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BOUNDEDNESS OF STEIN'S SQUARE FUNCTIONS AND BOCHNER-RIESZ MEANS ASSOCIATED TO OPERATORS ON HARDY SPACES

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Abstract. Let (X, d, μ) be a metric measure space endowed with a distance d and a non-negative Borel doubling measure μ. Let L be a non-negative self-adjoint operator of order m on $L^2(X)$. Assume that the semigroup e^{-tL} generated by L satisfies the Davies-Gaffney estimate of order m and L satisfies the Plancherel type estimate. Let $H^p_L(X)$ be the Hardy space associated with L. We show the boundedness of Stein's square function $\mathcal{G}_\delta(L)$ arising from Bochner-Riesz means associated to L from Hardy spaces $H^p_L(X)$ to $L^p(X)$, and also study the boundedness of Bochner-Riesz means on Hardy spaces $H^p_L(X)$ for $0 < p \leqslant 1$.

Keywords: non-negative self-adjoint operator; Stein's square function; Bochner-Riesz means; Davies-Gaffney estimate; molecule Hardy space

MSC 2010: 42B15, 42B25, 47F05

1. Introduction

Let L be a non-negative self-adjoint operator acting on $L^2(X)$, where X is a doubling measure space. It admits a spectral resolution

$$L = \int_0^\infty \lambda \, \mathrm{d}E(\lambda).$$

For a complex number $\delta = \sigma + \mathrm{i}\tau$, $\sigma > -1$, by the spectral theorem we can define the Bochner-Riesz means $S^\delta_R(L) = (I - L/R^m)^\delta_+$ of order δ of a function f as

$$S^\delta_R(L)f(x) = \int_0^R \Big(1 - \frac{\lambda}{R^m}\Big)^\delta \mathrm{d}E(\lambda)f(x), \quad x \in X, \ R > 0, \tag{1.1}$$

where m is a positive constant and $m \geqslant 2$.

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Due to the above, we can also consider the following square function associated to an operator L:

$$\mathcal{G}_\delta(L)f(x) = c_{m\delta}\Big(\int_0^\infty \Big|\frac{\partial}{\partial R}S_R^{\delta+1}(L)f(x)\Big|^2 R\,\mathrm{d}R\Big)^{1/2}, \quad x \in X, \tag{1.2}$$

where $c_{m\delta} = 1/(m(\delta+1))$.

Note that when L is the Laplacian $-\Delta$ on $\mathbb{R}^D$, the square function $\mathcal{G}_\delta(\Delta)$ is introduced by E. M. Stein in his study of Bochner-Riesz means [21]. It is known that the L^p boundedness of $\mathcal{G}_\sigma(\Delta)$ for $1 < p \leqslant 2$ holds if and only if $\sigma > D(1/p-1/2)-1/2$ (see [14], [15] and [21]). For the range $p > 2$, the condition $\sigma > \max\{1/2, D(1/2 - 1/p)\} - 1$ is known to be necessary and sufficient in dimensions $D = 1$ and 2. In dimensions $D \geqslant 3$, there are some partial results, see for instance, for $\sigma > D(1/2 - 1/p) - 1/2$ in [14] and [15]. For $0 < p \leqslant 1$, if $\sigma > D(1/p - 1/2) - 1/2$, then $\mathcal{G}_\sigma(\Delta)$ is bounded from H^p to L^p (see [16]). Boundedness of the square function $\mathcal{G}_\delta(\Delta)$ has been studied extensively because of its important role in the Bochner-Riesz analysis and we refer the reader to [5], [14], [15], [16] and [21] and the references therein.

Recently, in the abstract framework of a space of homogeneous type (X, d, μ) with dimension $n > 0$ (see Section 2 below), P. Chen, X. T. Duong and L. X. Yan ([5]) studied and obtained the L^p boundedness of Stein's square function $\mathcal{G}_\delta(L)$ when the semigroup e^{-tL}, generated by $-L$ on $L^2(X)$, has the kernels $p_t(x, y)$ which satisfy the Gaussian upper bounds (see, for example, [18])

$$|p_t(x,y)| \leqslant \frac{C}{V(x,t^{1/m})}\exp\Big(-\frac{d(x,y)^{m/(m-1)}}{ct^{1/(m-1)}}\Big)$$

for all $t > 0$ and $x, y \in X$, where C, c are constants. They showed that under the assumption of the Plancherel type estimate (see also [6], [10]), that is, for some $2 \leqslant q \leqslant \infty$ and any $t > 0$ and all Borel functions F such that $\operatorname{supp} F \subseteq [0, t]$,

$$\int_X |K_{F(\sqrt[m]{L})}(x,y)|^2\,\mathrm{d}\mu(x) \leqslant \frac{C}{V(y,t^{-1})}\|F(t\cdot)\|_{L^q}^2, \tag{1.3}$$

where $K_{F(\sqrt[m]{L})}(x,y)\colon X \times X \to \mathbb{C}$ denotes the kernel of the operator $F(\sqrt[m]{L})$, if $p \in (1, \infty)$ and $\sigma > (n + 1 - 2/q)|1/p - 1/2| - 1/2$, then $\mathcal{G}_\sigma(L)$ is bounded on $L^p(X)$ (see Theorem 1.1, [5]).

Sometimes it is not clear whether, or it is even not true that, a non-negative self-adjoint operator on $L^2(X)$ admits Gaussian upper bounds. This occurs, for example, for Schrödinger operators with bad potentials [20] or elliptic operators of higher order with bounded measurable coefficients [8]. So we consider the following weaker assumptions:

(H1) The operator L generates an analytic semigroup $\{e^{-tL}\}_{t>0}$ on $L^2(X)$ which satisfies the Davies-Gaffney estimate (of order m). That is, there exist constants $C, c > 0$ such that for any open subsets $U_1, U_2 \subset X$,

$$(1.4)\quad |\langle e^{-tL} f_1, f_2\rangle| \leqslant C \exp\Big(-\frac{\operatorname{dist}(U_1,U_2)^{m/(m-1)}}{ct^{1/(m-1)}}\Big)\|f_1\|_{L^2(X)}\|f_2\|_{L^2(X)}, \quad \forall\, t>0,$$

for every $f_i \in L^2(X)$ with $\operatorname{supp} f_i \subset U_i$, $i = 1, 2$, where $\operatorname{dist}(U_1, U_2) := \inf_{\substack{x\in U_1\\ y\in U_2}} d(x, y)$.

Motivated by the works [5] and [11] we study the boundedness of Stein's square function $\mathcal{G}_\delta(L)$ from the Hardy spaces $H^p_L(X)$ to $L^p(X)$. Moreover, we get the boundedness of Bochner-Riesz means $S^\delta_R(L)$ on the Hardy spaces $H^p_L(X)$ for $0 < p \leqslant 1$. For our purposes we introduce the Hardy spaces $H^p_L(X)$ as follows. Definition 1.1 below is inspired by [9].

Definition 1.1. Let L be a non-negative self-adjoint operator on $L^2(X)$ which satisfies the Davies-Gaffney estimate (1.4). Consider the following quadratic operator associated to L:

$$(1.5)\quad S_h f(x) = \Big(\int_0^\infty\int_{d(x,y)<t} |(t^m L)e^{-t^m L} f(y)|^2 \frac{d\mu(y)}{V(x,t)}\frac{dt}{t}\Big)^{1/2}, \quad x\in X,\ f \in L^2(X).$$

For each $0 < p \leqslant 1$, the space $H^p_L(X)$ is defined as the completion of $\{f \in L^2(X)\colon S_h f \in L^p(X)\}$ in the norm

$$\|f\|_{H^p_L(X)} = \|S_h f\|_{L^p(X)}.$$

Note that S. Hofmann, G. Z. Lu, D. Mitrea, M. Mitrea and L. X. Yan [12] developed a theory of Hardy spaces adapted to non-negative self-adjoint operators L on $L^2(X)$ which satisfy the Davies-Gaffney estimate (of order 2) in the framework of spaces of homogeneous type. X. T. Duong and J. Li [9] studied even non-self-adjoint operators and introduced Hardy spaces associated with operators which have a bounded holomorphic functional calculus on $L^2(X)$ and satisfy the Davies-Gaffney estimate (of order 2). For more details about Hardy spaces, we refer the reader to [1], [13].

