Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 65 (2015), No. 1, 119–134

Persistent URL: `http://dml.cz/dmlcz/144216`

Czechoslovak Mathematical Journal, 65 (140) (2015), 119–134

THE L^p-HELMHOLTZ PROJECTION IN FINITE CYLINDERS

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(Received September 12, 2013)

Abstract. In this article we prove for $1 < p < \infty$ the existence of the L^p-Helmholtz projection in finite cylinders Ω. More precisely, Ω is considered to be given as the Cartesian product of a cube and a bounded domain V having C^1-boundary. Adapting an approach of Farwig (2003), operator-valued Fourier series are used to solve a related partial periodic weak Neumann problem. By reflection techniques the weak Neumann problem in Ω is solved, which implies existence and a representation of the L^p-Helmholtz projection as a Fourier multiplier operator.

Keywords: Helmholtz projection; Helmholtz decomposition; weak Neumann problem; periodic boundary conditions; finite cylinder; cylindrical space domain; L^p-space; operator-valued Fourier multiplier; $\mathcal{R}$-boundedness; reflection technique; fluid dynamics

MSC 2010: 35Q30, 35J20, 35J25, 42B15, 46E40

1. Introduction

Let $n_1, n_2 \in \mathbb{N}_0$ be such that $n_1 + n_2 \geqslant 2$. Given a bounded domain $V \subset \mathbb{R}^{n_2}$ with C^1-boundary, we consider the domain $\Omega := (0, \pi)^{n_1} \times V$. The aim of this article is to prove the existence of the L^p-Helmholtz projection $\mathbb{P}_p \in \mathcal{L}(L^p(\Omega))$ for $1 < p < \infty$. It is well-known that $\mathbb{P}_2$ exists for any domain Ω in the Hilbert space case $p = 2$ and that $\mathbb{P}_p$ exists for the entire range $1 < p < \infty$ if Ω is a bounded C^1-domain, a half space or the whole space, for instance see [5]. However, $\mathbb{P}_p$ fails to exist in general. In particular, bounded domains with corners and some $1 < p < \infty$ are known such that $\mathbb{P}_p$ does not exist (see e.g. [14], Remark 1.3, and the references given there). Finite cylinders and cubes as considered in this paper may as well be treated with refined techniques that are successfully applied to bounded C^1-domains. However, the multiplier method we pursue avoids any cut-off technique and seems to be more suitable, since known results for the domain V can be transfered to Ω efficiently.

As the existence of $\mathbb{P}_p$ is equivalent to unique solvability of a corresponding weak Neumann problem in Ω, we subsequently focus on the latter. However, we investigate a partial periodic weak Neumann problem in the larger domain $\widetilde{\Omega} := (0, 2\pi)^{n_1} \times V$ first. More precisely, periodic boundary conditions with respect to $\partial(0, 2\pi)^{n_1} \times V$ and Neumann boundary conditions with respect to $(0, 2\pi)^{n_1} \times \partial V$ are imposed. Having succeeded in establishing unique solvability in a weak sense here, a reflection argument is involved to deduce unique solvability for the weak Neumann problem in Ω and thus the assertion of the main theorem of this article given by Theorem 2.1.

The special shape of $\widetilde{\Omega}$ together with the periodicity assumption allows for a Fourier series approach with respect to $(0, 2\pi)^{n_1}$. First, for each Fourier coefficient a parameter-dependent Neumann problem in V is uniquely solved. The question whether this already ensures unique solvability of the original problem in Ω is linked closely to the question whether the parameter-dependent solution operators define a discrete operator-valued Fourier multiplier. To verify the latter we apply a multiplier result which requests $\mathcal{R}$-bounds for the parameter-dependent family of solution operators.

In [9] by means of Fourier transform the result of Theorem 2.1 is proved for infinite layers and for infinite cylinders $\mathbb{R}^{n_1} \times V$ with V as above. Here, $\mathcal{R}$-boundedness of the parameter-dependent family of solution operators is inferred from an equivalent condition involving arbitrary Muckenhoupt weights. The $\mathcal{R}$-bounds established there will serve as a baseline for this article (see Theorem 4.1). However, in contrast to [9] no partial derivatives but discrete shifts of the parameter-dependent family of solution operators have to be $\mathcal{R}$-bounded (see Corollary 4.3).

Results on resolvent estimates, maximal regularity, and boundedness of the $\mathcal{H}^\infty$-calculus for the Stokes operator in $L^p_\sigma(\Omega)$ are serialized in [11], [12], and [13]. Again, Ω is assumed to be an infinite layer, an infinite cylinder or the union of finitely many of these with a bounded domain. As the idea is once more to apply operator-valued Fourier multipliers, these results are available to some extent in our setting, too. A similar approach to the L^p-Helmholtz projection involving both Fourier transform and Fourier series for layers and infinite rectangular cylinders $\mathbb{R}^{n_1} \times (0, \pi)^{n_2}$ can be found in [16]. Here the projection is constructed in a direct manner, that is, without the help of the corresponding weak Neumann problem. In Remark 4.5 we discuss other possible domains V and thus possible extensions of the results obtained here to a class of unbounded domains Ω. Another representation of the L^p-Helmholtz projection in layers by means of singular Green operators is deduced in the series [1], [2]. For general unbounded domains of class C^1 the existence of the Helmholtz projection in $L^2 \cap L^p$ for $p > 2$ and in $L^2 + L^p$ for $1 < p < 2$ instead of L^p is proved in [10].

2. Preliminaries and the main theorem

In the subsequent lines let $G \subset \mathbb{R}^n$ be a domain and E a Banach space. For $m \in \mathbb{N}_0 \cup \{\infty\}$ we denote by $C^m(G, E)$ the space of all m-times continuously differentiable functions. The space of C^m-functions compactly supported in G will be denoted by $C_0^m(G, E)$. Furthermore, $C_0^m(\overline{G}, E)$ denotes the space of functions which occur as restrictions of functions in $C_0^m(\mathbb{R}^n, E)$ to functions defined on G. The space $C^\infty_{\mathrm{per}}(\mathbb{R}^n, E)$ consists of all functions $u \in C^\infty(\mathbb{R}^n, E)$ which are 2π-periodic with respect to each coordinate direction. For $1 < p < \infty$ we denote by $L^p(G, E)$ the Lebesgue-Bochner spaces, which are known to be UMD spaces, provided E has the UMD property. In particular, $L^p(G, \mathbb{R})$ is a UMD space. Accordingly, if E enjoys property (α), then $L^p(G, E)$ is known to enjoy property (α). See [15] for the definitions of the UMD property and property (α).

