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ALAN DOW

Abstract. We prove that ♦$^*$ implies there is a zero-dimensional Hausdorff Lindelöf space of cardinality $2^\aleph_1$ which has points $G_\delta$. In addition, this space has the property that it need not be Lindelöf after countably closed forcing.

Keywords: Lindelöf; forcing

Classification: 54D20, 54A25

1. Introduction

The set-theoretic principle ♦$^*$ was formulated by Jensen ([2, p. 128] and [9, VI #16, p. 181]).

Definition 1.1. ♦$^*$ is the statement that there are countable $A_\alpha \subset P(\alpha)$, for $\alpha \in \omega_1$, such that for every $A \subset \omega_1$ there is a cub $C \subset \omega_1$ such that $A \cap \alpha \in A_\alpha$ for all $\alpha \in C$.

Definition 1.2 ([10]). A Lindelöf space is indestructible if it remains Lindelöf after any countably closed forcing. A Lindelöf space is destructible if it is not indestructible.

Notice that ♦$^*$ implies CH but is consistent with $2^{\aleph_1}$ being arbitrarily large ([9, VII (H18)–(H20) p. 249]). As is well-known, Shelah proved, using forcing, that it is consistent with CH to have Hausdorff zero-dimensional Lindelöf spaces with points $G_\delta$ which had cardinality $\aleph_2$ (see [5]). In establishing the consistency with CH of there being no such spaces with cardinality strictly between $\aleph_1$ and $2^{\aleph_1}$, Shelah also established the relevance of the notion of a space being destructible (see [5]). I. Gorelić [4] produced another forcing construction to establish the consistency of the existence of Lindelöf spaces with points $G_\delta$ which had cardinality $2^{\aleph_1}$ while allowing $2^{\aleph_1}$ to be as large as desired. F. Tall [10] points out that each of these examples is indestructible. R. Knight [8] extended the Shelah style construction in models of GCH with special $L$-like combinatorial structures (flat morasses) and constructed an example of cardinality $\aleph_\omega$. Close inspection of Lemma 3.5.2 of [8] shows that this example is also indestructible. Finally, let us mention that Juhasz [6] constructed a non-Hausdorff example in ZFC which (see [10]) is destructible.

In this note we will prove
Theorem 1.3. \( \diamondsuit^* \) implies there is a space that is zero-dimensional Hausdorff Lindelöf destructible of cardinality \( 2^{\aleph_1} \) and that has points \( G_\delta \).

This is the first consistent example of a Lindelöf Hausdorff destructible space with points \( G_\delta \).

Question 1. Does every Lindelöf Hausdorff destructible space have cardinality at least \( 2^{\aleph_1} \)?

2. A Lindelöf tree

We build our space \( X \) using the structure \( 2^{\leq \omega_1} \). For each \( t \in 2^{\leq \omega_1} \) let \([t]\) denote the set \( \{ s \in 2^{\leq \omega_1} : t \subseteq s \} \). For any \( t \in 2^{\leq \omega_1} \) such that \( \text{dom}(t) \) is a successor, let \( t^\dagger \) be the other immediate successor of the immediate predecessor of \( t \), i.e. \( t \) and \( t^\dagger \) are the two immediate successors of \( t \cap t^\dagger \). For distinct functions \( \rho, \psi \) in the tree \( 2^{\leq \omega_1} \), we will let \( \rho \wedge \psi \) denote the maximal element of \( 2^{\leq \omega_1} \) which is an initial segment of each of them. Let \( \sigma \) denote the standard topology on \( 2^{\leq \omega_1} \) that has the family

\[
\{ \emptyset \} \cup \{ [\rho \upharpoonright \xi + 1] : \xi \in \omega_1, \rho \in 2^{\omega_1} \} \cup \\
\{ [t \upharpoonright \xi + 1] \setminus ([t^-0] \cup [t^-1]) : \xi \in \text{dom}(t), t \in 2^{\leq \omega_1} \}
\]

as a subbase. Of course \( t \) is isolated and \([t]\) is clopen for all \( t \) such that \( \text{dom}(t) \in \omega_1 \) is not a limit.

This next lemma is very well-known but since it is crucial to our construction, we include a proof.

Lemma 2.1. The topology \( \sigma \) on \( 2^{\leq \omega_1} \) is compact zero-dimensional and Hausdorff. Also, for each \( \alpha \in \omega_1, 2^{\leq \alpha} \) is a compact first-countable subspace.

