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## Universally divergent Fourier series via Landau’s extremal functions

GERD HERZOG, PEER CHR. KUNSTMANN

*Abstract.* We prove the existence of functions  $f \in A(\mathbb{D})$ , the Fourier series of which being universally divergent on countable subsets of  $\mathbb{T} = \partial\mathbb{D}$ . The proof is based on a uniform estimate of the Taylor polynomials of Landau’s extremal functions on  $\mathbb{T} \setminus \{1\}$ .

*Keywords:* Fourier series; universal functions; Landau’s extremal functions

*Classification:* 42A16, 30B30, 47B38

### 1. Introduction

Let  $\mathbb{D}$  denote the unit disc in  $\mathbb{C}$  and  $\mathbb{T} = \partial\mathbb{D}$ . For  $f \in L^1(\mathbb{T})$  let

$$\sum_{k=-\infty}^{\infty} \widehat{f}(k)e_k$$

denote the corresponding Fourier series, i.e.

$$e_k(t) = \exp(ikt), \quad \widehat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(\exp(it)) \exp(-ikt) dt \quad (k \in \mathbb{Z}).$$

There is a tremendous amount of classical convergence and divergence results on Fourier series [14]. Moreover, several results on universally divergent Fourier (and trigonometric) series are known, see for example [3], [5], [7], [10], [11] and the references given there. Roughly speaking,  $f$  has a universally divergent Fourier series if the set of restrictions

$$\left\{ \left( \sum_{k=-n}^n \widehat{f}(k)e_k \right) \Big|_E : n \in \mathbb{N} \right\}$$

in a function space  $Y$ , of functions over a subset  $E$  of  $\mathbb{T}$ , is dense in  $Y$ . In [10] Müller proved that given a *countable* set  $E \subseteq \mathbb{T}$ , then the set of functions  $f \in C(\mathbb{T})$

having the property that

$$\left\{ \left( \sum_{k=-n}^n \widehat{f}(k)e_k \right) \Big|_E : n \in \mathbb{N} \right\} \text{ is dense in } \mathbb{C}^E$$

is a dense  $G_\delta$ -subset of  $C(\mathbb{T})$ . Here  $\mathbb{C}^E$  is the Fréchet space of all functions  $w : E \rightarrow \mathbb{C}$  endowed with the topology of pointwise convergence.

In this paper, we prove that there are even functions  $f \in C(\mathbb{T})$  with universal divergent Fourier series in the sense of Müller such that in addition  $\widehat{f}(-k) = 0$  ( $k \in \mathbb{N}$ ). While Müller’s proof is based on the Localization Principle of Fourier series we will use Landau’s extremal functions [9, §2].

To formulate our results let  $H(\mathbb{D})$  denote the Fréchet space of all analytic functions on  $\mathbb{D}$  endowed with the compact open topology, let  $A(\mathbb{D}) = C(\overline{\mathbb{D}}) \cap H(\mathbb{D})$  denote the disc-algebra endowed with the maximum norm  $\|\cdot\|_\infty$ , and for  $f \in H(\mathbb{D})$  let

$$S_n(f, z) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} z^k \quad (n \in \mathbb{N}_0).$$

For a given countable set  $E \subseteq \mathbb{T}$  consider the continuous linear operators

$$L_n : A(\mathbb{D}) \rightarrow \mathbb{C}^E, \quad L_n(f) = S_n(f, \cdot)|_E.$$

We will prove

**Theorem 1.** *The set of all  $f \in A(\mathbb{D})$  with the property*

$$\{L_n(f) : n \in \mathbb{N}_0\} \text{ is dense in } \mathbb{C}^E$$

*is a dense  $G_\delta$ -subset of  $A(\mathbb{D})$ .*

**Remark.** According to a result of Fejér [4], no  $q$ -to-one function  $f \in A(\mathbb{D})$  ( $q \in \mathbb{N}$ ) can share the property in Theorem 1, since for those functions

$$|S_n(f, 1)| \leq \left( 1 + \sqrt{\frac{q}{2}} \right) \|f\|_\infty \quad (n \in \mathbb{N}_0).$$