There is an equivalent characterization of the Hardy spaces $H^p_L(X)$ in terms of a molecular decomposition (see Theorem 3.3 below). In order to prove boundedness of an operator on $H^p_L(X)$, one only needs to understand the action of the operator on an individual molecule. P. Chen [4] obtained the boundedness of Bochner-Riesz means $S^\delta_R(L)$ on $H^p_L(X)$ for L satisfying the Davies-Gaffney estimate (of order 2) provided that L satisfies the so called Stein-Tomas restriction type condition. We generalize this result on $H^p_L(X)$ to L satisfying the Davies-Gaffney estimate (of order m, $m \geqslant 2$) provided that L satisfies a variation of Plancherel type estimates (see Theorem 1.2 below). Following the work of P. C. Kunstmann and M. Uhl [17], we introduce a variation of the Plancherel type condition (1.3) for L which fulfils the Davies-Gaffney estimate: there exist $C > 0$ and $q \in [2, \infty]$ such that for any $t > 0$, $y \in X$ and all bounded Borel functions $F\colon [0, \infty) \to \mathbb{C}$ with $\operatorname{supp} F \subseteq [0, t]$,

$$\|F(\sqrt[m]{L})\chi_{B(y,1/t)}\|_{L^2(X)\to L^2(X)} \leqslant C\|F(t\cdot)\|_{L^q}. \tag{1.6}$$

Having this replacement at hand, we are able to state our main results.

Theorem 1.2. *Let L be a non-negative self-adjoint operator on $L^2(X)$ satisfying the Davies-Gaffney estimate* (1.4) *and the Plancherel type condition* (1.6) *for some $q \in [2, \infty]$. Let $\delta = \sigma + \mathrm{i}\tau$ with $\sigma > 0$ and let $\mathcal{G}_\delta(L)$ be an operator given in* (1.2). *If $p \in (0, 1]$ and*

$$\sigma > n\Big(\frac{1}{p} - \frac{1}{2}\Big) - \frac{1}{q},$$

then there exists a constant $C = C(\sigma, \tau, p) > 0$ such that

$$\|\mathcal{G}_\delta(L)f\|_{L^p(X)} \leqslant C\|f\|_{H^p_L(X)}.$$

Theorem 1.3. *Let L be a non-negative self-adjoint operator on $L^2(X)$ satisfying the Davies-Gaffney estimate* (1.4) *and the Plancherel type condition* (1.6) *for some $q \in [2, \infty]$. If $p \in (0, 1]$, then for all $\delta > \max\{n(1/p - 1/2) - 1/q, 0\}$ we have*

$$\Big\|\Big(I - \frac{L}{R^m}\Big)_+^\delta\Big\|_{H^p_L(X)\to H^p_L(X)} \leqslant C$$

uniformly in $R > 0$.

Theorem 1.3, which is actually Corollary 5.3, follows from a spectral multiplier result as those in [11], [17] which will be stated in Section 5 as Theorem 5.1. The assertion of Theorem 1.3 generalizes results from [4].

This article is organized as follows. In Section 2, we prove some preliminary results concerning operators satisfying the Davies-Gaffney estimate. In Section 3, we state molecular decompositions of Hardy spaces $H^p_L(X)$ associated to an operator L, and then get the characterization of the Hardy spaces. In Section 4, we state a criterion for $H^p_L - L^p$ boundedness for singular integrals (cf. [3], [12]), and prove some estimates on Stein's square functions by using the Davies-Gaffney estimate (1.4) and the Plancherel estimate (1.6). We then apply the criterion for $H^p_L - L^p$ boundedness for singular integrals to prove Theorem 1.2. In Section 5, we get the boundedness of $S_R^\delta(L)$ on the Hardy spaces $H_L^p(X)$ for $0 < p \leqslant 1$.

2. Preliminaries

Throughout the whole article we assume that (X, d, μ) is a metric measure space endowed with a distance d and a nonnegative Borel measure μ on X such that the doubling condition

$$V(x, 2r) \leqslant CV(x, r) < \infty \tag{2.1}$$

holds for all $x \in X$ and for all $r > 0$, where $B(x, r) = \{y \in X\colon d(x, y) < r\}$ and $V(x, r) = \mu(B(x, r))$. A more general definition and further studies of these spaces can be found in [7].

It follows from the doubling property that the strong homogeneity property

$$V(x, \lambda r) \leqslant C\lambda^n V(x, r) \tag{2.2}$$

holds for some $C, n > 0$ uniformly for all $\lambda \geqslant 1$ and $x \in X$. In the sequel the value n always refers to the constants in (2.2) which will be also called the *dimension* of (X, d, μ). Of course, n is not uniquely determined and for any $n' > n$ the inequality (2.2) is still valid. However, the smaller n is, the stronger will be the multiplier theorems we are able to obtain. Therefore, we are interested in taking n as small as possible. Besides, there also exist C and n_0 such that

$$V(y, r) \leqslant C\Big(1 + \frac{d(x, y)}{r}\Big)^{n_0} V(x, r) \tag{2.3}$$

uniformly for all $x, y \in X$ and $r > 0$. In fact, property (2.3) with $n_0 = n$ is a direct consequence of the triangle inequality for the metric d and the strong homogeneity property (2.2). But, in general, n_0 can be taken to be smaller. For example, for the Lebesgue measure on $\mathbb{R}^D$ or the Lie groups with polynomial growth, n_0 can be taken to be 0.

Proposition 2.1. *Assume that the non-negative self-adjoint operator L satisfies the Davies-Gaffney estimate* (1.4). *Then for every $K \in \mathbb{N}$, the family of operators*

$$\{(tL)^K \mathrm{e}^{-tL}\}_{t>0}$$

satisfies the Davies-Gaffney estimate (1.4) *with c, $C > 0$ depending on K, n and n_0 in* (2.2) *and* (2.3) *only.*

Proof. The proof is similar to that of [12], Proposition 3.1, or [17], Lemma 2.7, so we omit the details here. □

As a consequence of Proposition 2.1, we have the following proposition.

Proposition 2.2. *Assume that the non-negative self-adjoint operator L satisfies the Davies-Gaffney estimate* (1.4). *Then for every $K_1, K_2 \in \mathbb{N}$, the family of operators*

$$\{(tL)^{K_1}(\mathrm{e}^{-tL})^{K_2}\}_{t>0}$$

satisfies the Davies-Gaffney estimate (1.4) *with c, $C > 0$ depending on K_1, K_2, n and n_0 in* (2.2) *and* (2.3) *only.*

3. Molecular decompositions of the Hardy spaces $H_L^p(X)$

Let us denote by $\mathcal{D}(T)$ the domain of an operator T. Recall that $B = B(x_B, r_B)$ is the ball of radius r_B centered at x_B. Given $\lambda > 0$, we will write λB for the ball with the same center as B and with radius $r_{\lambda B} = \lambda r_B$. We set

$$U_0(B) := B, \quad \text{and} \quad U_j(B) := 2^j B \setminus 2^{j-1} B \quad \text{for } j = 1, 2, \ldots . \tag{3.1}$$

We next describe the notion of a (p, m, M, ε)*-molecule* associated with an operator L which satisfies (H1).

Definition 3.1. Let $0 < p \leqslant 1$, $\varepsilon > 0$ and $M \in \mathbb{N}$. A function $a(x) \in L^2(X)$ is called a (p, m, M, ε)*-molecule* associated with L if there exist a function $b \in \mathcal{D}(L^M)$ and a ball B such that

(i) $a = L^M b$;

(ii) for every $k = 0, 1, 2, \ldots, M$ and $j = 0, 1, 2, \ldots$, we have

$$\|(r_B^m L)^k b\|_{L^2(U_j(B))} \leqslant r_B^{mM} 2^{-j\varepsilon} V(2^j B)^{1/2-1/p},$$

where the annuli $U_j(B)$ are defined in (3.1).

Next, we give the definition of the molecular Hardy spaces associated with L (cf. [9]).

Definition 3.2. Given $0 < p \leqslant 1$, $\varepsilon > 0$ and $M \in \mathbb{N}$, $M > \frac{1}{2}n(2-p)/mp$, we say that $f = \sum_j \lambda_j a_j$ is a *molecular (p, m, M, ε)-representation* of f if $\{\lambda_j\}_{j=0}^{\infty} \in l^p$, each a_j is a (p, m, M, ε)-molecule, and the sum converges in $L^2(X)$. Set

$$\mathbb{H}^p_{L,\mathrm{mol},M}(X) := \{f\colon f \text{ has a molecular } (p, m, M, \varepsilon)\text{-representation}\},$$

with the "norm" (it is true norm only when $p = 1$) given by

$$\|f\|_{\mathbb{H}^p_{L,\mathrm{mol},M}(X)} = \inf\left\{ \left(\sum_{j=0}^{\infty} |\lambda_j|^p \right)^{1/p} : f = \sum_{j=0}^{\infty} \lambda_j a_j \text{ is a molecular } (p, m, M, \varepsilon)\text{-representation} \right\}.$$

The space $H^p_{L,\mathrm{mol},M}(X)$ is then defined as the completion of $\mathbb{H}^p_{L,\mathrm{mol},M}(X)$ with quasi-metric d defined by $d(f,g) = \|f - g\|_{H^p_{L,\mathrm{mol},M}(X)}$ for all $f, g \in H^p_{L,\mathrm{mol},M}(X)$.

