Now, subtracting equations we also get the following information:

$$
(3,3)'-(4,4)'=-4(4(a^2+c^2+ad+d^2)-S)(4c^2-4ad+S)=0.
$$

Therefore, we can divide the study into two new cases depending on whether S is equal to $4(a^2+c^2+ad+d^2)$ or to $4(ad-c^2)$. In the first case, we get a contradiction due to $S = 4(a^2+c^2+ad+d^2) > 0$ and the system (6.12) reduces to the unique equation

$$(a^2+c^2+ad+d^2)((a^2+c^2+ad+d^2)+(a+d)^2) = 0.$$

If we assume that $S \neq 4(a^2+c^2+ad+d^2)$ but $S = 4(ad-c^2)$, from the equation $(1,2)' = 0$ using the fact that $S > 0$, we also know that necessarily $c = 0$. Finally, putting $c = 0$ and $S = 4ad$ in (6.12), we reduce the system to

$$\begin{aligned}(2,2)'' &= -16(a-d)(3a^3+7a^2d-3ad^2+d^3) = 0,\\ (3,3)'' &= (4,4)'' = 16(a-d)^2((a+d)^2+4ad) = 0.\end{aligned}$$

From the equation $(3,3)'' = 0$ and using the fact that $S = 4ad > 0$, we get $a = d$, the necessary and sufficient condition to fulfill the previous system. Thus, we have obtained a solution of the system (6.8). This is the solution (6.9) with $b = 0 = c$. Proposition 6.6 is proved. □

If we substitute $F = f - \gamma k_1$, $H = h + \gamma k_2$ and $S = \gamma^2 R^2 > 0$ into (6.9) we get just the proof of the last statement of Theorem 6.1. □

7. Semi-direct products $\mathbb{R}^3 \rtimes \mathbb{R}$

Let $\mathfrak{r}^3$ be the Lie algebra of $\mathbb{R}^3$ with a scalar product $\langle , \rangle_3$. The algebra of all derivations D of $\mathfrak{r}^3$ is $\mathfrak{gl}(3,\mathbb{R})$. This means that the matrix form of D depends on 9 arbitrary parameters with respect to any fixed orthonormal basis of $\mathfrak{r}^3$. Moreover, if D is fixed, then we can make three convenient rotations in the coordinate planes to obtain a particular orthonormal basis $\{f_1, f_2, f_3\}$ for which the matrix form of D is a sum of a diagonal matrix and a skew-symmetric matrix. In other words, we have the general matrix form

$$D\colon \left\{ \begin{pmatrix} a & b & c \\ -b & f & h \\ -c & -h & p \end{pmatrix} : \; a, b, c, f, h, p \in \mathbb{R} \right\}$$

depending just on 6 parameters. Moreover, we have

$$[f_1, f_2] = 0, \quad [f_1, f_3] = 0, \quad [f_2, f_3] = 0. \tag{7.1}$$

According to the general scheme, we consider the algebra $\mathfrak{g} = \mathfrak{r}^3 + \mathbb{R}$, where the multiplication table is given by (7.1) and

$$
\begin{aligned}
(7.2)\qquad &[f_4, f_1] = af_1 + bf_2 + cf_3, \quad [f_4, f_2] = -bf_1 + ff_2 + hf_3,\\
&[f_4, f_3] = -cf_1 - hf_2 + pf_3, \quad \langle f_i, f_4\rangle = k_i, \quad i = 1, 2, 3.
\end{aligned}
$$

Here $a, b, c, f, h, p, k_1, k_2, k_3$ are arbitrary parameters where $\sum_{i=1}^{3} k_i^2 < 1$.

This gives rise to a simply connected group space $(G = \mathbb{R}^3 \rtimes \mathbb{R}, g)$.

Theorem 7.1. *There are two families of metrics which make* $(H \rtimes \mathbb{R}, g)$ *a weakly Einstein manifold. The first family consists of Einstein, locally symmetric spaces, and the corresponding Lie algebra is determined by* (7.2), *where* $p = f = a$, *and* a, b, c, h, k_1, k_2, k_3 *are arbitrary. The second family corresponds to Example* 1.2.