Let $m \in \mathbb{N}_0$. The E-valued Sobolev space $W^{m,p}(G, E)$ of order m consists of all $u \in L^p(G, E)$ such that all distributional derivatives up to order m define functions in $L^p(G, E)$. For $G = \mathcal{Q}_n := (0, 2\pi)^n$ the E-valued periodic Sobolev space $W^{m,p}_{\mathrm{per}}(\mathcal{Q}_n, E)$ of order m consists of all $u \in W^{m,p}(\mathcal{Q}_n, E)$ admitting the L^p-equality

$$\partial_j^l u|_{x_j=0} = \partial_j^l u|_{x_j=2\pi}, \quad j = 1, \ldots, n;\ 0 \leqslant l < m.$$

Note that for $m \in \mathbb{N}$ we have

$$W^{m,p}(\mathcal{Q}_n, E) \hookrightarrow L^p(\mathcal{Q}_{n-1}, C^{m-1}([0, 2\pi], E))$$

thanks to the Sobolev embedding. Hence, all traces in the definition of $W^{m,p}_{\mathrm{per}}(\mathcal{Q}_n, E)$ are well-defined by continuity. For convenience we set $W^{0,p}_{\mathrm{per}}(\mathcal{Q}_n, E) = L^p(\mathcal{Q}_n, E)$.

We further consider the subset of functions of mean value zero denoted by $W^{1,p}_{(0),\mathrm{per}}(\mathcal{Q}_n, E) := W^{1,p}_{\mathrm{per}}(\mathcal{Q}_n, E) \cap L^p_{(0)}(\mathcal{Q}_n, E)$, that is, the set of all functions $f \in W^{1,p}_{\mathrm{per}}(\mathcal{Q}_n, E)$ such that $\hat{f}(0) = 0$. If $E = \mathbb{R}$, we drop the additional indication in the definitions above and write as usual $L^p(G)$, for instance.

We turn our attention to the function spaces of hydrodynamics as presented in [14], Section III.1. First recall the homogeneous Sobolev space

$$\widehat{W}^{1,p}(G) := \{u \in L^1_{\mathrm{loc}}(\overline{G})/\mathbb{R} \colon \nabla u \in L^p(G)^n\}$$

equipped with the norm $\|\nabla u\|_{L^p(G)^n}$. Setting

$$\mathcal{D}_\sigma(G) := \{u \in C_0^\infty(G)^n \colon \operatorname{div} u = 0 \text{ in } G\}$$

for $1 < p < \infty$, we consider the space $L^p_\sigma(G)$ given by the completion of $\mathcal{D}_\sigma(G)$ in the L^p-norm. If G locally coincides with a Lipschitz domain, the existence of generalized normal traces γ_ν of vector fields on the boundary of G allows for the representation

$$L^p_\sigma(G) = \{u \in L^p(G)^n\colon \operatorname{div} u = 0 \text{ in } G,\ \gamma_\nu u = 0\},$$

where $\operatorname{div} u = 0$ in G has to be understood in the sense of distributions. Let further

$$G_p(G) := \{\nabla u\colon\ u \in L^1_{\mathrm{loc}}(G)/\mathbb{R},\ \nabla u \in L^p(G)^n\}.$$

The existence of the most useful L^p-Helmholtz decomposition

$$L^p(G) = L^p_\sigma(G) \oplus G_p(G)$$

is equivalent to the existence of the L^p-Helmholtz projection, i.e., to the existence of a unique bounded linear projection operator $\mathbb{P}_p = \mathbb{P}_p^2$ having range $L^p_\sigma(G)$ and kernel $G_p(G)$. Let p' denote the Hölder conjugate of p. As is well-known (see e.g. [14], Lemma 1.2), the existence of $\mathbb{P}_p$ is equivalent to unique solvability of the corresponding weak Neumann problem

$$\int_G \nabla u \nabla \varphi \,\mathrm{d}x = \int_G f \nabla \varphi \,\mathrm{d}x, \quad \varphi \in \widehat{W}^{1,p'}(G) \tag{2.1}$$

for each $f \in L^p(G)^n$. Thus, investigating (2.1) with G given by $(0,\pi)^{n_1} \times V$ we prove our main theorem, which reads as follows.

Theorem 2.1. *Let $1 < p < \infty$. Let $n_1, n_2 \in \mathbb{N}_0$ be such that $n := n_1 + n_2 \geqslant 2$ and let $\Omega := (0,\pi)^{n_1} \times V$, where $V \subset \mathbb{R}^{n_2}$ is a bounded domain with C^1-boundary. Then there exists a unique bounded linear projection operator*

$$\mathbb{P} = \mathbb{P}_p\colon\ L^p(\Omega)^n \to L^p_\sigma(\Omega) \subset L^p(\Omega)^n$$

with range $R(\mathbb{P}) = L^p_\sigma(\Omega)$ and kernel $N(\mathbb{P}) = G_p(\Omega)$.

3. $\mathcal{R}$-boundedness and Fourier multipliers

In the proof of Theorem 2.1 we will make use of operator-valued Fourier multiplier results. Here the UMD property and property (α) of Banach spaces as well as the notion of $\mathcal{R}$-boundedness of operator families are employed. For convenience of the reader we comment briefly on the latter. Given Banach spaces X and Y we write $\mathcal{L}(X,Y)$ for the space of bounded linear operators from X to Y and abbreviate $\mathcal{L}(X) := \mathcal{L}(X,X)$.

Definition 3.1. Let X and Y be Banach spaces. A family $\mathcal{T} \subset \mathcal{L}(X,Y)$ is called *$\mathcal{R}$-bounded*, if there exist $C > 0$ and $p \in [1,\infty)$ such that for all $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$, and all independent symmetric $\{-1,1\}$-valued random variables ε_j on a probability space $(G,\mathcal{M},P)$ for $j = 1,\dots,N$ we have that

$$\Big\|\sum_{j=1}^{N} \varepsilon_j T_j x_j\Big\|_{L^p(G,Y)} \leqslant C\Big\|\sum_{j=1}^{N} \varepsilon_j x_j\Big\|_{L^p(G,X)}. \tag{3.1}$$

The smallest $C > 0$ such that (3.1) is satisfied is called the $\mathcal{R}$-bound of $\mathcal{T}$ and denoted by $\mathcal{R}_p(\mathcal{T})$.