As a direct consequence of the definition, we note that

$$H^p_{L,\mathrm{mol},M_2}(X) \subset H^p_{L,\mathrm{mol},M_1}(X)$$

whenever $0 < p \leqslant 1$ and the integer $M_i \in \mathbb{N}$, $i = 1, 2$ with $[\frac{1}{2}n(2-p)/mp] < M_1 < M_2 < \infty$. We shall see that any choice of $\varepsilon > 0$ and $M > \frac{1}{2}n(2-p)/mp$ leads to the same spaces $H^p_{L,\mathrm{mol},M}(X)$; this follows from the more general fact that the "square function" and the "molecular" H^p spaces are equivalent whenever $\varepsilon > 0$ and the parameter M is large enough. One can show the following theorem, which is proved as Theorem 3.15 of [9] in the special case when $m = 2$. In fact, the parameter $m = 2$ is not essential, similarly we can obtain the conclusion for more general cases. We omit the details here.

Theorem 3.3. *Let the non-negative self-adjoint operator L satisfy the Davies-Gaffney estimate* (1.4)*. Assume that $0 < p \leqslant 1$, $\varepsilon > 0$ and $M > [\frac{1}{2}n(2-p)/mp]$, $M \in \mathbb{N}$. Then $H^p_L(X) = H^p_{L,\mathrm{mol},M}(X)$ with equivalent norms $\|f\|_{H^p_{L,\mathrm{mol},M}(X)} \approx \|f\|_{H^p_L(X)}$, where the implicit constants depend only on p, M, ε and on the constants in the Davies-Gaffney estimate and the doubling condition.*

4. Boundedness of Stein's square functions from $H^p_L(X)$ to $L^p(X)$

In this section we will prove Theorem 1.2. First, we state a criterion for $H^p_L - L^p$ boundedness for singular integrals.

Proposition 4.1. *Let L be a nonnegative self-adjoint operator which satisfies the Davies-Gaffney estimate* (1.4)*. Let $0 < p \leqslant 1$. Assume that T is a non-negative sublinear operator which is bounded on $L^2(X)$. If for some $M_0 > n(2-p)/(2p)$ and $C > 0$ the estimate*

$$\|Ta\|_{L^2(U_j(B))} \leqslant C 2^{-jM_0} V(B)^{1/2-1/p} \tag{4.1}$$

is satisfied for each (p,m,M,ε)-molecule a and all $j\geqslant 0$, then T is bounded from $H^p_L(X)$ to $L^p(X)$.

P r o o f. The proof of this proposition is standard (cf. [3], [12]). For the sake of completeness, we provide it here.

Suppose that $f\in H^p_L(X)$. By Theorem 3.3 and density, we can write $f=\sum_j \lambda_j a_j$ in the $L^2(X)$ sense, where a_j are (p,m,M,ε)-molecules and $\Big(\sum_{j=0}^{\infty}|\lambda_j|^p\Big)^{1/p}\approx \|f\|_{H^p_L(X)}$. We claim that

$$|T(f)|\leqslant \sum_{j=0}^{\infty}|\lambda_j||T(a_j)|. \tag{4.2}$$

Indeed, for every $\eta>0$ we have that, if $f^N=\sum_{j>N}\lambda_j a_j$, then

$$\begin{aligned}\mu\Big\{|T(f)|-\sum_{j=0}^{\infty}|\lambda_j||T(a_j)|>\eta\Big\}&\leqslant \limsup_{N\to\infty}\mu\{|T(f^N)|>\eta\}\\&\leqslant C_T\eta^{-2}\limsup_{N\to\infty}\|f^N\|^2_{L^2(X)}=0,\end{aligned}\tag{4.3}$$

from which (4.2) follows, where C_T is the L^2-bound of T. Thus we have

$$\|T(f)\|^p_{L^p(X)}\leqslant \sum_{j=0}^{\infty}|\lambda_j|^p\|T(a_j)\|^p_{L^p(X)}. \tag{4.4}$$

By Hölder inequalities and (4.1), one has

$$\begin{aligned}\|T(a_j)\|^p_{L^p(X)}&=\sum_{k=0}^{\infty}\int_{U_k(B)}(Ta_j(x))^p\,\mathrm{d}\mu(x)\\&\leqslant\sum_{k=0}^{\infty}V(2^kB)^{1-p/2}\|Ta_j\|^p_{L^2(U_k(B))}\\&\leqslant\sum_{k=0}^{\infty}2^{kn(1-p/2)}V(B)^{1-p/2}2^{-kM_0p}V(B)^{p/2-1}\\&=\sum_{k=0}^{\infty}2^{kn(1-p/2)-kM_0p}\leqslant C.\end{aligned}\tag{4.5}$$

This together with (4.4) yields

$$\|T(f)\|^p_{L^p(X)}\leqslant C\sum_{j=0}^{\infty}|\lambda_j|^p\leqslant C\|f\|^p_{H^p_L(X)}. \tag{4.6}$$

Then the proof is complete. □

Lemma 4.2. *Suppose that L satisfies the Davies-Gaffney estimate* (1.4) *and the Plancherel estimate* (1.6) *for some $q \in [2,\infty]$. Then for any $v \geqslant 2/q$, $\varepsilon > 0$, there exists a constant $C = C(v,\varepsilon)$ such that*

$$\|F(\sqrt[m]{L})\chi_{B(y,1/t)}\|_{L^2(X)\to L^2(X,(1+td(\cdot,y))^v\,\mathrm{d}\mu)} \leqslant C\|F_{(t)}(\lambda)\|_{W^q_{v/2+\varepsilon}}$$

for every $t > 0$, $y \in X$, and all bounded Borel functions $F\colon [0,\infty) \to \mathbb{C}$ with $\operatorname{supp} F \subseteq [t/4,t]$, *where $F_{(t)}(\lambda) = F(t\lambda)$ and $\|F\|_{W^q_v} = \|(I-\mathrm{d}^2/\mathrm{d}x^2)^{v/2}F\|_{L^q}$.*

Proof. For a proof, see Lemma 4.10 of [17]. □

Proposition 4.3. *Let the non-negative self-adjoint operator L satisfy the Davies-Gaffney estimate* (1.4) *and the Plancherel estimate* (1.6) *for some $q \in [2,\infty]$. Let $\delta = \sigma + \mathrm{i}\tau$ with $\sigma > 0$, let $\mathcal{G}_\sigma(L)$ be an operator given in* (1.2). *Suppose that $0 < p \leqslant 1$ and $M \in \mathbb{N}$, $M > n(2-p)/(2mp)$. If*

$$\sigma > n\Big(\frac{1}{p} - \frac{1}{2}\Big) - \frac{1}{q},$$

then there exist constants $v_0 > n(2-p)/(2p)$ and $C = C(\sigma,\tau) > 0$ such that for any ball B

$$(\alpha) \qquad \|\mathcal{G}_\delta(L)(I-\mathrm{e}^{-r_B^m L})^M f\|_{L^2(U_j(B))} \leqslant C2^{-jv_0}\|f\|_{L^2(B)}$$

for all integers $j \geqslant 0$ and for all $f \in L^2(X)$ with $\operatorname{supp} f \subset B$;

$$(\beta) \qquad \|\mathcal{G}_\delta(L)(I-\mathrm{e}^{-r_B^m L})^M f\|_{L^2(U_j(B))} \leqslant C2^{-|j-i|v_0}2^{in}\|f\|_{L^2(U_i(B))}$$

for all integers $j, i \geqslant 0$ and for all $f \in L^2(X)$ with $\operatorname{supp} f \subset U_i(B)$.