In the remainder of the section we will prove the announced theorem. We replace the basis $\{f_i\}$ by the new basis $\{e_i\}$, as in the formula (2.3). Then we get an orthonormal basis for which

$$
\begin{aligned}
(7.3)\quad &[e_1, e_2] = 0, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0, \quad [e_4, e_1] = \frac{1}{R}(ae_1 + be_2 + ce_3),\\
&[e_4, e_2] = \frac{1}{R}(-be_1 + fe_2 + he_3), \quad [e_4, e_3] = \frac{1}{R}(-ce_1 - he_2 + pe_3).
\end{aligned}
$$

Now we are going to calculate, in the new basis, the expression for the condition for (G, g) to be a weakly Einstein manifold. From [1] we know

Lemma 7.2.

$$
\begin{aligned}
(7.4)\qquad &\nabla_{e_1} e_1 = \frac{a}{R} e_4, \quad \nabla_{e_2} e_2 = \frac{f}{R} e_4, \quad \nabla_{e_3} e_3 = \frac{p}{R} e_4, \quad \nabla_{e_4} e_4 = 0,\\
&\nabla_{e_1} e_2 = 0 = \nabla_{e_2} e_1, \quad \nabla_{e_1} e_3 = 0 = \nabla_{e_3} e_1, \quad \nabla_{e_2} e_3 = 0 = \nabla_{e_3} e_2,\\
&\nabla_{e_1} e_4 = -\frac{a}{R} e_1, \quad \nabla_{e_4} e_1 = \frac{b}{R} e_2 + \frac{c}{R} e_3, \quad \nabla_{e_2} e_4 = -\frac{f}{R} e_2,\\
&\nabla_{e_4} e_2 = -\frac{b}{R} e_1 + \frac{h}{R} e_3, \quad \nabla_{e_3} e_4 = -\frac{p}{R} e_3, \quad \nabla_{e_4} e_3 = -\frac{c}{R} e_1 - \frac{h}{R} e_2.
\end{aligned}
$$

Similarly to Lemma 2.2 we can now derive

Lemma 7.3 ([1]). *The components of the curvature operator are*

$$\begin{aligned}(7.5)\quad \mathcal{R}(e_1,e_2) &= \frac{af}{R^2}A_{12}, & \mathcal{R}(e_1,e_4) &= \frac{a^2}{R^2}A_{14} + \frac{b(f-a)}{R^2}A_{24} + \frac{c(p-a)}{R^2}A_{34},\\ \mathcal{R}(e_1,e_3) &= \frac{ap}{R^2}A_{13}, & \mathcal{R}(e_2,e_4) &= \frac{b(f-a)}{R^2}A_{14} + \frac{f^2}{R^2}A_{24} + \frac{h(p-f)}{R^2}A_{34},\\ \mathcal{R}(e_2,e_3) &= \frac{fp}{R^2}A_{23}, & \mathcal{R}(e_3,e_4) &= \frac{c(p-a)}{R^2}A_{14} + \frac{h(p-f)}{R^2}A_{24} + \frac{p^2}{R^2}A_{34}.\end{aligned}$$

Further, we obtain easily

Lemma 7.4 ([1]). *The matrix of the Ricci tensor of type* $(1,1)$ *expressed with respect to the basis* $\{e_1,e_2,e_3,e_4\}$ *is of the form*

$$(7.6)\qquad \begin{pmatrix} -\frac{a(a+f+p)}{R^2} & \frac{b(a-f)}{R^2} & \frac{c(a-p)}{R^2} & 0\\ \frac{b(a-f)}{R^2} & -\frac{f(a+f+p)}{R^2} & \frac{h(f-p)}{R^2} & 0\\ \frac{c(a-p)}{R^2} & \frac{h(f-p)}{R^2} & -\frac{p(a+f+p)}{R^2} & 0\\ 0 & 0 & 0 & -\frac{a^2+f^2+p^2}{R^2} \end{pmatrix}.$$

Now we obtain the following analogue of Lemmas 2.3 and 2.4:

Lemma 7.5.

$$\begin{aligned}(7.7)\quad |\mathcal{R}|^2 = \sum_{i,j,k,l=1}^{4} \alpha_{ijkl}^2 = \frac{4}{R^4}(&a^4+f^4+p^4+a^2f^2+a^2p^2+f^2p^2+2b^2(a-f)^2\\ &+2c^2(a-p)^2+2h^2(f-p)^2).\end{aligned}$$

Lemma 7.6. *The condition* (2.7) *is equivalent to the system of algebraic equations*

$$\begin{aligned}(7.8)\qquad (1,2) &= b(a-f)(a^2+f^2) - ch(a-p)(f-p) = 0,\\ (1,3) &= c(a-p)(a^2+p^2) - bh(a-f)(f-p) = 0,\\ (2,3) &= h(f-p)(f^2+p^2) - bc(a-f)(a-p) = 0,\\ (1,1) &= -2h^2(f-p)^2 + (a-f)(a+f)(f^2+p^2) + a^4 - p^4 = 0,\\ (2,2) &= -2c^2(a-p)^2 + (f-p)(f+p)(a^2+p^2) - a^4 + f^4 = 0,\\ (4,4) &= 2b^2(a-f)^2 + 2c^2(a-p)^2 + 2h^2(f-p)^2 + a^4 - a^2f^2 + f^4\\ &\quad - a^2p^2 - f^2p^2 + p^4 = 0.\end{aligned}$$

Proposition 7.7. *The only solutions of the system of algebraic equations* (7.8) *are, up to a re-numeration of the triplet* $\{e_1, e_2, e_3\}$*, the following ones:*

(1) $p = f = a$, *and* a, b, c, h, k_1, k_2, k_3 *are arbitrary. In this situation, the corresponding spaces are Einstein with all Ricci eigenvalues equal to* $-3a^2/R^2$.