Proof: One standard method of proof is to construct a canonical embedding of \( 2^{\leq \omega_1} \) into \( 2^{2^{\omega_1}} \) and show that the range is closed in the product topology. However, we will give a more direct proof. Certainly \( \sigma \) is zero-dimensional since the members of the generating subbase are easily shown to also be closed. If \( s, t \) are distinct elements of \( 2^{\leq \omega_1} \), we show they have disjoint neighborhoods. If \( t \subseteq s \), then, for any \( \xi \in \text{dom}(t) \), \( t \in [t \upharpoonright \xi + 1] \setminus ([t^-0] \cup [t^-1]) \) and \( s \in ([t^-0] \cup [t^-1]) \). Otherwise, we may assume that \( y = s \wedge t \) is strictly below each of \( s \) and \( t \), and note that \([y^-0]\) and \([y^-1]\) are disjoint and each contains one of \( s, t \).

Now assume that \( \mathcal{U} \) is a cover by basic open sets. Let \( T_\mathcal{U} \) denote the set of all \( t \in 2^{\leq \omega_1} \) such that there is no finite subcollection of \( \mathcal{U} \) whose union contains \([t]\). If \( \emptyset \notin T_\mathcal{U} \) then \( \mathcal{U} \) has a finite subcover. So assume that \( T_\mathcal{U} \) is not empty. Observe that if \( t \in T_\mathcal{U} \), then \( t \upharpoonright \xi \in T_\mathcal{U} \) for all \( \xi \in \text{dom}(t) \). For each \( \rho \in 2^{\omega_1} \), there is a \( \xi \in \omega_1 \) such that \( [\rho \upharpoonright \xi + 1] \in \mathcal{U} \), so we have that \( T_\mathcal{U} \) is a subtree of \( 2^{\omega_1} \) with no uncountable branch. Similarly, \( T_\mathcal{U} \) has no maximal elements, since if each of \([t^-0]\) and \([t^-1]\) are covered by a finite union from \( \mathcal{U} \), then certainly, \([t] = \{t\} \cup [t^-0] \cup [t^-1]\) is as well. Choose any maximal chain \( \{t_\xi : \xi \in \alpha\} \subset T_\mathcal{U} \) and let \( t = \bigcup \{t_\xi : \xi \in \alpha\} \). Since \( T \) has no maximal elements, \( t \) is on a limit level
and \( U \) contains a finite cover of \([t]\). But in addition, there is some \( \xi < \alpha \) such that \([t_\xi] \setminus ([t^-0] \cup [t^-1]) \) is in \( U \). This is a contradiction, since it shows that \( U \) has a finite cover of \([t_\xi] \) – contradicting that \( t_\xi \in T_U \).

It is obvious that \( 2^{\leq \alpha} \) is a closed subset of \( 2^{\leq \omega_1} \), and, for each non-isolated \( t \in 2^{\leq \alpha} \), the collection \( \{ [t \uparrow \xi + 1] \setminus ([t^-0] \cup [t^-1]) : \xi \in \text{dom}(t) \} \) is a neighborhood base at \( t \).

Next we consider Lindelöf subspaces.

**Lemma 2.2.** If \( Y \subset 2^{<\omega_1} \) satisfies that \( Y \cap 2^\alpha \) is countable for all \( \alpha \in \omega_1 \), then the complement of \( Y \) in \( 2^{\leq \omega_1} \) is Lindelöf in the topology induced by \( \sigma \).

**Proof:** Assume that \( U \) is a cover of \( 2^{\leq \omega_1} \setminus Y \) by basic clopen sets. Let us again set \( T_U \) to be the set of \( t \in 2^{<\omega_1} \) such that \( U \) contains a countable cover of \([t] \setminus Y \). As in the proof of Lemma 2.1, \( T_U \) (if non-empty) is downwards closed, has no maximal elements, and no uncountable branches. Now let us show that \( T_U \) is branching. Suppose that \( T_U \cap [t] \) is a chain. Then it is a countable chain (with supremum in \( Y \)), and let \( \{ t_\gamma : \gamma \in \alpha \} \) be an enumeration in increasing order and let \( t_\alpha \) denote the union. For each \( \gamma \in \alpha \), we have that \( t_{\gamma + 1}^1 \) is not in \( T_U \), and so there is a countable \( U_\gamma \subset U \) whose union covers \( \{ t_\gamma \cup [t_{\gamma + 1}^1] \setminus Y \) Furthermore there is a countable \( U_\alpha \subset U \) that covers \([t_\alpha] \setminus Y \). It should be clear that \( \bigcup \{ U_\gamma : \gamma \leq \alpha \} \) covers \([t] \).