Proof. We first show that the operator $\mathcal{G}_\delta(L)$ is bounded on $L^2(X)$ (see [5]). For every $R > 0$ and $\lambda > 0$, we recall that $S_R^\delta(\lambda) = (1-\lambda/R^m)_+^\delta$, and

$$F_R^\delta(\lambda) = c_\delta R\frac{\partial}{\partial R}S_R^{\delta+1}(\lambda)$$

with $c_{m\delta} = 1/(m(\delta+1))$. It follows from the spectral theory in [22] that for any $f \in L^2(X)$,

$$\begin{aligned}
(4.7) \qquad \|\mathcal{G}_\delta(L)f\|_{L^2(X)} &= \Big\{\int_0^\infty \langle\overline{F_R^\delta}(L)F_R^\delta(L)f,f\rangle\frac{\mathrm{d}R}{R}\Big\}^{1/2}\\
&= \Big\{\Big\langle\int_0^\infty |F_R^\delta|^2(L)\frac{\mathrm{d}R}{R}f,f\Big\rangle\Big\}^{1/2}\\
&= \Big\{\int_{\lambda^{1/2}}^\infty \Big(1-\frac{\lambda}{R^m}\Big)^{2\sigma}\frac{\lambda^2}{R^{2m+1}}\mathrm{d}R\Big\}^{1/2}\|f\|_{L^2(X)}\\
&= B_\sigma\|f\|_{L^2(X)},
\end{aligned}$$

where

$$B_\sigma^2 = \int_{\lambda^{1/m}}^{\infty} \Big(1 - \frac{\lambda}{R^m}\Big)^{2\sigma} \frac{\lambda^2}{R^{2m+1}} \,\mathrm{d}R = \int_1^\infty s^{-(2m+1)}(1 - s^{-m})^{2\sigma} \,\mathrm{d}s < \infty$$

and the integral above converges if $\sigma > -1/2$.

To complete the proof of this proposition, we need some preliminary results. We shall be working with an auxiliary nontrivial function φ with compact support. The choice of φ in the statements is not unique. Let $\varphi \in C_c^\infty(0, \infty)$ be a non-negative function satisfying

$$\text{(4.8)} \qquad \operatorname{supp} \varphi \subseteq \Big[\frac{1}{4}, 1\Big], \quad \sum_{l=-\infty}^{\infty} \varphi(2^{-l}\lambda) = 1 \quad \text{for any } \lambda > 0.$$

Since $\operatorname{supp} F_R^\delta(\lambda^m) \subset [0, R]$ and $\operatorname{supp} \varphi \subseteq [1/4, 1]$, we have that for every $\lambda > 0$,

$$F_R^\delta(\lambda^m) = \sum_{l=-\infty}^{\infty} \varphi(2^{-l}\lambda/R) F_R^\delta(\lambda^m) = \sum_{l=-\infty}^{1} \varphi(2^{-l}\lambda/R) F_R^\delta(\lambda^m).$$

This decomposition implies that the sequence $\sum_{l=-N}^{1} \varphi(2^{-l}\sqrt[m]{L}/R) F_R^\delta(L)$ converges strongly in $L^2(X)$ to $F_R^\delta(L)$ (see, for instance, Reed and Simon [19], Theorem VIII.5). For every $l \leqslant 1$ and $r > 0$, we set for $\lambda > 0$,

$$\text{(4.9)} \qquad F_{R,l,r}^\delta(\lambda) = \varphi(2^{-l}\lambda/R) F_R^\delta(\lambda^m)(1 - \mathrm{e}^{-(r\lambda)^m})^M.$$

We may write

$$\text{(4.10)} \qquad F_R^\delta(L)(I - \mathrm{e}^{-r^m L})^M f = \lim_{N\to\infty} \sum_{l=-N}^{1} F_{R,l,r}^\delta(\sqrt[m]{L}) f,$$

where the sequence converges strongly in $L^2(X)$.

For a ball B, we let r_B be the radius of B. For every $j = 1, 2, 3, \ldots$, we recall that $U_j(B) = 2^j B \setminus 2^{j-1} B$ is defined in (3.1). Then the following result holds.

Lemma 4.4. *Suppose that $F_{R,l,r_B}^\delta(\sqrt[m]{L})$ are defined as above. Let $\sigma > n(1/p - 1/2) - 1/q$ with some $q \in [2, \infty]$ and let $\max\{1/q, n(1/p - 1/2)\} < v < \sigma + 1/q$ and $v < mM$. Then there exists a constant $C = C(v, \sigma) > 0$ such that*

$$\text{(4.11)} \qquad \begin{aligned} &\|\chi_{U_j(B)} F_{R,l,r_B}^\delta(\sqrt[m]{L}) \chi_B\|_{L^2(X)\to L^2(X)} \\ &\quad \leqslant C 2^{ml} \mathrm{e}^{c|\tau|} \max\{1, (2^l R r_B)^{n/2}\} (2^l R 2^{j-1} r_B)^{-v} \min\{1, (2^l R r_B)^{mM}\} \end{aligned}$$

for all $j = 2, 3, \ldots$, *and*

$$\text{(4.12)}\quad \|\chi_{U_j(B)} F^{\delta}_{R,l,r_B}(\sqrt[m]{L})\chi_{U_i(B)}\|_{L^2(X)\to L^2(X)} \leqslant C2^{ml}\mathrm{e}^{c|\tau|}2^{in}\max\{1,(2^lRr_B)^{n/2}\}(2^lR2^{|j-i|}r_B)^{-v}\min\{1,(2^lRr_B)^{mM}\}$$

for all $|j - i| > 4$.

P r o o f of Lemma 4.4. Consider a ball $B \subset X$ with center $y \in X$ and radius r_B. Due to $\operatorname{supp} F^{\delta}_{R,l,r_B}(\lambda) \subset [2^lR/4, 2^lR]$, we use Lemma 4.2 to obtain that for any $l \in \mathbb{Z}$,

$$\|F^{\delta}_{R,l,r_B}(\sqrt[m]{L})\chi_{B(y,2^{-l}R^{-1})}\|_{L^2(X)\to L^2(X,(1+2^lRd(\cdot,y))^{2v}\,\mathrm{d}\mu)} \leqslant C\|F^{\delta}_{R,l,r_B}(2^lR\lambda)\|_{W^q_v}.$$

Let $j \geqslant 2$. For each $x \in U_j(B)$ we have, due to $d(x,y) \geqslant 2^{j-1}r_B$, the estimate $(1 + 2^lRd(x,y))^{2v} > (2^lR2^{j-1}r_B)^{2v}$. Hence we get

$$\begin{aligned}\text{(4.13)}\quad \|\chi_{U_j(B)} F^{\delta}_{R,l,r_B}(\sqrt[m]{L})\chi_{B(y,2^{-l}R^{-1})}\|_{L^2(X)\to L^2(X)} &\leqslant C(2^lR2^{j-1}r_B)^{-v} \\ &\quad\times \|\chi_{U_j(B)} F^{\delta}_{R,l,r_B}(\sqrt[m]{L})\chi_{B(y,2^{-l}R^{-1})}\|_{L^2(X)\to L^2(X,(1+2^lRd(\cdot,y))^{2v}\,\mathrm{d}\mu)} \\ &\leqslant C(2^lR2^{j-1}r_B)^{-v}\|F^{\delta}_{R,l,r_B}(2^lR\lambda)\|_{W^q_v}.\end{aligned}$$

Case 1. $r_B \leqslant 2^{-l}R^{-1}$. From (4.13) we have

$$\text{(4.14)}\quad \|\chi_{U_j(B)} F^{\delta}_{R,l,r_B}(\sqrt[m]{L})\chi_B\|_{L^2(X)\to L^2(X)} \leqslant C(2^lR2^{j-1}r_B)^{-v}\|F^{\delta}_{R,l,r_B}(2^lR\lambda)\|_{W^q_v}.$$

Case 2. $r_B > 2^{-l}R^{-1}$. In this case we follow Lemma 2.2 of [17] to select a finite number of points $y_1, \ldots, y_K \in B(y, r_B)$ such that

(i) $d(y_j, y_k) > 2^{-l-1}R^{-1}$ for all $j, k \in \{1, \ldots, K\}$ with $j \neq k$;

(ii) $B(y, r_B) \subset \bigcup_{m=1}^{K} B(y_m, 2^{-l}R^{-1})$;

(iii) $K \lesssim (2^lRr_B)^n$;

(iv) each $x \in B(y, r_B)$ is contained in at most M balls of $B(y_m, 2^{-l}R^{-1})$, where M depends only on the constants in (2.2).