(2) $c = h = 0$, $a = f = -p$, *and* a, b, k_1, k_2, k_3 *are arbitrary. In this situation, the corresponding spaces are neither Einstein nor locally symmetric. Moreover, this case corresponds to Example* 1.2.

Proof. Suppose first $a - f = 0$. Thus, the equation $(4,4) = 0$ of (7.8) reduces to

$$(4,4)' = (a-p)^2(2c^2 + 2h^2 + (a+p)^2) = 0.$$

Now, we get only two possibilities which both are solutions of the system (7.8): $a = p$ or $a = -p$, $c = h = 0$. We call them solutions 1 and 2, respectively.

For the solution 1 we obtain from (7.6) that all four Ricci eigenvalues are equal to $-3a^2/R^2$. Then the corresponding spaces are Einstein and by [9] they are locally symmetric.

For the solution 2 we obtain from (7.6) that the Ricci eigenvalues are $\varrho_1 = \varrho_2 = -a^2/R^2$, $\varrho_3 = a^2/R^2$, $\varrho_4 = -3a^2/R^2$ and, from (7.4) and (7.5) that the corresponding spaces are not locally symmetric due to $(\nabla_{e_1}\mathcal{R})(e_1, e_3)e_3 \neq 0$. Besides, it is easy to check that the curvature tensor (7.5) takes on the form

$$\begin{aligned} &\mathcal{R}(e_1,e_2) = \frac{a^2}{R^2}A_{12}, \quad \mathcal{R}(e_1,e_3) = -\frac{a^2}{R^2}A_{13}, \quad \mathcal{R}(e_1,e_4) = \frac{a^2}{R^2}A_{14}, \\ &\mathcal{R}(e_2,e_3) = -\frac{a^2}{R^2}A_{23}, \quad \mathcal{R}(e_2,e_4) = \frac{a^2}{R^2}A_{24}, \quad \mathcal{R}(e_3,e_4) = \frac{a^2}{R^2}A_{34}. \end{aligned} \tag{7.9}$$

Then, the space of the curvature operators is obviously spanned by the six operators A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, A_{34}. Hence the Lie algebra generated by these operators is $\mathfrak{so}(4)$. We see that the action of the holonomy algebra on the tangent space T_eG is irreducible and hence the corresponding Riemannian manifolds are irreducible. Moreover, in this case the formula (7.3) simplifies as follows

$$\begin{aligned} &[e_1,e_2] = 0, \quad [e_1,e_3] = 0, \quad [e_2,e_3] = 0, \quad [e_4,e_1] = \frac{1}{R}(ae_1 + be_2), \\ &[e_4,e_2] = \frac{1}{R}(-be_1 + ae_2), \quad [e_4,e_3] = \frac{1}{R}(-ae_3). \end{aligned} \tag{7.10}$$

Then making the change of the basis

$$e_1' = e_4, \quad e_2' = e_3, \quad e_3' = e_2, \quad e_4' = e_1,$$

and putting $\alpha = -a/R$, $\beta = b/R$, we get that the multiplication table (7.10) for the new basis $\{e_1', e_2', e_3', e_4'\}$ becomes exactly the same as in Example 1.2.

Let now assume that $a - f \neq 0$. Because the system is symmetric with respect to all re-numerations of the basis $\{e_1, e_2, e_3\}$ which implies corresponding permutations and some changes of sign of the symbols a, b, c, f, h, p, we can also assume that $a - p \neq 0$ and $p - f \neq 0$. Moreover, note that $bch = 0$ if and only if $b = c = h = 0$ due to the equations $(1,2) = 0$, $(1,3) = 0$ and $(2,3) = 0$. In addition, if we put $b = c = h = 0$ in the equation $(4,4) = 0$ we get $a^2 = \frac{1}{2}((f^2 + p^2) \pm \mathrm{i}\sqrt{3}(f - p) \times (f + p))$. Therefore, necessarily $f = -p$ and $a = -f$, a contradiction. Now we can also assume that $bch \neq 0$ and from the equation $(1,2) = 0$ we obtain $b = -ch(a-p)(-f+p)/(a-f)(a^2+f^2)$. Substituting this value of b into the equation $(1,3) = 0$ we get