While the property of $\mathcal{R}$-boundedness is independent of $p \in [1,\infty)$, the $\mathcal{R}$-bound $\mathcal{R}_p(\mathcal{T})$ is not. However, for our purposes there is no need to distinguish the p-dependent $\mathcal{R}$-bounds. Hence, we omit the index p and merely write $\mathcal{R}(\mathcal{T})$. The following lemma collects two useful properties of $\mathcal{R}$-bounded families. The first one shows that $\mathcal{R}$-bounds essentially behave like uniform norm bounds, the second one is known as the contraction principle of Kahane (see e.g. [15], Proposition 2.5, or [8], Lemma 3.5).

Lemma 3.2.

a) *Let X, Y, and Z be Banach spaces and let $\mathcal{T},\mathcal{S} \subset \mathcal{L}(X,Y)$ and $\mathcal{U} \subset \mathcal{L}(Y,Z)$ be $\mathcal{R}$-bounded. Then $\mathcal{T}+\mathcal{S} \subset \mathcal{L}(X,Y)$, $\mathcal{T}\cup\mathcal{S} \subset \mathcal{L}(X,Y)$, and $\mathcal{U}\mathcal{T} \subset \mathcal{L}(X,Z)$ are $\mathcal{R}$-bounded as well and we have*

$$\mathcal{R}(\mathcal{T}+\mathcal{S}),\quad \mathcal{R}(\mathcal{T}\cup\mathcal{S}) \leqslant \mathcal{R}(\mathcal{S}) + \mathcal{R}(\mathcal{T}),\quad \mathcal{R}(\mathcal{U}\mathcal{T}) \leqslant \mathcal{R}(\mathcal{U})\mathcal{R}(\mathcal{T}).$$

b) *Let $p \in [1,\infty)$. Then for all $N \in \mathbb{N}$, $x_j \in X$, ε_j as in Definition 3.1, and for all $a_j, b_j \in \mathbb{C}$ with $|a_j| \leqslant |b_j|$ for $j = 1,\dots,N$, we have*

$$\Big\|\sum_{j=1}^{N} a_j\varepsilon_j x_j\Big\|_{L^p(G,X)} \leqslant 2\Big\|\sum_{j=1}^{N} b_j\varepsilon_j x_j\Big\|_{L^p(G,X)}. \tag{3.2}$$

We turn to operator-valued Fourier multipliers and related multiplier theorems. Let $n \in \mathbb{N}$, let $1 < p < \infty$, and let X and Y be Banach spaces. Given any function $f \in L^p(\mathcal{Q}_n, X)$ and $k \in \mathbb{Z}^n$, the k-th Fourier coefficient $\hat{f}(k) \in X$ of f is defined as $\hat{f}(k) := (2\pi)^{-n} \int_{\mathcal{Q}_n} \mathrm{e}^{-\mathrm{i}kx} f(x)\,\mathrm{d}x$. Given $M\colon \mathbb{Z}^n \to \mathcal{L}(X,Y)$, the relation

$$(T_M f)\hat{}\,(k) = M(k)\hat{f}(k), \quad k \in \mathbb{Z}^n$$

for Fourier coefficients $\hat{f}(k)$ of f defines a linear operator T_M between the spaces of X- and Y-valued trigonometric polynomials $\mathcal{T}(\mathcal{Q}_n, X)$ and $\mathcal{T}(\mathcal{Q}_n, Y)$. If $C > 0$ exists such that

$$\|T_M f\|_{L^p(\mathcal{Q}_n, Y)} \leqslant C\|f\|_{L^p(\mathcal{Q}_n, X)}, \quad f \in \mathcal{T}(\mathcal{Q}_n, X),$$

then M is called a *discrete operator-valued* (L^p-)*Fourier multiplier.* In that case T_M extends to $T_M \in \mathcal{L}(L^p(\mathcal{Q}_n, X), L^p(\mathcal{Q}_n, Y))$ by density and T_M is called the *Fourier multiplier operator associated with* M.

For the following important multiplier theorem we will need partial discrete derivatives of M defined as $\Delta^{e_j} M(k) := M(k) - M(k - e_j)$. Here e_j denotes the j-th unit vector in $\mathbb{R}^n$. For arbitrary $\gamma \in \{0,1\}^n$ we set

$$\Delta^0 M = M, \quad \Delta^\gamma M := \Delta^{\gamma_1 e_1} \ldots \Delta^{\gamma_n e_n} M. \tag{3.3}$$

Instead of $\gamma \in \{0,1\}^n$ we henceforth also write $0 \leqslant \gamma \leqslant 1$ or merely $\gamma \leqslant 1$.

Theorem 3.3. *Let $1 < p < \infty$, let X and Y be* UMD *Banach spaces having property (α), and let $\mathcal{T} \subset \mathcal{L}(X,Y)$ be $\mathcal{R}$-bounded. If $M\colon \mathbb{Z}^n \to \mathcal{L}(X,Y)$ satisfies $\{M(k);\ k \in \mathbb{Z}^n\} \subset \mathcal{T}$ and*

$$\{k^\gamma \Delta^\gamma M(k)\colon\ k \in \mathbb{Z}^n \setminus [-1,1]^n,\ 0 \leqslant \gamma \leqslant 1,\ \gamma \neq 0\} \subset \mathcal{T}, \tag{3.4}$$

then M defines a Fourier multiplier.

There are many contributions to Theorem 3.3 as stated above. For the one-dimensional case see [3], for higher dimensions [6], [7], and [18]. The latter allows to neglect the unite cube $[-1,1]^n$ in case $\gamma \neq 0$. See [16] for a comprehensive discussion on Fourier multiplier theorems in $L^p(\mathcal{Q}_n, X)$. The next lemma simplifies the verification of Sobolev regularity of functions in the range of multiplier operators ([16], Lemma 3.11).