Observe that for all $j \geqslant 2$ and $m \in \{1, 2, \ldots, K\}$,

$$U_j(B(y, r_B)) \subset \bigcup_{\eta=j-1}^{j+1} U_\eta(B(y_m, r_B)).$$

By (4.13),

$$\begin{aligned}(4.15)\qquad &\|\chi_{U_j(B(y,r_B))}F^\delta_{R,l,r_B}(\sqrt[m]{L})\chi_{B(y_m,2^{-l}R^{-1})}\|_{L^2(X)\to L^2(X)}\\ &\leqslant C\sum_{\eta=j-1}^{j+1}\|\chi_{U_\eta(B(y_m,r_B))}F^\delta_{R,l,r_B}(\sqrt[m]{L})\chi_{B(y_m,2^{-l}R^{-1})}\|_{L^2(X)\to L^2(X)}\\ &\leqslant C\sum_{\eta=j-1}^{j+1}(2^lR2^{\eta-1}r_B)^{-v}\|F^\delta_{R,l,r_B}(2^lR\lambda)\|_{W^q_v}\\ &\leqslant C(2^lR2^{j-1}r_B)^{-v}\|F^\delta_{R,l,r_B}(2^lR\lambda)\|_{W^q_v}.\end{aligned}$$

Consider $g,h\in L^2(X)$ with $\operatorname{supp} g\subset B$, $\|g\|_{L^2(X)}=1$ and $\operatorname{supp} h\subset U_j(B)$, $\|h\|_{L^2(X)}=1$. From (4.15) we obtain that for every $j\geqslant 2$,

$$\begin{aligned}&|\langle h,\chi_{U_j(B(y,r_B))}F^\delta_{R,l,r_B}(\sqrt[m]{L})\chi_{B(y,r_B)}g\rangle|^2\\ &\qquad\leqslant\|\chi_{B(y,r_B)}F^\delta_{R,l,r_B}(\sqrt[m]{L})^*\chi_{U_j(B(y,r_B))}h\|^2_{L^2(X)}\|g\|^2_{L^2(X)}\\ &\qquad\leqslant\sum_{m=1}^{K}\|\chi_{U_j(B(y,r_B))}F^\delta_{R,l,r_B}(\sqrt[m]{L})\chi_{B(y_m,2^{-l}R^{-1})}\|^2_{L^2(X)\to L^2(X)}\\ &\qquad\leqslant\sum_{m=1}^{K}C(2^lR2^{j-1}r_B)^{-2v}\|F^\delta_{R,l,r_B}(2^lR\lambda)\|^2_{W^q_v}.\end{aligned}$$

Taking the supremum over all such g,h and recalling $\sqrt{K}\leqslant C(2^lRr_B)^{n/2}$, we deduce

$$\begin{aligned}(4.16)\qquad &\|\chi_{U_j(B(y,r_B))}F^\delta_{R,l,r_B}(\sqrt[m]{L})\chi_{B(y,r_B)}\|_{L^2(X)\to L^2(X)}\\ &\qquad\leqslant C(2^lRr_B)^{n/2}(2^lR2^{j-1}r_B)^{-v}\|F^\delta_{R,l,r_B}(2^lR\lambda)\|_{W^q_v}.\end{aligned}$$

Now for any Sobolev space $W^q_v(\mathbb{R})$, if k is an integer greater than v, then

$$\begin{aligned}(4.17)\qquad &\|F^\delta_{R,l,r_B}(2^lR\lambda)\|_{W^q_v}\\ &\leqslant C\|(2^l\lambda)^m\varphi(\lambda)(1-2^{ml}\lambda^m)^\delta_+\|_{W^q_v}\|(1-\mathrm{e}^{-(2^lRr_B)^m\lambda^m})^M\|_{C^k[1/4,1]}\\ &\leqslant C2^{ml}\|\varphi(\lambda)(1-2^{ml}\lambda^m)^\delta_+\|_{W^q_v}\min\{1,(2^lRr_B)^{mM}\}.\end{aligned}$$

It is known that for $\sigma>-1/2$, $0<v<\sigma+1/q$

$$(4.18)\qquad \sup_{l\in\mathbb{Z}\,:\,l\leqslant 1}\|\varphi(\lambda)(1-2^{ml}\lambda^m)^\delta_+\|_{W^q_v(\mathbb{R})}\leqslant C_\sigma\mathrm{e}^{c|\tau|}$$

see Lemma 2.2 of [5]. This, in combination with (4.14), (4.16) and (4.17), yields

$$\begin{aligned}&\|\chi_{U_j(B)}F^\delta_{R,l,r_B}(\sqrt[m]{L})\chi_B\|_{L^2(X)\to L^2(X)}\\ &\qquad\leqslant C2^{ml}\mathrm{e}^{c|\tau|}\max\{1,(2^lRr_B)^{n/2}\}(2^lR2^jr_B)^{-v}\min\{1,(2^lRr_B)^{mM}\}.\end{aligned}$$

Then the proof of (4.11) is complete.

Next we have to check (4.12). Since L is a non-negative self-adjoint operator, one can swap i and j in the term on the left-hand side of (4.12). Hence, it will be enough to show the assertion for every $i, j \in \mathbb{N}$ with $j - i > 4$. By applying [2] Lemma 3.4, (4.11), and the doubling property, we get

$$\begin{aligned}
&\|\chi_{U_j(B)} F^{\delta}_{R,l,r_B}(\sqrt[m]{L})\chi_{U_i(B)}\|_{L^2(X)\to L^2(X)} \\
&\quad\leqslant C\int_X \|\chi_{U_j(B(y,r_B))} F^{\delta}_{R,l,r_B}(\sqrt[m]{L})\chi_{B(z,r_B)}\|_{L^2(X)\to L^2(X)} \\
&\qquad \times \|\chi_{B(z,r_B)}\chi_{U_i(B(y,r_B))}\|_{L^2(X)\to L^2(X)} \frac{\mathrm{d}\mu(z)}{V(z,r_B)} \\
&\quad\leqslant C\int_{B(y,2^{i+1}r_B)\setminus B(y,2^{i-2}r_B)} \\
&\qquad \sum_{\eta=j-i-3}^{\eta=j+i+1} \|\chi_{U_\eta(B(z,r_B))} F^{\delta}_{R,l,r_B}(\sqrt[m]{L})\chi_{B(z,r_B)}\|_{L^2(X)\to L^2(X)} \frac{\mathrm{d}\mu(z)}{V(z,r_B)} \\
&\quad\leqslant C\int_{B(y,2^{i+1}r_B)} \sum_{\eta=j-i-3}^{\eta=j+i+1} C(F)2^{-\eta v}2^{(i+1)n}\frac{\mathrm{d}\mu(z)}{V(z,2^{i+1}r_B)},
\end{aligned}$$

where $C(F) = C2^{ml}\mathrm{e}^{c|\tau|}\max\{1,(2^l R r_B)^{n/2}\}(2^l R r_B)^{-v}\min\{1,(2^l R r_B)^{mM}\}$. In the remaining steps we covered $U_j(B(y,r_B))$ by dyadic annuli around the point z with the same radius r_B. With help of

$$\sum_{\eta=j-i-3}^{\eta=j+i+1} 2^{-\eta v} = 2^{3v}2^{-(j-i)v}\sum_{\eta=0}^{\eta=2i+4} 2^{-\eta v} \leqslant C2^{-(j-i)v},$$

we finish our estimates as follows:

$$\begin{aligned}
(4.19)\quad &\|\chi_{U_j(B)} F^{\delta}_{R,l,r_B}(\sqrt[m]{L})\chi_{U_i(B)}\|_{L^2(X)\to L^2(X)} \\
&\quad\leqslant C(F)2^{-(j-i)v}2^{(i+1)n}\int_{B(y,2^{i+1}r_B)}\frac{\mathrm{d}\mu(z)}{V(z,2^{i+1}r_B)} \\
&\quad\leqslant C2^{ml}\mathrm{e}^{c|\tau|}2^{in}\max\{1,(2^l R r_B)^{n/2}\}(2^l R 2^{(j-i)} r_B)^{-v}\min\{1,(2^l R r_B)^{mM}\}.
\end{aligned}$$

Thus, the proof of Lemma 4.4 is completed. □

Back to the p r o o f of Proposition 4.3. Let B be a ball with the radius r_B of B and all f supported in B. Fix v_0 in Lemma 4.4. For $j = 0, 1$, we use the L^2 boundedness of $\mathcal{G}_\delta(L)(I - \mathrm{e}^{-r_B^m L})^M$ to get that

$$(4.20)\qquad \|\mathcal{G}_\delta(L)(I-\mathrm{e}^{-r_B^m L})^M f\|_{L^2(U_j(B))} \leqslant C\|f\|_{L^2(B)}.$$

For $j \geqslant 2$, from the definition of $\mathcal{G}_\delta(L)$ and (4.9), we use the Minkowski inequality to obtain that

$$\|\mathcal{G}_\delta(L)(I-\mathrm{e}^{-r_B^m L})^M f\|_{L^2(U_j(B))} \leqslant \sum_{l\leqslant 1}\left(\int_0^\infty\int_{U_j(B)} |F^\delta_{R,l,r_B}(\sqrt[m]{L})f|^2\,\mathrm{d}\mu\,\frac{\mathrm{d}R}{R}\right)^{1/2}.$$

One may write

$$\begin{aligned}&\int_0^\infty\int_{U_j(B)} |F^\delta_{R,l,r_B}(\sqrt[m]{L})f|^2\,\mathrm{d}\mu\,\frac{\mathrm{d}R}{R}\\&\qquad=\left(\int_0^{2^{-l}r_B^{-1}}+\int_{2^{-l}r_B^{-1}}^\infty\right)\int_{U_j(B)}|F^\delta_{R,l,r_B}(\sqrt[m]{L})f|^2\,\mathrm{d}\mu(x)\frac{\mathrm{d}R}{R}=I+II.\end{aligned}$$