Just as Müller’s Theorem extends to  $L^p$ -spaces, we can extend Theorem 1 to certain Banach spaces of analytic functions, such as the Hardy spaces  $H^p(\mathbb{D})$  ( $1 \leq p < \infty$ ), or the little Bloch space

$$\begin{aligned} \mathcal{B}_0 &= \{f \in H(\mathbb{D}) : \lim_{|z| \rightarrow 1^-} (1 - |z|^2)|f'(z)| = 0\}, \\ \|f\| &= |f(0)| + \max_{|z| < 1} (1 - |z|^2)|f'(z)| \end{aligned}$$

(compare [1]), for example.

**Theorem 2.** Let  $(X, \|\cdot\|)$  be a Banach space with  $A(\mathbb{D}) \subseteq X \subseteq H(\mathbb{D})$  such that

- (1)  $A(\mathbb{D})$  is dense in  $X$ ,
- (2)  $\|\cdot\|$  is weaker than  $\|\cdot\|_\infty$  on  $A(\mathbb{D})$ ,
- (3) convergence in  $X$  implies convergence in  $H(\mathbb{D})$ .

Let  $E \subseteq \mathbb{T}$  be countable, and let

$$\tilde{L}_n : X \rightarrow \mathbb{C}^E, \quad \tilde{L}_n(f) = S_n(f, \cdot)|_E.$$

Then, the set of all  $f \in X$  with the property

$$\{\tilde{L}_n(f) : n \in \mathbb{N}_0\} \text{ is dense in } \mathbb{C}^E$$

is a dense  $G_\delta$ -subset of  $X$ .

In [10] Müller also discusses the problem of replacing pointwise convergence by uniform convergence for compact subsets  $E$  of  $\mathbb{T}$ . We note here that the general topological category argument [10, Lemma 2] is also applicable in our setting. Combined with Theorem 1 it proves that there are many and even uncountable compact subsets  $E \subseteq \mathbb{T}$  such that the set of all  $f \in A(\mathbb{D})$  with the property

$$\{S_n(f, \cdot)|_E : n \in \mathbb{N}_0\} \text{ is dense in } C(E)$$

is a dense  $G_\delta$ -subset of  $A(\mathbb{D})$ . Here  $C(E)$  denotes the Banach space of all continuous functions  $w : E \rightarrow \mathbb{C}$  endowed with the maximum norm.

It follows from [8, Cor. 3.3] that there are countable compact subsets  $E \subseteq \mathbb{T}$  with a single accumulation point such that, for no function  $f \in A(\mathbb{D})$ , the set

$$\{S_n(f, \cdot)|_E : n \in \mathbb{N}_0\}$$

is dense in  $C(E)$ . On other hand, recent results in [2] show that, for any compact subset  $E \subseteq \mathbb{T}$  of arc length measure 0, the set of all  $f \in H^p(\mathbb{D})$ ,  $p \in [1, \infty)$ , with the property

$$\{S_n(f, \cdot)|_E : n \in \mathbb{N}_0\} \text{ is dense in } C(E)$$

is a dense  $G_\delta$ -subset of  $H^p(\mathbb{D})$ .

**Remark.** After submitting this paper Jürgen Müller informed us that he and George Costakis have independently found Theorem 1. Their proof is based on Fejér polynomials

$$F_n(z) = \sum_{k=0, k \neq n}^{2n} \frac{z^k}{n-k},$$

instead of Landau's extremal functions and has not been published up to now.