Lemma 3.4. *Let $1 < p < \infty$, $m \in \mathbb{N}_0$, and let $M\colon \mathbb{Z}^n \to \mathcal{L}(X,Y)$. Then the following assertions are equivalent:*

(i) $T_M \in \mathcal{L}(L^p(\mathcal{Q}_n, X), W^{m,p}_{\mathrm{per}}(\mathcal{Q}_n, Y))$,

(ii) $M_\alpha\colon k \mapsto k^\alpha M(k)$ *defines a Fourier multiplier for each* $|\alpha| = m$.

The following version of Parseval's formula is crucial for the application of the theory of Fourier multipliers in variational problems.

Proposition 3.5. *Let X be a Banach space and let X' denote its dual space. Let $f \in L^1(\mathcal{Q}_n, X)$ and $g(x) := \sum\limits_{k\in\mathbb{Z}^n} \hat{g}(k)\mathrm{e}^{\mathrm{i}kx}$ with $\hat{g}(k) \in X'$ for $k \in \mathbb{Z}^n$ and $(\hat{g}(k))_{k\in\mathbb{Z}^n} \in l^1(\mathbb{Z}^n, X')$. Then*

$$\frac{1}{(2\pi)^n}\int_{\mathcal{Q}_n} \big(f(x), \overline{g(x)}\big)_X \,\mathrm{d}x = \sum_{k\in\mathbb{Z}^n} (\hat{f}(k), \overline{\hat{g}(k)})_X.$$

P r o o f. Thanks to the assumptions we have

$$\begin{aligned}\frac{1}{(2\pi)^n}\int_{\mathcal{Q}_n} \big(f(x), \overline{g(x)}\big)_X \,\mathrm{d}x &= \frac{1}{(2\pi)^n}\int_{\mathcal{Q}_n} \Big(f(x), \overline{\sum_{k\in\mathbb{Z}^n} \hat{g}(k)\mathrm{e}^{\mathrm{i}kx}}\Big)_X \,\mathrm{d}x \\ &= \sum_{k\in\mathbb{Z}^n} \frac{1}{(2\pi)^n}\int_{\mathcal{Q}_n} \big(\mathrm{e}^{-\mathrm{i}kx} f(x), \overline{\hat{g}(k)}\big)_X \,\mathrm{d}x = \sum_{k\in\mathbb{Z}^n} \big(\hat{f}(k), \overline{\hat{g}(k)}\big)_X.\end{aligned}$$

□

As in this case $(\hat{g}(k))_{k\in\mathbb{Z}^n}$ is rapidly decreasing, Proposition 3.5 applies to $g \in C^\infty_{\mathrm{per}}(\mathbb{R}^n, X')$. During the reflection procedure later on, however, we intend to apply Proposition 3.5 to Lipschitz continuous functions of tensor product type. We make this result available in the next lemma.

Lemma 3.6. *Let X be a Banach space and $\eta \in X$. For $j = 1, \ldots, n$ let $h_j\colon [0, 2\pi] \to \mathbb{C}$ define periodic and Lipschitz continuous functions. Then $g\colon \overline{\mathcal{Q}_n} \to X;\ g := \Big[\bigotimes\limits_{j=1}^n h_j\Big] \otimes \eta$ fulfills $(\hat{g}(k))_{k\in\mathbb{Z}^n} \in l^1(\mathbb{Z}^n, X)$.*

P r o o f. For each $j = 1, \ldots, n$ Bernstein's theorem in one variable ([4]) gives $(\hat{h}_j(k))_{k\in\mathbb{Z}} \in l^1(\mathbb{Z})$. Now the claim follows thanks to the tensor product structure of g. Indeed, due to $\hat{g}(k) = \prod\limits_j \hat{h}_j(k_j)\eta$ for each $N \in \mathbb{N}$ we have

$$\begin{aligned}\sum_{|k|_\infty \leqslant N} \|\hat{g}(k)\|_X &= \sum_{|k_1|\leqslant N} \ldots \sum_{|k_n|\leqslant N} |\hat{h}_1(k_1)| \ldots |\hat{h}_n(k_n)| \cdot \|\eta\|_X \\ &= \|\eta\|_X \prod_{j=1}^n \sum_{|k_j|\leqslant N} |\hat{h}_j(k_j)| < \|\eta\|_X \prod_{j=1}^n \|(\hat{h}_j(k_j))\|_{l^1(\mathbb{Z})} < \infty.\end{aligned}$$

□

Let $\widetilde{\Omega} := (0, 2\pi)^{n_1} \times V$. We subsequently investigate a weak realization of the partial periodic Neumann problem

$$
\begin{aligned}
\Delta u = F \quad &\text{in } \widetilde{\Omega}, \\
\partial_\nu u = 0 \quad &\text{on } (0, 2\pi)^{n_1} \times \partial V, \\
u|_{x_j = 2\pi} - u|_{x_j = 0} = 0, \quad &j = 1, \dots, n_1, \\
\partial_j u|_{x_j = 2\pi} - \partial_j u|_{x_j = 0} = 0, \quad &j = 1, \dots, n_1.
\end{aligned} \tag{4.1}
$$

More precisely, given $f \in L^p(\widetilde{\Omega})^n$ we consider the variational problem

$$
\int_{\widetilde{\Omega}} \nabla u \nabla \varphi \, \mathrm{d}x = \int_{\widetilde{\Omega}} f \nabla \varphi \, \mathrm{d}x, \quad \varphi \in C^\infty_{\mathrm{per}}(\mathbb{R}^{n_1}) \otimes C^\infty_0(\overline{V}). \tag{4.2}
$$

Our aim is to find a unique (up to constants) solution u in a suitable L^p-subspace such that $u|_{x_j = 2\pi} - u|_{x_j = 0} = 0$ for $j = 1, \dots, n_1$. In what follows we adopt the strategy pursued in [9]. We write $x = (x', x'')$ as well as $f = (f', f'')$ and $\varphi(x) = \Phi(x')\psi(x'')$, where $\Phi \in C^\infty_{\mathrm{per}}(\mathbb{R}^{n_1})$ and $\psi \in C^\infty_0(\overline{V})$. Calculating Fourier coefficients with respect to x', Parseval's formula for Fourier series as presented in Proposition 3.5 yields

$$
\sum_{k \in \mathbb{Z}^{n_1}} \int_V \big(|k|^2 \hat{u}(k) \overline{\psi} + \nabla'' \hat{u}(k) \overline{\nabla'' \psi} - \hat{f}'(k) \overline{\mathrm{i} k \psi} - \hat{f}''(k) \overline{\nabla'' \psi} \big) \, \mathrm{d}x'' \overline{\widehat{\Phi}(k)} = 0.
$$