For the term I, we note that $0<R<2^{-l}r_B^{-1}$, and then $\max\{1,(r_B2^lR)^n\}=1$ $\min\{1,(2^lRr_B)^{2mM}\}=(2^lRr_B)^{2mM}$. In view of the inequality (4.11), we have

$$\begin{aligned}I&\leqslant C\mathrm{e}^{c|\tau|}\|f\|^2_{L^2(B)}\int_0^{2^{-l}r_B^{-1}} 2^{2ml}(2^jr_B2^lR)^{-2v_0}(2^lRr_B)^{2mM}\frac{\mathrm{d}R}{R}\\&\leqslant C\mathrm{e}^{c|\tau|}2^{2ml}2^{-2jv_0}\|f\|^2_{L^2(B)}.\end{aligned}$$

Consider the term II. Since $r_B2^lR>1$, we have $(r_B2^lR)^n<(r_B2^lR)^{n(2/p-1)}$. In view of the inequality (4.11) again, one obtains

$$\begin{aligned}II&\leqslant C\mathrm{e}^{c|\tau|}\|f\|^2_{L^2(B)}\int_{2^{-l}r_B^{-1}}^\infty 2^{2ml}(2^jr_B2^lR)^{-2v_0}(2^lRr_B)^{n}\frac{\mathrm{d}R}{R}\\&\leqslant C\mathrm{e}^{c|\tau|}\|f\|^2_{L^2(B)}\int_{2^{-l}r_B^{-1}}^\infty 2^{2ml}(2^jr_B2^lR)^{-2v_0}(2^lRr_B)^{n(2/p-1)}\frac{\mathrm{d}R}{R}\\&\leqslant C\mathrm{e}^{c|\tau|}2^{2ml}2^{-2jv_0}\|f\|^2_{L^2(B)}.\end{aligned}$$

Therefore, a simple calculation shows that for every $j\geqslant 2$,

$$\begin{aligned}\left(\int_{U_j(B)}|\mathcal{G}_\delta(L)(I-\mathrm{e}^{-r_B^mL})^Mf|^2\,\mathrm{d}\mu\right)^{1/2}&\leqslant C\mathrm{e}^{c|\tau|}\|f\|_{L^2(B)}2^{-jv_0}\sum_{l\leqslant 1}2^{ml}\\&=C2^{-jv_0}\|f\|_{L^2(B)}.\end{aligned}\tag{4.21}$$

Then (α) of Proposition 4.3 is proved.

In the following, we will check (β). Let f be supported in $U_i(B)$. For $|j-i|\leqslant 4$, by using the L^2 boundedness of $\mathcal{G}_\delta(L)(I-\mathrm{e}^{-r_B^mL})^M$, we get

$$\|\mathcal{G}_\delta(L)(I-\mathrm{e}^{-r_B^mL})^Mf\|_{L^2(U_j(B))}\leqslant C\|f\|_{L^2(U_i(B))}.\tag{4.22}$$

For $|j-i|>4$, we also use the Minkowski inequality to obtain that

$$\|\mathcal{G}_\delta(L)(I-\mathrm{e}^{-r_B^m L})^M f\|_{L^2(U_j(B))} \leqslant \sum_{l\leqslant 1}\biggl(\int_0^\infty \int_{U_j(B)} |F^\delta_{R,l,r_B}(\sqrt[m]{L})f|^2 \,\mathrm{d}\mu \frac{\mathrm{d}R}{R}\biggr)^{1/2}. \tag{4.23}$$

With help of (4.12), by using an argument in a way similar to the proof of (α), we get

$$\begin{aligned}\int_0^\infty\int_{U_j(B)} |F^\delta_{R,l,r_B}(\sqrt[m]{L})f|^2 \,\mathrm{d}\mu \,\frac{\mathrm{d}R}{R} &= \biggl(\int_0^{2^{-l}r_B^{-1}} + \int_{2^{-l}r_B^{-1}}^\infty\biggr)\int_{U_j(B)} |F^\delta_{R,l,r_B}(\sqrt[m]{L})f|^2 \,\mathrm{d}\mu(x)\,\frac{\mathrm{d}R}{R}\\ &\leqslant C\mathrm{e}^{c|\tau|}2^{2ml}2^{-2|j-i|v_0}2^{2in}\|f\|^2_{L^2(U_i(B))}.\end{aligned} \tag{4.24}$$

Inserting (4.24) into (4.23) yields that for every $|j-i|>4$,

$$\begin{aligned}\biggl(\int_{U_j(B)} |\mathcal{G}_\delta(L)(I-\mathrm{e}^{-r_B^m L})^M f|^2 \,\mathrm{d}\mu\biggr)^{1/2} &\leqslant C\mathrm{e}^{c|\tau|}2^{in}\|f\|_{L^2(U_i(B))}2^{-|j-i|v_0}\sum_{l\leqslant 1}2^{ml}\\ &= C2^{-|j-i|v_0}2^{in}\|f\|_{L^2(U_i(B))}.\end{aligned}$$

Then (β) of Proposition 4.3 is proved. The proof is complete. □

P r o o f of Theorem 1.2. We apply Proposition 4.1 to show that for every $p\in(0,1]$ and $\sigma>n(1/p-1/2)-1/q$ there exists a constant $C=C(p)>0$ such that for every $f\in H^p_L(X)$,

$$\|\mathcal{G}_\sigma(L)f\|_{L^p(X)} \leqslant C\|f\|_{H^p_L(X)}. \tag{4.25}$$

So we only need to check (4.1) in Proposition 4.1. Let $\varepsilon\in(n+n(1/p-1/2),n+v_0)$ be fixed, define $\tilde\varepsilon=\varepsilon-n$, where v_0 is the constant given in Proposition 4.3. Let a be an (p,m,M,ε)-molecule. First, we have that for $j=0,1,2$,

$$\|\mathcal{G}_\sigma(L)a\|_{L^2(U_j(B))} \leqslant \|\mathcal{G}_\sigma(L)a\|_{L^2(X)} \leqslant C\|a\|_{L^2(X)} \leqslant CV(B)^{1/2-1/p}.$$

Now assume that $j\geqslant 3$. By the spectral theorem, we write

$$\begin{aligned}I &= m\biggl(r_B^{-m}\int_{r_B}^{\sqrt[m]{2}r_B} s^{m-1}\,\mathrm{d}s\biggr)\cdot I\\ &= mr_B^{-m}\int_{r_B}^{\sqrt[m]{2}r_B} s^{m-1}(I-\mathrm{e}^{-s^m L})^M\,\mathrm{d}s\\ &\quad+\sum_{u=1}^M muC_{u,M}r_B^{-m}\int_{r_B}^{\sqrt[m]{2}r_B} s^{m-1}\mathrm{e}^{-us^m L}\,\mathrm{d}s,\end{aligned} \tag{4.26}$$

where $C_{u,M} = (-1)^{u+1}/uC_M^u$. However, $\partial_s \mathrm{e}^{-us^m L} = -mus^{m-1}L\mathrm{e}^{-us^m L}$ and therefore,

$$(4.27)\quad muL\int_{r_B}^{\sqrt[m]{2}r_B} s^{m-1}\mathrm{e}^{-us^m L}\,\mathrm{d}s = \mathrm{e}^{-ur_B^m L} - \mathrm{e}^{-2ur_B^m L} = \mathrm{e}^{-ur_B^m L}(I-\mathrm{e}^{-ur_B^m L}) = \mathrm{e}^{-ur_B^m L}(I-\mathrm{e}^{-r_B^m L})\sum_{i=0}^{u-1}\mathrm{e}^{-ir_B^m L}.$$

Set $P_{m,M,r_B}(L) = r_B^{-m}\int_{r_B}^{\sqrt[m]{2}r_B} s^{m-1}(I-\mathrm{e}^{-s^m L})^M\,\mathrm{d}s$. Inserting the equation (4.27) into (4.26), we obtain the formula

$$I = mP_{m,M,r_B}(L) + \sum_{u=1}^{M} C_{u,M} r_B^{-m} L^{-1}(I-\mathrm{e}^{-r_B^m L})\sum_{i=u}^{2u-1}\mathrm{e}^{-ir_B^m L}.$$