Now we have established that \( T_U \) is branching and has no maximal elements. Set \( t_0 = \emptyset \) and by recursion on \( s \in 2^{<\omega} \), choose \( t_s \in T_U \) so that for \( s \in 2^{<\omega} \), \( t_s \subset (t_{s-0} \setminus t_{s-1}) \) and \( t_{s-0} \perp t_{s-1} \). Let \( \delta \in \omega_1 \) so that \( \{ t_s : s \in 2^{<\omega} \} \subset 2^{<\delta} \). Choose any \( x \in 2^\omega \) so that \( t_x = \bigcup_n t_x \upharpoonright n \in 2^{<\delta} \setminus Y \). By construction, \( \text{dom}(t_x) \) is a limit ordinal. Choose any \( \xi \in \text{dom}(t_x) \) so that \( [t_x \uparrow \xi + 1] \setminus ([t_x^-0] \cup [t_x^-1]) \) is contained in some \( U \in U \). Fix \( n \) so that \( \xi \subset \text{dom}(t_x \upharpoonright n) \), and choose any \( s \in 2^{<\omega} \) so that \( x \upharpoonright n \subset s \) and \( s \not\subset x \). Finally we can conclude that \( T_U \) must be empty, since we have that \([t_s] \subset U \).

3. **Points** \( G_\delta \)

Let \( \{ A_\alpha : \alpha \in \omega_1 \} \) be a sequence as in Definition 1.1 witnessing the statement \( \Diamond^* \).

**Definition 3.1.** For each limit \( \alpha \in \omega_1 \) let \( S_\alpha = \{ t \in 2^\alpha : t^{-1}(1) \in A_\alpha \} \). For \( 0 < \alpha \) not a limit, let \( S_\alpha \) be the empty set, and let \( S_0 = \{ \emptyset \} \).

**Lemma 3.2.** For each \( \rho \in 2^{\omega_1} \), there is a club \( C_\rho \subset \omega_1 \) such that \( C_\rho \subset \{ \alpha : \rho \upharpoonright \alpha \in S_\alpha \} \).

**Proof:** This is just a restatement of the fact that the sequence \( \{ A_\alpha : \alpha \in \omega_1 \} \) is a \( \Diamond^* \) sequence.

For each \( \rho \in 2^{\omega_1} \) fix a cub \( C_\rho \) as in Lemma 3.2.
Proposition 3.3. For each $\rho \in 2^{\omega_1}$, there is a countable-to-one function $f_\rho : \omega_1 \to 2^\omega$ so that for each $x \in 2^\omega$, there is a $\delta_x \in C_\rho \cup \{0\}$ and $\delta_x < \gamma_x \in C_\rho$ so that $f_\rho^{-1}(x)$ is equal to the interval $[\delta_x, \gamma_x)$.

Proof: First let $\{\delta_x : x \in 2^\omega\}$ be any enumeration of $C_\rho \cup \{0\}$. For each $x \in 2^\omega$, define $\gamma_x$ to be $\min(C_\rho \setminus [0, \delta_x])$. Assume that $\delta_x < \delta_y$. Then it is obvious that $\gamma_x \leq \delta_y$. Now define $f_\rho$ so that $f_\rho([\delta_x, \gamma_x)) = \{x\}$ for all $x \in 2^\omega$. \hfill \Box

Now we are ready to prove our main theorem.

Proof of Theorem 1.3: Fix the sequence $\{S_\alpha : \alpha \in \omega_1\}$ as in Definition 3.1, and let $Y$ equal the union of this family. Our space $X$ will have as its base set $(2^{\omega_1} \times 2^\omega) \cup 2^{<\omega_1} \setminus Y$. We will use the fact (Lemma 2.2) that $2^{<\omega_1} \setminus Y$ is Lindelöf when using the topology $\sigma$. Recall that for each $\rho \in 2^{\omega_1}$ and $\xi \in \omega_1$, $[\rho \restriction \xi + 1] \setminus Y$ is a clopen set. In this proof, for any $s \in 2^{<\omega}$, we will use $[s]_{2^\omega}$ to denote the set $\{x \in 2^\omega : s \subset x\}$.