Plugging in $\Phi_1(x') := \sin(kx')$ and $\Phi_2(x') := \cos(kx')$ for a fixed $k \in \mathbb{Z}^{n_1}$, a suitable complex linear combination takes us to the variational problems

$$
\int_V \big(|k|^2 \hat{u}(k) \overline{\psi} + \nabla'' \hat{u}(k) \overline{\nabla'' \psi} \big) \, \mathrm{d}x'' = \int_V \big(\hat{f}'(k) \overline{\mathrm{i} k \psi} + \hat{f}''(k) \overline{\nabla'' \psi} \big) \, \mathrm{d}x'' \tag{4.3}
$$

for all $\psi \in C^\infty_0(\overline{V})$ and all $k \in \mathbb{Z}^{n_1}$. In what follows we state results on the parameter-dependent problem

$$
\int_V \big(|k|^2 v \psi + \nabla'' v \nabla'' \psi \big) \, \mathrm{d}x'' = \int_V \big(- \mathrm{i} k g' \psi + g'' \nabla'' \psi \big) \, \mathrm{d}x''. \tag{4.4}
$$

In order to improve readability we rewrite (4.4) as

$$
\begin{aligned}
(|k|^2 - \Delta) v = -\mathrm{i} k g' + g'' \nabla \quad &\text{in } V, \\
\nu(\nabla v - g'') = 0 \quad &\text{on } \partial V,
\end{aligned}
$$

where $k \in \mathbb{Z}^{n_1}$, $v \in W^{1,p}(V)$ and $g = (g', g'') \in L^p(V)^n$. We define

$$g_V := \frac{1}{|V|} \int_V g(x'')\,\mathrm{d}x'' \tag{4.5}$$

and set

$$L^p_{(0)}(V) := \{g \in L^p(V)\colon\ g_V = 0\} \quad \text{and} \quad W^{1,p}_{(0)}(V) := W^{1,p}(V) \cap L^p_{(0)}(V).$$

Finally, let

$$\mathcal{D}\colon\ L^p(V)^n \to L^p_{(0)}(V)^{n_1} \times L^p(V)^{n_2}; \quad (g', g'') \mapsto (g' - g'_V, g'').$$

Observe that (4.4) coincides with the usual weak Neumann problem on V in the case $k = 0$. Let $v_0 = Q_0(g' - g'_V, g'') = Q_0\mathcal{D}g$ denote its unique solution $v_0 \in \widehat{W}^{1,p}(V)$ with right-hand side $(g' - g'_V, g'')$. The following result for the case $k \neq 0$ from [9], Theorem 3.6, is crucial for our further calculations.

Theorem 4.1. *Let $1 < p < \infty$. Let $V \subset \mathbb{R}^{n_2}$ be a bounded domain with C^1-boundary. Then for each $k \in \mathbb{Z}^{n_1} \setminus \{0\}$ and each $g \in L^p(V)^n$ such that $g' \in L^p_{(0)}(V)^{n_1}$, there exists a unique solution $v \in W^{1,p}_{(0)}(V)$ of (4.4). Let*

$$Q(k)\colon\ L^p_{(0)}(V)^{n_1} \times L^p(V)^{n_2} \to L^p(V)^n; \quad Q(k)\begin{pmatrix} g' \\ g'' \end{pmatrix} := \begin{pmatrix} kv \\ \nabla v \end{pmatrix}.$$

Then the set

$$\{Q(k)\mathcal{D}\colon\ k \in \mathbb{Z}^{n_1} \setminus \{0\}\} \subset \mathcal{L}(L^p(V)^n) \tag{4.6}$$

is $\mathcal{R}$-bounded.

To deal with the multiplier condition in Theorem 3.3 we need the following discrete product rule (see e.g. [17], Lemma 3.3.6). Note that in contrast to the classical product rule for differentiable functions, here we have to keep control of the shifts appearing in one of the factors of each term.

Lemma 4.2. *Let X be a Banach space, $S(k) \in \mathbb{R}$ and $T(k) \in X$ for $k \in \mathbb{Z}^n$. For each $\alpha \in \mathbb{N}_0^n$ we then have*

$$\Delta^\alpha(ST)(k) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} (\Delta^{\alpha-\beta}S)(k-\beta)(\Delta^\beta T)(k), \quad k \in \mathbb{Z}^n.$$

As both notations are standard in literature, in what follows we retain the notation Δ^γ for the shift operator with respect to $k \in \mathbb{Z}^{n_1}$ as defined in (3.3), although a similar notation Δ for the Laplacian with respect to $x'' \in V$ appears simultaneously.

Corollary 4.3. *The set*

$$\{k^\gamma \Delta^\gamma (Q(k)\mathcal{D})\colon\ k \in \mathbb{Z}^{n_1} \setminus [-1,1]^{n_1},\ 0 \leqslant \gamma \leqslant 1\} \subset \mathcal{L}(L^p(V)^n) \tag{4.7}$$

is $\mathcal{R}$-bounded.

P r o o f. Given $g \in L^p(V)^n$ for $k \in \mathbb{Z}^{n_1} \setminus [-1,1]^{n_1}$, let v_k denote the solution of (4.4) with parameter k.