## 2. Landau's extremal functions

In the sequel let

$$a_k := (-1)^k \binom{-1/2}{k} \quad (k \in \mathbb{N}_0),$$

that is

$$a_0 = 1, \quad a_k = \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdot 6 \cdots 2k} \quad (k \in \mathbb{N}),$$

and

$$K_n(z) := a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \quad (n \in \mathbb{N}_0).$$

In [9] Landau proved that  $0 \notin K_n(\overline{\mathbb{D}})$ , that

$$\gamma_n := \sup\{|S_n(f, 1)| : f \in A(\mathbb{D}), \|f\|_\infty \leq 1\} \quad (n \in \mathbb{N}_0)$$

is a maximum which is attained at

$$f(z) = R_n(z) := \frac{z^n K_n(1/z)}{K_n(z)} = \frac{a_n + a_{n-1}z + a_{n-2}z^2 + \cdots + a_0 z^n}{a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n},$$

and that

$$S_n(R_n, 1) = \gamma_n = \sum_{k=0}^n a_k^2 \sim \frac{\log(n)}{\pi} \quad (n \rightarrow \infty).$$

In particular we have

$$S_n(R_n, 1) = \gamma_n \rightarrow \infty \quad (n \rightarrow \infty).$$

Some immediate facts on the rational functions  $R_n$  are

$$|R_n(z)| = 1 \quad (z \in \mathbb{T}), \quad \|R_n\|_\infty = 1.$$

To utilize the functions  $R_n$  in the proof of Theorem 1 we prove the following theorem, which seems to us of some interest on its own. It asserts that the functions  $S_n(R_n, \cdot)$ ,  $n \in \mathbb{N}_0$ , have a common majorant on  $\overline{\mathbb{D}} \setminus \{1\}$ . We shall use this on  $\mathbb{T} \setminus \{1\}$ .

**Theorem 3.** *For each  $z \in \overline{\mathbb{D}} \setminus \{1\}$  we have*

$$|S_n(R_n, z)| \leq \frac{4}{|1-z|} \quad (n \in \mathbb{N}_0).$$

PROOF: We fix  $n \in \mathbb{N}_0$ , and consider

$$S_n(R_n, z) = \sum_{k=0}^n b_k z^k.$$

Observe that here  $b_k = b_{k,n}$  depend on  $n$ , but we shall ignore this in notation. Since the  $a_k$  are real, it is clear that  $b_k \in \mathbb{R}$ , and from  $\|R_n\|_\infty = 1$  we get by

Cauchy's formula  $|b_k| \leq 1$  ( $k = 0, \dots, n$ ). In [13], Wintner proved that in fact  $b_k \in [0, 1]$  ( $k = 0, \dots, n$ ). From

$$(a_0 + a_1z + a_2z^2 + \dots + a_nz^n)R_n(z) = a_n + a_{n-1}z + a_{n-2}z^2 + \dots + a_0z^n$$

we obtain (cf. [13])

$$\sum_{k=0}^m a_{m-k}b_k = a_{n-m} \quad (m = 0, \dots, n).$$

By setting  $P_k := \text{diag}([1, \dots, 1], -k) \in \mathbb{R}^{(n+1) \times (n+1)}$  (the band matrix with 1's in the  $k$ -th subdiagonal and 0's else),  $b := (b_0, b_1, \dots, b_n)^\top \in \mathbb{R}^{n+1}$  and  $a := (a_n, a_{n-1}, \dots, a_0)^\top \in \mathbb{R}^{n+1}$ , the linear system above can be written as

$$\left( \sum_{k=0}^n a_k P_k \right) b = a.$$

We claim that

$$\left( \sum_{k=0}^n a_k P_k \right)^{-1} = a_0 P_0 + (a_1 - a_0)P_1 + (a_2 - a_1)P_2 + \dots + (a_n - a_{n-1})P_n.$$

In order to see this we start with the Taylor expansion

$$(1+x)^\alpha = \sum_{k=0}^\infty \binom{\alpha}{k} x^k \quad (|x| < 1)$$

for  $\alpha = -\frac{1}{2}$  and  $\alpha = \frac{1}{2}$  and take the Cauchy product

$$1 = (1+x)^{-1/2}(1+x)^{1/2} = \sum_{k=0}^\infty \sum_{\nu=0}^k \binom{-1/2}{\nu} \binom{1/2}{k-\nu} x^k.$$