We define a clopen base for the topology $\tau$. For each $t \in 2^{<\omega_1}$, we use the notation $[t]_X$ to denote

$$[t]_X = [t] \cap (2^{<\omega_1} \setminus Y) \cup ([t] \cap 2^{\omega_1}) \times 2^\omega.$$ 

Again, for each $\rho \in 2^{\omega_1}$ and each $\xi \in \omega_1$, the set $[\rho \restriction \xi + 1]_X$ is declared to be a clopen set in $\tau$ (i.e. $[\rho \restriction \xi + 1]_X$ and its complement are in $\tau$). Let us observe that for $t \in Y$, $[t]_X$ is equal to $[t\setminus 0]_X \cup [t\setminus 1]_X$ and so is also clopen.

Next, for each $\rho \in 2^{\omega_1}$ and each $x \in 2^\omega$, let $f_\rho^{-1}(\{x\})$ be denoted as $[\delta_x^\rho, \gamma_x^\rho)$ as per Proposition 3.3. For $s \in 2^{<\omega}$, and $\gamma \in C_\rho$, we define

$$U(\rho, s, \gamma) = ([\rho] \times [s]_{2^\omega}) \cup \{[\rho \restriction \delta_x^\rho]_X \setminus [\rho \restriction \gamma_x^\rho]_X : x \in [s]_{2^\omega} \text{ and } \gamma \leq \delta_x^\rho\}.$$

When the choice of $\rho$ is clear from the context, we will use $\delta_x, \gamma_x$ as referring to $\delta_x^\rho, \gamma_x^\rho$. The topology $\tau$ will also contain each such $U(\rho, s, \gamma)$. Notice that, for each $\gamma \in C_\rho$ and each $n \in \omega$, the family $\{U(\rho, s, \gamma) : s \in 2^n\}$ is a partition of the clopen set $[\rho \restriction \gamma]_X$, and so each is clopen.

Claim 1. For each $t \in 2^{<\omega_1} \cap X$, the family

$$\{[t \restriction \xi + 1]_X \setminus ([t\setminus 0]_X \cup [t\setminus 1]_X) : \xi \in \text{dom}(t)\}$$

is a neighborhood base for $t$.

To show this we must consider some $\rho, s, \gamma$ such that $t \in U(\rho, s, \gamma)$ and $\gamma \in C_\rho$. There is a unique $x \in 2^\omega$ such that $t \in [\rho \restriction \delta_x]_X \setminus [\rho \restriction \gamma_x]_X$. Since $\rho \restriction \delta_x \in Y$, we know that $t \not= \rho \restriction \delta_x$. Since $[\rho \restriction \delta_x]_X \setminus [\rho \restriction \gamma_x]_X$ contains $[t \restriction \delta_x + 1]_X \setminus (\text{dom}(t) \cup [t\setminus 1]_X)$, we have proven the claim.

Claim 2. For each $\rho \in 2^{\omega_1}$ and $z \in 2^\omega$, the point $(\rho, z)$ is the only element of the intersection of the family $\{U(\rho, z \restriction n, \gamma_z) : n \in \omega\}$.
It is clear that for any $\gamma \in C_\rho$, $U(\rho, s, \gamma) \cap \{\rho\} \times [s]_{2^{\omega}}$ is equal to $\{\rho\} \times [s]_{2^{\omega}}$. Now suppose that $\psi \in 2^{\omega_1} \setminus \{\rho\}$ and $t \in X \cap 2^{< \omega_1}$. Let $\rho \restriction \xi_\psi = \psi \cap \rho$ and $\rho \restriction t \triangleleft t \cap \rho$. Choose any $s \in 2^{< \omega}$ so that $z \in [s]_{2^{\omega}}$ and neither of $f_\rho(\xi_\psi)$, $f_\rho(\xi_\psi)$ are in $[s]_{2^{\omega}} \setminus \{z\}$. But now, if $\gamma_z \leq \xi$ then $f_\rho(\xi) \neq z$. Therefore, for all $x \in [s]_{2^{\omega}}$ with $\gamma_z \leq \gamma_x$, we have that $\{\xi_\psi \restriction x\}$ is disjoint from $[\delta_x, \gamma_x \restriction x]$ and therefore $[\rho \restriction \delta_x \restriction X \setminus [\rho \restriction \gamma_x \restriction X$ is disjoint from $\{t\} \cup (\{\psi\} \times 2^{\omega})$. This completes the proof of the claim.