The case $\gamma = 0$ being already proved in Theorem 4.1, let henceforth $0 \leqslant \gamma \leqslant 1$, $\gamma \neq 0$ and let $\kappa\colon\ \mathbb{Z}^n \to \mathbb{Z}^n$ denote the identity. Then the product rule yields

$$\begin{aligned} k^\gamma \Delta^\gamma (Q(k)\mathcal{D})g &= k^\gamma \Delta^\gamma \begin{pmatrix} kv_k \\ \nabla v_k \end{pmatrix} \\ &= k^\gamma \begin{pmatrix} (k-\gamma)\Delta^\gamma v_k \\ \nabla \Delta^\gamma v_k \end{pmatrix} - k^\gamma \sum_{\alpha \leqslant \gamma, \alpha \neq \gamma} \begin{pmatrix} (\Delta^{\gamma-\alpha}\kappa)_{k-\alpha}(\Delta^\alpha v)_k \\ 0 \end{pmatrix}. \end{aligned}$$

Since $0 \neq \gamma - \alpha$, we have

$$(\Delta^{\gamma-\alpha}\kappa)_{k-\alpha} = \begin{cases} e_j, & \text{if } \gamma - \alpha = e_j, \\ 0, & \text{else.} \end{cases}$$

Thus, we infer

$$\begin{aligned} k^\gamma \Delta^\gamma (Q(k)\mathcal{D})g &= k^\gamma \begin{pmatrix} (k-\gamma)\Delta^\gamma v_k \\ \nabla \Delta^\gamma v_k \end{pmatrix} - k^\gamma \sum_{j=1,\dots,n_1,\gamma_j=1} \begin{pmatrix} e_j(\Delta^{\gamma-e_j} v)_k \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} (k-\gamma)k^\gamma \Delta^\gamma v_k \\ \nabla k^\gamma \Delta^\gamma v_k \end{pmatrix} - \sum_{j=1,\dots,n_1,\gamma_j=1} \begin{pmatrix} k_j e_j k^{\gamma-e_j}(\Delta^{\gamma-e_j} v)_k \\ 0 \end{pmatrix}. \end{aligned}$$

Observe the existence of $C > 0$ such that $|k_j| \leqslant |k| \leqslant C|k-\gamma|$ for all $k \in \mathbb{Z}^{n_1} \setminus [-1,1]^{n_1}$ and all $j = 1,\dots,n_1$. Setting $w_k^\gamma := \Delta^\gamma v_k$, thanks to Lemma 3.2 it only remains to prove that the sets

$$\left\{ L^p(V)^n \to L^p(V)^n;\ g \mapsto \begin{pmatrix} (k-\gamma)k^\gamma w_k^\gamma \\ \nabla k^\gamma w^\gamma(k) \end{pmatrix} \colon\ k \in \mathbb{Z}^{n_1} \setminus [-1,1]^{n_1} \right\}$$

are $\mathcal{R}$-bounded. This assertion is proved by induction. Recall that the case $\gamma = 0$ has already been proved in Theorem 4.1. Employing the product rule one more time we find

$$\begin{aligned} (|k-\gamma|^2 - \Delta)k^\gamma w_k^\gamma &= k^\gamma (|k-\gamma|^2 - \Delta)\Delta^\gamma v_k \\ &= k^\gamma \Delta^\gamma ((|\kappa|^2 - \Delta)v)_k - k^\gamma \sum_{\alpha \leqslant \gamma, \alpha \neq \gamma} (\Delta^{\gamma-\alpha}(|\kappa|^2 - \Delta))_{k-\alpha}(\Delta^\alpha v)_k. \end{aligned}$$

Taking into account that v_k defines the solution of (4.4) with parameter k, for the first addend on the right we find

$$
\begin{aligned}
k^\gamma \Delta^\gamma((|\kappa|^2 - \Delta)v)_k &= k^\gamma(\Delta^\gamma(-\mathrm{i}\kappa g'))_k + k^\gamma(\Delta^\gamma g''\nabla)_k \\
&= -\mathrm{i}k^\gamma(\Delta^\gamma \kappa)_k g' = \begin{cases} -\mathrm{i}k_j e_j g' & \text{if } \gamma = e_j, \\ 0 & \text{else.} \end{cases}
\end{aligned}
$$

The second addend equals $-\sum_{|\alpha|\leqslant\gamma,\alpha\neq\gamma} k^{\gamma-\alpha}(\Delta^{\gamma-\alpha}(|\kappa|^2-\Delta))_{k-\alpha}k^\alpha w_k^\alpha$. Since we have

$$
(\Delta^{\gamma-\alpha}(|\kappa|^2-\Delta))_{k-\alpha} = \begin{cases} -2k_j + 1 & \text{if } \gamma - \alpha = e_j, \\ 0 & \text{else,} \end{cases}
$$

the sum reduces to $\sum_{j=1,\dots,n_1,\gamma_j=1} k_j(1-2k_j)k^{\gamma-e_j}w_k^{\gamma-e_j}$. Altogether, $k^\gamma w_k^\gamma$ solves (4.4) with right-hand side $(G^{k,\gamma}+H^{k,\gamma},0) \in L^p_{(0)}(V)^{n_1} \times L^p(V)^{n_2}$ and parameter $k-\gamma$, where

$$
G_j^{k,\gamma} = \begin{cases} 0 & \text{if } k_j - \gamma_j = 0 \text{ or } \gamma_j = 0, \\ \mathrm{i}\dfrac{1-2k_j}{k_j-\gamma_j}k^{\gamma-e_j}w_k^{\gamma-e_j}e_j & \text{else} \end{cases}
$$

and

$$
H_j^{k,\gamma} = \begin{cases} \dfrac{k_j}{k_j-\gamma_j}g_j' & \text{if } \gamma = e_j, \\ 0 & \text{else} \end{cases}
$$

for $j = 1,\dots,n_1$. Thus, Theorem 4.1 applies to $k^\gamma w_k^\gamma$ and $(G^{k,\gamma}+H^{k,\gamma},0)$. Since we have

$$
\left|\frac{1-2k_j}{k_j-\gamma_j}\right| \leqslant C \quad \text{and} \quad \left|\frac{k_j}{k_j-\gamma_j}\right| \leqslant C
$$

for all $k \in \mathbb{Z}^{n_1}$ such that $k_j - \gamma_j \neq 0$, thanks to the induction hypothesis and Lemma 3.2 the sets

$$
\{L^p(V)^n \to L^p(V)^n;\ g \mapsto (G^{k,\gamma}+H^{k,\gamma},0)\colon\ k \in \mathbb{Z}^{n_1} \setminus [-1,1]^{n_1}\}
$$

are $\mathcal{R}$-bounded. Applying Lemma 3.2 one more time completes the proof. $\square$