Calculating I^M by means of the binomial formula leads to

$$\begin{aligned} I &= m^M(P_{m,M,r_B}(L))^M \\ &+ \sum_{l=1}^{M-1} r_B^{-ml}L^{-l}(I-\mathrm{e}^{-r_B^m L})^l(P_{m,M,r_B}(L))^{M-l}\sum_{u=1}^{(2M-1)l} C(l,u,M)\mathrm{e}^{-ur_B^m L} \\ &+ \sum_{u=1}^{(2M-1)M} C(M,u,M) r_B^{-mM}L^{-M}(I-\mathrm{e}^{-r_B^m L})^M\mathrm{e}^{-ur_B^m L} \end{aligned}$$

for some constants $C(l,u,M)\in\mathbb{R}$, $l=1,2,\ldots,M$. Recall that $F_R^\delta(\lambda) = c_\delta R\times(\partial/\partial R)S_R^{\delta+1}(\lambda)$; applying the above identity, we note that $a = L^M b$ to obtain

$$\begin{aligned} &F_R^\delta(L)a(x) \\ &= m^M r_B^{-m}\int_{r_B}^{\sqrt[m]{2}r_B} s^{m-1}P_{m,M,r_B}^{M-1}(L)F_R^\delta(L)(I-\mathrm{e}^{-s^m L})^M a(x)\,\mathrm{d}s \\ &+ \sum_{l=1}^{M-1}\Bigg\{\sum_{u=1}^{(2M-1)l} C(l,u,M) r_B^{-m(M+1)} \\ &\times\int_{r_B}^{\sqrt[m]{2}r_B} s^{m-1}(r_B^m L)^{M-l}(I-\mathrm{e}^{-r_B^m L})^l P_{m,M,r_B}^{M-l-1}(L)\mathrm{e}^{-ur_B^m L}F_R^\delta(L)(I-\mathrm{e}^{-s^m L})^M b(x)\,\mathrm{d}s\Bigg\} \\ &+ \sum_{u=1}^{(2M-1)M} C(M,u,M) r_B^{-mM}\mathrm{e}^{-ur_B^m L}F_R^\delta(L)(I-\mathrm{e}^{-r_B^m L})^M b(x). \end{aligned}$$

Putting this into the definition of $\mathcal{G}_\delta(L)$ in (1.2), we have

$$(4.28)\qquad \mathcal{G}_\delta(L)a(x) = \left(\int_0^\infty |F_R^\delta(L)a(x)|^2\frac{\mathrm{d}R}{R}\right)^{1/2} \leqslant \sum_{l=0}^{M} G_{l,M,r_B}^m(x),$$

where

$$G^m_{0,M,r_B}(x) = m^M r_B^{-m} \int_{r_B}^{\sqrt[m]{2} r_B} s^{m-1} \times \left(\int_0^\infty |P^{M-1}_{m,M,r_B}(L) F^\delta_R(L)(I - \mathrm{e}^{-s^m L})^M a(x)|^2 \frac{\mathrm{d}R}{R} \right)^{1/2} \mathrm{d}s,$$

and for $l = 1, 2, \dots, M-1$,

$$G^m_{l,M,r_B}(x) = \sum_{u=1}^{(2M-1)l} C(l,u,M) r_B^{-m(M+1)} \int_{r_B}^{\sqrt[m]{2} r_B} s^{m-1} \left(\int_0^\infty |(r_B^m L)^{M-l} \mathrm{e}^{-u r_B^m L} \times (I - \mathrm{e}^{-r_B^m L})^l P^{M-l-1}_{m,M,r_B}(L) F^\delta_R(L)(I - \mathrm{e}^{-s^m L})^M b(x)|^2 \frac{\mathrm{d}R}{R} \right)^{1/2} \mathrm{d}s,$$

and

$$G^m_{M,M,r_B}(x) = \sum_{u=1}^{(2M-1)M} C(M,u,M) r_B^{-mM} \times \left(\int_0^\infty |\mathrm{e}^{-u r_B^m L} F^\delta_R(L)(I - \mathrm{e}^{-r_B^m L})^M (b)(x)|^2 \frac{\mathrm{d}R}{R} \right)^{1/2}.$$

Now we shall estimate $\{G_{l,M,r_B}\}_{l=0}^M$ by examining l in three different cases.

Subcase 1. $l = 0$. It follows from condition (1.4) that the operator $P^{M-1}_{m,M,r_B}(L)$ satisfies L^2 off-diagonal estimates, that is, there exist constants $c, C > 0$ such that for every $i, j = 0, 1, 2, \dots$

$$\begin{aligned} \|P^{M-1}_{m,M,r_B}(L) f\|_{L^2(U_j(B))} &\leqslant C \exp(-\mathrm{dist}(U_j(B), U_i(B))^{m/(m-1)} / c r_B^{m/(m-1)}) \|f\|_{L^2(U_i(B))} \\ &\leqslant C \mathrm{e}^{-c 2^{|j-i|}} \|f\|_{L^2(U_i(B))}. \end{aligned}$$

Hence, one can write

$$\begin{aligned} \|G^m_{0,M,r_B}\|_{L^2(U_j(B))} &\leqslant C r_B^{-m} \sum_{i=0}^\infty \int_{r_B}^{\sqrt[m]{2} r_B} s^{m-1} \\ &\quad \times \left(\int_0^\infty \int_{U_j(B)} |P^{M-1}_{m,M,r_B}(L)([F^\delta_R(L)(I - \mathrm{e}^{-s^m L})^M a] \chi_{U_i(B)})(x)|^2 \,\mathrm{d}\mu(x) \frac{\mathrm{d}R}{R} \right)^{1/2} \mathrm{d}s \\ &\leqslant C r_B^{-m} \sum_{i=0}^\infty \mathrm{e}^{-c 2^{|j-i|}} \int_{r_B}^{\sqrt[m]{2} r_B} s^{m-1} \left(\int_0^\infty \|F^\delta_R(L)(I - \mathrm{e}^{-s^m L})^M a\|^2_{L^2(U_i(B))} \frac{\mathrm{d}R}{R} \right)^{1/2} \mathrm{d}s \\ &\leqslant C r_B^{-m} \sum_{i=0}^\infty \mathrm{e}^{-c 2^{|j-i|}} \int_{r_B}^{\sqrt[m]{2} r_B} s^{m-1} \|\mathcal{G}_\delta(L)(I - \mathrm{e}^{-s^m L})^M a\|_{L^2(U_i(B))} \,\mathrm{d}s. \end{aligned}$$

In order to use Proposition 4.3, we note that for every $s \in [r_B, \sqrt[m]{2}r_B]$, $U_0(B) = B \subset B(x_B, s)$ and $U_i(B) \subset U_{i-1}(B(x_B, s)) \cup U_i(B(x_B, s))$ for $i \geqslant 1$. By the Minkowski inequality, for every $s \in [r_B, \sqrt[m]{2}r_B]$,

$$\begin{aligned}(4.29)\qquad &\|\mathcal{G}_\delta(L)(I - \mathrm{e}^{-s^m L})^M a\|_{L^2(U_i(B)}\\ &\leqslant \|\mathcal{G}_\delta(L)(I - \mathrm{e}^{-s^m L})^M (a\chi_{B(x_B,s)})\|_{L^2(U_i(B))}\\ &\quad + \sum_{\eta=1}^{\infty} \|\mathcal{G}_\delta(L)(I - \mathrm{e}^{-s^m L})^M (a\chi_{U_\eta(B(x_B,s))})\|_{L^2(U_i(B))}.\end{aligned}$$

Due to (α) in Proposition 4.3,

$$\begin{aligned}(4.30)\qquad &\|\mathcal{G}_\delta(L)(I - \mathrm{e}^{-s^m L})^M (a\chi_{B(x_B,s)})\|_{L^2(U_i(B))}\\ &\qquad\leqslant C2^{-iv_0}\|a\|_{L^2(B)} \leqslant C2^{-iv_0}V(B)^{1/2-1/p}.\end{aligned}$$

The series in (4.29) can be estimated with help of (β) in Proposition 4.3,

$$\begin{aligned}(4.31)\qquad &\sum_{\eta=1}^{\infty} \|\mathcal{G}_\delta(L)(I - \mathrm{e}^{-s^m L})^M (a\chi_{U_\eta(B(x_B,s))})\|_{L^2(U_i(B))}\\ &\leqslant C\sum_{\eta=1}^{\infty} 2^{-|\eta-i|v_0}2^{\eta n}\|a\|_{L^2(U_\eta(B))}\\ &\leqslant C\sum_{\eta=1}^{\infty} 2^{-|\eta-i|v_0}2^{\eta n}2^{-\eta\varepsilon}V(2^\eta B)^{1/2-1/p}\\ &\leqslant C2^{-i(\varepsilon-n)}V(B)^{1/2-1/p};\end{aligned}$$

in the last step we used the fact that

$$\begin{aligned}\sum_{\eta=1}^{\infty} 2^{-|\eta-i|v_0}2^{-\eta(\varepsilon-n)} &= 2^{-i(\varepsilon-n)}\left(\sum_{m=-\infty}^{0} 2^{m(\varepsilon-n)}2^{-|m|v_0} + \sum_{m=1}^{i-1} 2^{m(\varepsilon-n)}2^{-mv_0}\right)\\ &\leqslant C2^{-i(\varepsilon-n)}\left(\sum_{m=-\infty}^{0} 2^{-|m|v_0} + \sum_{m=1}^{\infty} 2^{m(\varepsilon-n)}2^{-mv_0}\right)\\ &\leqslant C2^{-i(\varepsilon-n)}.\end{aligned}$$