We conclude that

$$\sum_{\nu=0}^k \binom{-1/2}{\nu} \binom{1/2}{k-\nu} = \delta_{0k} \quad (k \in \mathbb{N}_0).$$

For  $k \in \mathbb{N}$  we get by a well-known formula

$$a_k - a_{k-1} = (-1)^k \left[ \binom{-1/2}{k} + \binom{-1/2}{k-1} \right] = (-1)^k \binom{1/2}{k}.$$

Then

$$I + \sum_{k=1}^n (a_k - a_{k-1})P_k = I + \sum_{k=1}^n (-1)^k \binom{1/2}{k} P_k = \sum_{k=0}^n (-1)^k \binom{1/2}{k} P_k$$

and

$$\sum_{k=0}^n a_k P_k = I + \sum_{k=1}^n (-1)^k \binom{-1/2}{k} P_k = \sum_{k=0}^n (-1)^k \binom{-1/2}{k} P_k.$$

We take the product, respect  $P_k P_l = P_{k+l}$  (setting  $P_j = 0$  for  $j > n$ ), and arrive at

$$\begin{aligned} & \left( \sum_{k=0}^n a_k P_k \right) \left( I + \sum_{k=1}^n (a_k - a_{k-1}) P_k \right) \\ &= \left( \sum_{k=0}^n (-1)^k \binom{-1/2}{k} P_k \right) \left( \sum_{k=0}^n (-1)^k \binom{1/2}{k} P_k \right) \\ &= \sum_{k,l=0}^n (-1)^{k+l} \binom{-1/2}{k} \binom{1/2}{l} P_{k+l} \\ &= \sum_{k=0}^n (-1)^k \left( \sum_{\nu=0}^k \binom{-1/2}{\nu} \binom{1/2}{k-\nu} \right) P_k = I, \end{aligned}$$

and our claim is proved. We now rewrite

$$\begin{aligned} \left( \sum_{k=0}^n a_k P_k \right)^{-1} &= a_0 P_0 + (a_1 - a_0) P_1 + (a_2 - a_1) P_2 + \cdots + (a_n - a_{n-1}) P_n \\ &= I - \left( \frac{a_0}{2} P_1 + \frac{a_1}{4} P_2 + \cdots + \frac{a_{n-1}}{2n} P_n \right) =: I - Q. \end{aligned}$$

Now  $a$  is an increasing vector with positive entries, therefore each vector

$$\frac{a_k}{2k+2} P_{k+1} a \quad (k = 0, \dots, n-1)$$

is an increasing vector. Thus, with  $c := Qa$  we have  $b = a - c$  where  $a$  and  $c$  are increasing, and from  $a_k, b_k \in [0, 1]$ ,  $c_k \geq 0$  ( $k = 0, \dots, n$ ) we get that all entries of  $c$  are in  $[0, 1]$ . This proves that  $b$  has variation

$$\sum_{k=0}^{n-1} |b_{k+1} - b_k| \leq 2.$$

Next, we consider

$$\begin{aligned} (1-z)S_n(R_n, z) &= (1-z) \sum_{k=0}^n b_k z^k = \sum_{k=0}^n b_k z^k - \sum_{k=1}^{n+1} b_{k-1} z^k \\ &= b_0 - b_n z^{n+1} + \sum_{k=1}^n (b_k - b_{k-1}) z^k, \end{aligned}$$

thus, for each  $z \in \overline{\mathbb{D}}$

$$|(1 - z)S_n(R_n, z)| \leq b_0 + b_n + \sum_{k=1}^n |b_k - b_{k-1}| \leq 1 + 1 + 2 = 4.$$

□

**Remark.** We note that the constant 4 in Theorem 3 is most certainly not optimal. Numerical experiments suggest that the vector  $b$  is always convex, and that

$$|S_n(R_n, z)| \leq \frac{2}{|1 - z|} \quad (n \in \mathbb{N}_0, z \in \overline{\mathbb{D}} \setminus \{1\}).$$

### 3. Universality criterion

To prove Theorem 1 we use the universality criterion of Grosse-Erdmann [6, Theorem 1.57].