In what follows we denote functions from $L^p((0,2\pi)^{n_1}, \widehat{W}^{1,p}(V))$ which are constant with respect to $x' \in (0,2\pi)^{n_1}$ merely by $\widehat{W}^{1,p}(V)$ since no confusion seems likely. Inspired by (4.5), for $f \in L^p(\widetilde{\Omega})^n$ we define

$$
f_V' = f_V'(x') := \frac{1}{|V|}\int_V f'(x',x'')\,\mathrm{d}x''. \tag{4.8}
$$

Theorem 4.4. *Let $1 < p < \infty$. Let $V \subset \mathbb{R}^{n_2}$ be a bounded domain with C^1-boundary, $\widetilde{\Omega} := (0, 2\pi)^{n_1} \times V$, and let $f \in L^p(\widetilde{\Omega})^n$. Then the weak realization* (4.2) *of the partial periodic Neumann problem* (4.1) *has a unique solution*

$$u \in \big(W^{1,p}(\widetilde{\Omega}) \cap W^{1,p}_{(0),\mathrm{per}}((0, 2\pi)^{n_1}, L^p(V))\big) + \widehat{W}^{1,p}(V).$$

Proof. Define

$$M_1(k) := \begin{cases} 0, & k = 0, \\ Q(k)\mathcal{D}, & k \neq 0 \end{cases} \quad \text{and} \quad M_2(k) := \begin{cases} Q_0\mathcal{D}, & k = 0, \\ 0, & k \neq 0. \end{cases}$$

Then

$$M_1 \colon \mathbb{Z}^{n_1} \to \mathcal{L}(L^p(V)^n) \quad \text{and} \quad M_2 \colon \mathbb{Z}^{n_1} \to \mathcal{L}(L^p(V)^n, \widehat{W}^{1,p}(V))$$

define Fourier multipliers. The assertion on M_1 follows thanks to the $\mathcal{R}$-boundedness result (4.7) of Corollary 4.3 from Theorem 3.3. The assertion on M_2 is a direct consequence of the UMD property of $L^p(V)^n$ implying that the Kronecker symbol δ_{k1} defines a Fourier multiplier on $L^p(V)^n$. By construction and Lemma 3.4

$$T_{M_1} \colon L^p(\widetilde{\Omega}) \to W^{1,p}(\widetilde{\Omega}) \cap W^{1,p}_{\mathrm{per}}((0, 2\pi)^{n_1}, L^p(V)).$$

Thus, for each $f \in L^p(\widetilde{\Omega})^n$ there exists

$$u \in \big(W^{1,p}(\widetilde{\Omega}) \cap W^{1,p}_{(0),\mathrm{per}}((0, 2\pi)^{n_1}, L^p(V))\big) + \widehat{W}^{1,p}(V)$$

such that $\hat{u}(k)$ for each $k \in \mathbb{Z}^{n_1}$ solves

$$\int_V \big(|k|^2 \hat{u}(k)\psi + \nabla'' \hat{u}(k) \nabla'' \psi\big)\, \mathrm{d}x'' = \int_V \big(-\mathrm{i}k(\hat{f}' - \hat{f}'_V)(k)\psi + \hat{f}''(k)\nabla''\psi\big)\, \mathrm{d}x''$$

for each $\psi \in C_0^\infty(\overline{V})$. To solve (4.2) it remains to treat

$$\int_V \big(|k|^2 \widehat{w}(k)\psi + \nabla'' \widehat{w}(k) \nabla'' \psi\big)\, \mathrm{d}x'' = -\int_V \mathrm{i}k\hat{f}'_V(k)\psi\, \mathrm{d}x''$$

for $k \in \mathbb{Z}^{n_1} \setminus \{0\}$. As f'_V is independent of x'', we immediately find a solution

$$w \in W^{1,p}_{(0),\mathrm{per}}((0, 2\pi)^{n_1}) \hookrightarrow W^{1,p}_{(0),\mathrm{per}}((0, 2\pi)^{n_1}, L^p(V)),$$

by $\widehat{w}(k) := -\mathrm{i}(k/|k|^2)\hat{f}'_V(k)$ for $k \neq 0$ and $\widehat{w}(0) := 0$. Thus,

$$u + w \in \big(W^{1,p}(\widetilde{\Omega}) \cap W^{1,p}_{(0),\mathrm{per}}((0, 2\pi)^{n_1}, L^p(V))\big) + \widehat{W}^{1,p}(V)$$

solves (4.2). To prove uniqueness of the solution let u_1 and u_2 solve (4.2). Calculating Fourier coefficients of $v := u_1 - u_2$, Proposition 3.5 takes us to

$$\int_V \left(|k|^2 \hat{v}(k)\psi + \nabla'' \hat{v}(k) \nabla'' \psi\right) \mathrm{d}x'' = 0, \quad \psi \in C_0^\infty(\overline{V})$$

for each $k \in \mathbb{Z}^n$. For $k = 0$ this equals $\int_V \nabla'' \hat{v}(0) \nabla'' \psi \, \mathrm{d}x'' = 0$ and well-known results on the weak Neumann problem on V yield $\hat{v}(0) = 0$. In the case $k \neq 0$ we employ the uniqueness result of Theorem 4.1 to find $\hat{v}(k) = 0$ for $k \in \mathbb{Z}^n \setminus \{0\}$. Thus $v = 0$ due to well-known uniqueness results on Fourier coefficients. □

In the following remark we briefly comment on possible extensions of Theorem 4.4 concerning the domain V, which lead to certain unbounded domains Ω.

Remark 4.5. a) The domain V may as well be assumed to be the whole space, the half space (or any space which leads to the whole space by carrying out finitely many reflections with respect to the coordinate axes) or a bent half space (see [9], Section 3, for the precise definition). Indeed, minor difficulties arise in this case for 0 is not an eigenvalue of the Neumann Laplacian. Setting $g_V := 0$ in (4.5), the assertion of Theorem 4.1 is proved in [9], Theorems 3.2 and 3.4, for these domains, too. This leads, e.g., to domains Ω representing infinite rectangular cylinders (cf. [16]), halves of infinite layers, halves of infinite rectangular cylinders or bent halves of infinite layers.

b) More generally, for $n_1, n_2, n_3 \in \mathbb{N}$ (non-physical) unbounded and non-smooth domains of shape $\Omega := \mathbb{R}^{n_1} \times (0, \pi)^{n_2} \times V$ can be treated. Here, the results of [9] and multiplier results both for Fourier transform and Fourier series have to be used.