$$(1,3)' = \frac{c(a-p)}{a^2+f^2}((a^2+f^2)(a^2+p^2) - h^2(f-p)^2) = 0.$$

Therefore, $h^2 = (a^2+f^2)(a^2+p^2)/(f-p)^2$ and the equation $(1,1) = 0$ becomes equivalent to

$$(1,1)' = a^4 + a^2(f^2+p^2) + (f^2+p^2)^2 + f^2p^2 = 0.$$

Thus, $a^2 = \frac{1}{2}(-(f^2+p^2) \pm \mathrm{i}\sqrt{(3f^2+p^2)(f^2+3p^2)})$ which is a contradiction due to $(3f^2+p^2)(f^2+3p^2) > 0$. Proposition 7.7 is proved. □

This completes the proof of the Theorem 7.1 and also that of the Main Theorem.

8. Isomorphisms of EPS spaces

In this section we study the possible isomorphisms among the EPS spaces $(G,g)_{\alpha,\beta}$ depending on the parameters α and β. Here we shall use the classification of 4-dimensional solvable algebras given by de Graaf [4]. According to this classification theorem, our algebras $\mathfrak{g}_{\alpha,\beta}$ from Example 1.2 for different β^2/α^2 are not isomorphic and hence the corresponding groups $G_{\alpha,\beta}$ are not isomorphic as well.

Theorem 8.1. *Two Lie algebras $\mathfrak{g}_{\alpha,\beta}$, $\mathfrak{g}_{\alpha',\beta'}$ are isomorphic if and only if $\beta/\alpha = \pm\beta'/\alpha'$. Therefore the corresponding simply connected Lie groups $G_{\alpha,\beta}$, $G_{\alpha',\beta'}$ are isomorphic if and only if $\beta/\alpha = \pm\beta'/\alpha'$.*

P r o o f. We divide the proof into two cases according as $\beta \neq 0$ or $\beta = 0$ in (1.3). For the first case we replace in (1.3) the basis $\{e_i\}$ by a new quadruplet $\{x_i\}$

$(i = 1, 2, 3, 4)$ putting

$$(8.1) \qquad \begin{aligned} x_1 &= e_2 + e_3 + e_4, \\ x_2 &= -e_2 + \Big(1 - \frac{\beta}{\alpha}\Big)e_3 + \Big(1 + \frac{\beta}{\alpha}\Big)e_4, \\ x_3 &= e_2 + \Big(1 - \frac{\beta(2\alpha+\beta)}{\alpha^2}\Big)e_3 + \Big(1 + \frac{\beta(2\alpha-\beta)}{\alpha^2}\Big)e_4, \\ x_4 &= -\frac{1}{\alpha}e_1, \end{aligned}$$

where $\alpha \neq 0$, $\beta \neq 0$ are constants. In fact, the determinant of the coefficients is $-(2\beta/\alpha^4)(4\alpha^2 + \beta^2) \neq 0$. Thus we get a basis for which

$$(8.2) \qquad \begin{aligned} &[x_4, x_1] = x_2, \quad [x_4, x_2] = x_3, \quad [x_4, x_3] = Ax_1 + (2 + A)x_2 + x_3, \\ &[x_1, x_2] = 0, \quad [x_1, x_3] = 0, \quad [x_2, x_3] = 0, \end{aligned}$$

where $A = -1 - \beta^2/\alpha^2$. Note that this multiplication table corresponds to the class M^6_{ab} of the classification of solvable Lie algebras of dimension 4 given by de Graaf [4] where $a = A$, $b = 2 + A$. Hence A is an algebraic invariant and β^2/α^2 is also an algebraic invariant.

On the other hand, if $\beta = 0$ in (1.3), we replace the basis $\{e_i\}$ by a new basis $\{x_i\}$ $(i = 1, 2, 3, 4)$ putting

$$(8.3) \qquad x_1 = e_4, \quad x_2 = e_2 + e_3, \quad x_3 = -e_2 + e_3, \quad x_4 = -\frac{1}{\alpha}e_1,$$

where $\alpha \neq 0$ is a constant. Then we get a basis for which

$$(8.4) \qquad \begin{aligned} &[x_4, x_1] = x_1, \quad [x_4, x_2] = x_3, \quad [x_4, x_3] = x_2, \\ &[x_1, x_2] = 0, \quad [x_1, x_3] = 0, \quad [x_2, x_3] = 0. \end{aligned}$$

This multiplication table corresponds to the class M^3_a of the classification of solvable Lie algebras of dimension 4 given by de Graaf [4] putting $a = -1$. □

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