Recall that $\tilde{\varepsilon} = \varepsilon - n < v_0$. In view of the inequalities (4.30) and (4.31), we have the estimate of (4.29)

$$\|\mathcal{G}_\delta(L)(I - \mathrm{e}^{-s^m L})^M a\|_{L^2(U_i(B)} \leqslant C2^{-i\tilde{\varepsilon}}V(B)^{1/2-1/p},$$

which yields that

$$\text{(4.32)}\quad \|G_{0,M,r_B}\|_{L^2(U_j(B))} \leqslant C\sum_{i=0}^{\infty} \mathrm{e}^{-c2^{|j-i|}} 2^{-i\tilde{\varepsilon}} V(B)^{1/2-1/p} \leqslant C2^{-j\tilde{\varepsilon}} V(B)^{1/2-1/p}.$$

Subcase 2. $l = M$. In this case we may write

$$\|G^m_{M,M,r_B}\|_{L^2(U_j(B))} \leqslant Cr_B^{-mM} \sum_{u=1}^{(2M-1)M} \sum_{i=0}^{\infty} \biggl(\int_0^\infty \int_{U_j(B)} |\mathrm{e}^{-ur_B^m L}([F_R^\delta(L)(I - \mathrm{e}^{-r_B^m L})^M b]\chi_{U_i(B)})(x)|^2 \,\mathrm{d}\mu(x) \frac{\mathrm{d}R}{R}\biggr)^{1/2}.$$

It follows from the condition (1.4) that the operators $\{\mathrm{e}^{-ur_B^m L}\}_{u=1}^{(2M-1)M}$ satisfy L^2 off-diagonal estimate, and then

$$\begin{aligned}\text{(4.33)}\quad \|G^m_{M,M,r_B}\|_{L^2(U_j(B))} &\leqslant Cr_B^{-mM} \sum_{i=0}^{\infty} \mathrm{e}^{-c2^{|j-i|}} \|\mathcal{G}_\delta(L)(I - \mathrm{e}^{-r_B^m L})^M b\|_{L^2(U_i(B))} \\ &\leqslant Cr_B^{-mM} \sum_{i=0}^{\infty} \mathrm{e}^{-c2^{|j-i|}} r_B^{mM} 2^{-i\tilde{\varepsilon}} V(B)^{1/2-1/p} \\ &\leqslant C2^{-j\tilde{\varepsilon}} V(B)^{1/2-1/p}.\end{aligned}$$

Subcase 3. $l = 1, 2, \ldots, M-1$. In these cases, one has

$$\begin{aligned}&\|G^m_{l,M,r_B}\|_{L^2(U_j(B))} \\ &\leqslant \sum_{u=1}^{(2M-1)l} C(l,u,M) r_B^{-m(M+1)} \sum_{i=0}^{\infty} \int_{r_B}^{\sqrt[m]{2}r_B} s^{m-1} \biggl(\int_0^\infty \int_{U_j(B)} |(r_B^m L)^{M-l} \mathrm{e}^{-ur_B^m L} \\ &\quad \times (I - \mathrm{e}^{-r_B^m L})^l P^{M-l-1}_{m,M,r_B}(L)([F_R^\delta(L)(I - \mathrm{e}^{-s^m L})^M b]\chi_{U_i(B)})(x)|^2 \,\mathrm{d}\mu(x) \frac{\mathrm{d}R}{R}\biggr)^{1/2} \mathrm{d}s.\end{aligned}$$

By Proposition 2.2, the operator family $\{(tL)^{M-l}\mathrm{e}^{-utL}\}_{t>0}$ satisfies L^2 off-diagonal estimates, and it is easy to prove that L^2 off-diagonal estimates also hold for $\{(tL)^{M-l}\mathrm{e}^{-utL}(I - \mathrm{e}^{-tL})^l\}_{t>0}$. So using arguments similar to Subcase 1, we conclude that

$$\|G^m_{l,M,r_B}\|_{L^2(U_j(B))} \leqslant C2^{-j\tilde{\varepsilon}} V(B)^{1/2-1/p}.$$

This, in combination with estimates (4.32) and (4.33), gives the desired estimate (4.1) for $T = \mathcal{G}_\delta(L)$. The proof of Theorem 1.2 is complete. □

5. Boundedness of Bochner-Riesz means $S_R^\delta(L)$ on $H_L^p(X)$

In this section we prove a result for Bochner-Riesz means $S_R^\delta(L)$. First, we will state a Hörmander type spectral multiplier result on $H_L^p(X)$. As a corollary, we get the boundedness of Bochner-Riesz means $S_R^\delta(L)$ on $H_L^p(X)$ for $0 < p \leqslant 1$, which generalizes the results from [4] for operators L satisfying the Davies-Gaffney estimates (of order m).

Theorem 5.1. *Let L be a non-negative self-adjoint operator which satisfies the Davies-Gaffney estimate* (1.4) *and the Plancherel estimate* (1.6) *for some $q \in [2,\infty]$. Suppose that $0 < p \leqslant 1$. If $v > \max\{n(1/p - 1/2), 1/q\}$ and $F\colon [0,\infty) \to \mathbb{C}$ is a bounded Borel function with*

$$\sup_{l\in\mathbb{Z}} \|\varphi F(2^l\cdot)\|_{W_v^q} < \infty,$$

where φ is the function given in (4.8), *then there exists a constant $C > 0$ such that for all $f \in H_L^p(X)$*

$$\|F(L)f\|_{H_L^p(X)} \leqslant C\Big(\sup_{l\in\mathbb{Z}} \|\varphi F(2^l\cdot)\|_{W_v^q} + |F(0)|\Big)\|f\|_{H_L^p(X)}.$$

The following proposition plays an important role in proving Theorem 5.1.

Proposition 5.2. *Let L be a non-negative self-adjoint operator on $L^2(X)$ satisfying the Davies-Gaffney estimate* (1.4). *Let F be a bounded Borel function. Suppose that $0 < p \leqslant 1$ and $M \in \mathbb{N}$, $M > \frac{1}{2}n(2-p)/mp$. Assume that there exist constants $M_0 > n(1/p - 1/2)$ and $C > 0$ such that for every $j = 2, 3\ldots,$*

$$\|F(L)(1 - \mathrm{e}^{-r_B^m L})^M f\|_{L^2(U_j(B))} \leqslant C2^{-jM_0}\|f\|_{L^2(B)}$$

for any ball B with radius r_B and for all $f \in L^2(X)$ with $\operatorname{supp} f \subset B$. Then the operator $F(L)$ extends to a bounded operator on $H_L^p(X)$. More precisely, there exists a constant $C > 0$ such that for all $f \in H_L^p(X)$

$$\|F(L)f\|_{H_L^p(X)} \leqslant C\|f\|_{H_L^p(X)}.$$

P r o o f. The proof is similar to that of Theorem 3.1 [11] or Theorem 4.6 [17]. We omit the details here. □

P r o o f of Theorem 5.1. The proof follows from a slight modification of an argument as in [17], Theorem 4.2. In fact, we can get the desired result by using Proposition 5.2 and Lemma 4.2. We omit the details here. □

A standard application of spectral multiplier theorems is Bochner-Riesz means. Let us recall that Bochner-Riesz means of order δ for a non-negative self-adjoint operator L is defined by the formula

$$S_R^\delta(L) = \Big(I - \frac{L}{R^m}\Big)_+^\delta, \quad R > 0.$$

If we set $F(\lambda) = (1-\lambda^m)_+^\delta$ in Theorem 5.1, then $F \in W_\alpha^q$ if and only if $\delta > \alpha - 1/q$. So we have the following corollary.

Corollary 5.3. *Let L be a non-negative self-adjoint operator on $L^2(X)$ satisfying the Davies-Gaffney estimate* (1.4) *and the Plancherel type condition* (1.6) *for some $q \in [2,\infty]$. If $p \in (0,1]$, then for all $\delta > \max\{n(1/p-1/2)-1/q, 0\}$ we have*

$$\Big\| \Big(I - \frac{L}{R^m}\Big)_+^\delta \Big\|_{H_L^p(X)\to H_L^p(X)} \leqslant C$$

uniformly in $R > 0$.

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