5. Analysis of the weak Neumann problem—Proof of Theorem 2.1

Now we are in the position to treat the variational problem (2.1) in $\Omega := (0, \pi)^{n_1} \times V$. This will be done by means of an appropriate reflection technique. To this end let $\phi \in L^p(\Omega)$ for $V \subset \mathbb{R}^{n_2}$ as before and $n := n_1 + n_2$. We define the extension $\mathfrak{E}\phi := \mathfrak{E}_{n_1} \ldots \mathfrak{E}_1 \phi$ to $\widetilde{\Omega} := \mathcal{Q}_{n_1} \times V$ by even extension iteratively throughout all coordinate directions. More precisely, for $i = 1, \ldots, n_1$ let $\mathfrak{E}_i$ extend

$$\phi^{i-1} := \mathfrak{E}_{i-1} \ldots \mathfrak{E}_1 \phi \in L^p((0, 2\pi)^{i-1} \times (0, \pi)^{n_1 - i + 1} \times V)$$

to

$$\phi^i := \mathfrak{E}_i \ldots \mathfrak{E}_1 \phi \in L^p((0, 2\pi)^i \times (0, \pi)^{n_1 - i} \times V)$$

such that ϕ^i is even with respect to $x_i = \pi$. This construction gives rise to an extension operator $\mathfrak{E} \in \mathcal{L}(L^p(\Omega), L^p(\widetilde{\Omega}))$. Let further $\mathfrak{R} \in \mathcal{L}(L^p(\widetilde{\Omega}), L^p(\Omega))$ denote the operator of restriction. Finally, consider $f \in L^p(\Omega)^n$ and let $\mathfrak{O}_i$ denote the extension of

$$f^{i-1} := \mathfrak{O}_{i-1} \dots \mathfrak{O}_1 f \in L^p((0,2\pi)^{i-1} \times (0,\pi)^{n_1-i+1} \times V)^n$$

to

$$f^i := \mathfrak{O}_i \dots \mathfrak{O}_1 f \in L^p((0,2\pi)^i \times (0,\pi)^{n_1-i} \times V)$$

such that the i-th component of f^i is odd, whereas all other components of f^i are even with respect to $x_i = \pi$. Then we easily verify the useful $L^p(\widetilde{\Omega})$-identity

$$\mathfrak{O}\nabla u = \nabla \mathfrak{E} u, \quad u \in W^{1,p}(\Omega). \tag{5.1}$$

Theorem 5.1. *Let $1 < p < \infty$ and $\Omega := (0,\pi)^{n_1} \times V$, where $V \subset \mathbb{R}^{n_2}$ is a bounded domain with C^1-boundary. Then for each $f \in L^p(\Omega)^n$ there exists a unique solution $u \in \widehat{W}^{1,p}(\Omega)$ of problem (2.1).*

P r o o f. Let $f \in L^p(\Omega)^n$ be given. In order to prove unique solvability of the variational problem (2.1), due to density it suffices to consider $\varphi \in \Big[\bigotimes_{j=1}^{n_1} C_0^\infty([0,\pi])\Big] \otimes C_0^\infty(\overline{V})$. Here we make use of the cylindrical structure of Ω and the fact that Ω is a bounded domain. From Theorem 4.4 we know the existence of a unique solution

$$U \in (W^{1,p}(\widetilde{\Omega}) \cap W^{1,p}_{(0),\mathrm{per}}((0,2\pi)^{n_1}, L^p(V))) + \widehat{W}^{1,p}(V)$$

to the variational problem of the partial periodic Neumann problem (4.2) with right-hand side $\mathfrak{O}f$. Thanks to Lemma 3.6 the class of test functions $C^\infty_{\mathrm{per}}(\mathbb{R}^{n_1}) \otimes C_0^\infty(\overline{V})$ in (4.2) may be enlarged such that the class $\Big[\bigotimes_{j=1}^{n_1} \mathrm{Lip}_{\mathrm{per}}([0,2\pi])\Big] \otimes C_0^\infty(\overline{V})$ is included. Here $\mathrm{Lip}_{\mathrm{per}}([0,2\pi])$ denotes the space of periodic Lipschitz continuous functions of one variable. This can be done without any loss of validity in the results of the previous sections. Since $\mathfrak{E}\varphi \in \Big[\bigotimes_{j=1}^{n_1} \mathrm{Lip}_{\mathrm{per}}([0,2\pi])\Big] \otimes C_0^\infty(\overline{V})$, in particular it holds that

$$\int_{\widetilde{\Omega}} \nabla U \nabla \mathfrak{E}\varphi \,\mathrm{d}x = \int_{\widetilde{\Omega}} \mathfrak{O} f \nabla \mathfrak{E}\varphi \,\mathrm{d}x \tag{5.2}$$

for all $\varphi \in \Big[\bigotimes_{j=1}^{n_1} C_0^\infty([0,\pi])\Big] \otimes C_0^\infty(\overline{V})$. Making use of the transformation formula, we easily see that

$$V^i(x) := U(x_1, \dots, x_{i-1}, 2\pi - x_i, x_{i+1}, \dots, x_{n_1}, x'')$$

for $i = 1, \ldots, n_1$ define solutions of (4.2) with right-hand side $\mathfrak{O}f$, too. Thus, $V^i = U$ for $i = 1, \ldots, n_1$ by uniqueness. Consequently, there exists $u \in \widehat{W}^{1,p}(\Omega)$ such that $U = \mathfrak{E}u$, which implies $\nabla U = \mathfrak{O}\nabla u$ by (5.1). Hence, from (5.1) and (5.2) we deduce

$$\int_\Omega \nabla u \nabla \varphi \,\mathrm{d}x = \int_\Omega f \nabla \varphi \,\mathrm{d}x,$$

i.e., u defines a solution of (2.1). The uniqueness of u follows along the same lines from the uniqueness of U. $\square$

Now well-known results prove our main theorem (cf. [14], Lemma 1.2).

Proof of Theorem 2.1. The existence of the Helmholtz projection $\mathbb{P}_p \in \mathcal{L}(L^p(\Omega))$ follows immediately from Theorem 5.1. $\square$

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