**Theorem 4.** *Let  $X$  be a complete metric space,  $Y$  a separable metric space, and  $T_n : X \rightarrow Y, n \in \mathbb{N}_0$ , continuous maps. Denoting  $\mathcal{U} = \mathcal{U}((T_n)_{n \in \mathbb{N}_0})$  as the set of all  $x \in X$  such that*

$$\{T_n x : n \in \mathbb{N}_0\} \text{ is dense in } Y,$$

*the following assertions are equivalent.*

- (1) *The family  $(T_n)_{n \in \mathbb{N}_0}$  is topologically transitive, i.e. for any pair  $U \subseteq X, V \subseteq Y$  of nonempty open sets, there is some  $n \in \mathbb{N}_0$  such that*

$$T_n(U) \cap V \neq \emptyset.$$

- (2) *The set  $\mathcal{U}$  is a dense  $G_\delta$ -subset of  $X$ .*
- (3) *The set  $\mathcal{U}$  is dense in  $X$ .*

In the proof of Theorem 1 we shall apply Theorem 4 to the situation

$$X = A(\mathbb{D}), \quad Y = \mathbb{C}^E, \quad T_n = L_n \quad (n \in \mathbb{N}_0)$$

and check that (1) holds. Here we already observe that the equivalence of (2) and (3) in Theorem 4 shows that Theorem 1 implies Theorem 2.

**PROOF OF THEOREM 2:** Assumption (3) of Theorem 2 implies that  $\tilde{L}_n : X \rightarrow \mathbb{C}^E$  is continuous ( $n \in \mathbb{N}_0$ ). By Theorem 1, the set  $\mathcal{U}((L_n)_{n \in \mathbb{N}_0})$  is dense in  $A(\mathbb{D})$ . Since  $A(\mathbb{D})$  is densely and continuously embedded in  $X$ , we obtain that  $\mathcal{U}((L_n)_{n \in \mathbb{N}_0})$  is dense in  $X$ . Hence also the superset  $\mathcal{U}((\tilde{L}_n)_{n \in \mathbb{N}_0})$  is dense in  $X$ , and is a dense  $G_\delta$ -subset of  $X$  by Theorem 4. □

**Open Problem.** It would be interesting to know whether  $\mathcal{U}((\tilde{L}_n)_{n \in \mathbb{N}_0})$  is a dense  $G_\delta$ -set also in case  $X = H^\infty(\mathbb{D})$ . Clearly  $\mathcal{U}((L_n)_{n \in \mathbb{N}_0}) \subseteq H^\infty(\mathbb{D})$ , but assumption (1) of Theorem 2 is not satisfied.

We prepare the proof of Theorem 1 and note that if, in the situation of Theorem 4,  $D$  is a dense subset of  $X$ , and if  $d$  and  $\rho$  are the metrics on  $X$  and  $Y$ , respectively, then topological transitivity of  $(T_n)_{n \in \mathbb{N}_0}$  is equivalent to the following condition:

$$\forall \varepsilon > 0 \quad \forall (x, y) \in D \times Y \quad \exists (n, z) \in \mathbb{N}_0 \times X : d(x, z) < \varepsilon \wedge \rho(T_n z, y) < \varepsilon.$$

In our situation, we let  $D$  denote the set of all polynomials, which is known to be a dense subset of  $X = A(\mathbb{D})$ , [12, p. 366]. Moreover, let  $E = \{z_k : k \in \mathbb{N}\}$  with  $z_k \neq z_j$  ( $k \neq j$ ) and let  $Y = \mathbb{C}^E$  be endowed with the usual Fréchet metric

$$\rho(v, w) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|v(z_k) - w(z_k)|}{1 + |v(z_k) - w(z_k)|}.$$

**4. Proof of Theorem 1**

Let  $\varepsilon > 0$ ,  $p \in D$ ,  $n_0 := \text{grad } p$  and  $w \in \mathbb{C}^E$ . Let  $m \in \mathbb{N}$  be such that

$$\sum_{k=m+1}^{\infty} \frac{1}{2^k} < \varepsilon.$$

It is sufficient to find a function  $f \in A(\mathbb{D})$  and some  $n \geq n_0$  such that

$$\|f\|_{\infty} < \varepsilon \wedge L_n(f)(z_k) = S_n(f, z_k) = w(z_k) - p(z_k) =: \zeta_k \quad (k = 1, \dots, m).$$

Once such  $n$  and  $f$  are known we set  $g := f + p$  and obtain  $\|p - g\|_{\infty} < \varepsilon$  and

$$\begin{aligned} \rho(L_n(g), w) &= \rho((S_n(f, \cdot) + p)|_E, w) \\ &= \rho(S_n(f, \cdot)|_E, w - p|_E) \leq \sum_{k=m+1}^{\infty} \frac{1}{2^k} < \varepsilon. \end{aligned}$$

To construct  $f$  and  $n$  we set  $\zeta = (\zeta_1, \dots, \zeta_m)^\top$  and we make the ansatz

$$f(z) = \lambda_1 R_n(\overline{z_1} z) + \dots + \lambda_m R_n(\overline{z_m} z)$$

where  $\lambda_1, \dots, \lambda_m \in \mathbb{C}$  have to be chosen. Recalling that  $\|R_n(\overline{z_k} \cdot)\|_{\infty} = 1$  ( $k = 1, \dots, m$ ) we already find

$$\|f\|_{\infty} \leq \|\lambda\|_1$$

where  $\|\cdot\|_1$  denotes the  $l^1$ -norm on  $\mathbb{C}^m$ .

Now we consider the  $m \times m$ -matrix  $Q_n$  with entries

$$q_{kj}^{(n)} = S_n(R_n(\overline{z_j} \cdot), z_k) \quad (k, j \in \{1, \dots, m\}).$$

Note that if  $\lambda = (\lambda_1, \dots, \lambda_m)^\top$  solves  $Q_n \lambda = \zeta$ , then

$$S_n(f, z_k) = \zeta_k \quad (k = 1, \dots, m).$$

Moreover

$$q_{kk}^{(n)} = S_n(R_n(\overline{z_k} \cdot), z_k) = S_n(R_n, 1) = \gamma_n, \quad (k = 1, \dots, m),$$

and according to Theorem 3 we have, for  $k \neq j$ ,

$$|q_{kj}^{(n)}| = |S_n(R_n(\overline{z_j} \cdot), z_k)| \leq \frac{4}{|1 - \overline{z_j} z_k|} = \frac{4}{|z_k - z_j|} \leq c,$$

where

$$c := \max \left\{ \frac{4}{|z_k - z_j|} : j, k = 1, \dots, m, k \neq j \right\}$$

does not depend on  $n$ .

Since  $\gamma_n \rightarrow \infty$  ( $n \rightarrow \infty$ ) we thus find

$$I - \frac{Q_n}{\gamma_n} \rightarrow 0 \quad (n \rightarrow \infty).$$

So for large  $n$  we have by Neumann's series

$$Q_n^{-1} = \frac{1}{\gamma_n} \left( I - \left( I - \frac{Q_n}{\gamma_n} \right) \right)^{-1} = \frac{1}{\gamma_n} \sum_{r=0}^{\infty} \left( I - \frac{Q_n}{\gamma_n} \right)^r,$$

and we conclude that  $Q_n^{-1} \rightarrow 0$  as  $n \rightarrow \infty$ . In particular, we can choose  $n \in \mathbb{N}$  such that

$$\|\lambda\|_1 = \|Q_n^{-1} \zeta\|_1 < \varepsilon.$$

This ends the proof. □

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