Let $\Phi$ be the canonical map from $X$ (with topology $\tau$) onto $2^{< \omega_1} \setminus Y$ (with topology $\sigma$). That is, $\Phi(t) = t$ for all $t \in X \cap 2^{< \omega_1}$, and $\Phi((\rho, x)) = \rho$ for all $\rho \in 2^{\omega_1}$ and $x \in 2^{\omega}$. It is evident that point preimages under $\Phi$ are compact. It is immediate that $\Phi$ is continuous since $\Phi^{-1}[t] = [t]_X$ for all $t \in 2^{< \omega_1}$. This is also useful to show that $\Phi$ is closed. By [3, 1.4.13] it is sufficient to show that if $U \subset X$ is an open set containing a fiber $\Phi^{-1}(t)$ for some $t \in 2^{< \omega_1} \setminus Y$, then there is a neighborhood $W$ of $t$ such that $\Phi^{-1}(W)$ is contained in $U$. Let then, $t \in 2^{< \omega_1} \setminus Y$ and suppose that $U \subset X$ is an open set containing $\Phi^{-1}(t)$. This is obvious if $t \in 2^{< \omega_1}$, so suppose that $t = \rho \in 2^{\omega_1}$. Since $\Phi^{-1}(\rho)$ is simply $\{\rho\} \times 2^{\omega}$, it is clear that there is $\gamma \in C_\rho$ and $n \in \omega$ such that $U(\rho, s, \gamma) \subset U$ for each $s \in 2^n$. As remarked above, this implies that $[\rho \restriction \gamma \restriction X$ is contained in $U$. Since $[\rho \restriction \gamma \restriction X$ is a neighborhood of $\rho$ and, again, $[\rho \restriction \gamma \restriction X = \Phi^{-1}([\rho \restriction \gamma])$, this completes the proof that $\Phi$ is a closed mapping.

Now that we have established that there is a perfect map (continuous, closed, point-preimages compact) from $X$ onto a Lindelöf space, we conclude [3, 3.8.8] that $X$ is also Lindelöf.

Finally, it is immediate that the forcing notion $2^{< \omega_1}$ will introduce a new member $\psi$ of $2^{\omega_1}$. Since the forcing adds no new members to $2^{< \omega_1}$, the set $\{\psi \restriction \xi + 1 : \xi \in \omega_1\}$ is a subset of $X$ and has no complete accumulation point in $X$. We conclude that $X$ is not Lindelöf in the forcing extension. □

4. Remarks on consistency

Let us consider the following principle which is evidently weaker than $\diamond^*$.

**Definition 4.1.** $w \diamond^*$ is the statement that there is a subset $Y \subset 2^{< \omega_1}$ such that

1. for each $\alpha \in \omega_1$, $Y \cap 2^{< \alpha}$ contains no perfect set,
2. for each $\rho \in 2^{\omega_1}$, there is a cub $C_\rho \subset \omega_1$ such that $\{\rho \restriction \gamma : \gamma \in C_\rho\}$ is contained in $Y$.

Say that the set $Y$ is a $w \diamond^*$ sequence.

The hypothesis “CH and $w \diamond^*$” is sufficient to prove Theorem 1.3. It is probable that this is a weaker statement than $\diamond^*$ but, just as $\diamond^*$ sequence is destroyed by forcing with $2^{< \omega_1}$ (see [9, p. 300 J5]), so too is a $w \diamond^*$-sequence. This implies that $w \diamond^*$ fails in the models in which it has been shown that any Lindelöf points $G_\delta$ space of cardinality greater than $\omega_1$ must be destructible. In particular, such a model (see [10]) is obtained by countably closed forcing that collapses a supercompact cardinal to $\aleph_2$. It is reasonable to conjecture that in that model Lindelöf
spaces with points $G_\delta$ will have cardinality at most $\aleph_1$, and the approach till now has focused on trying to show that there are (in ZFC) no destructible Lindelöf spaces with points $G_\delta$. However there is a stronger property that any ZFC example of such space must have which we now define. A space with character at most $\omega_1$ would have to have this first property.

**Definition 4.2.** Say that a regular Lindelöf space with points $G_\delta$ is **reconstructible** if it is destructible and there is a countably closed poset so that in the forcing extension, it is no longer Lindelöf but it can be embedded into a regular Lindelöf space with points $G_\delta$.

It may not be as natural, but there is a similar, but weaker, property which is the property we are really after. We use the word elementarily in reference to the set-theoretic notion of elementary extensions of models.

**Definition 4.3.** Say that a regular Lindelöf space $X$ with points $G_\delta$ is **elementarily reconstructible** if there is a countably closed poset so that in the forcing extension, it is no longer Lindelöf and there is a regular Lindelöf space $Y$ with points $G_\delta$ that has a dense subspace $Z$ and a continuous mapping $f$ from $Z$ onto $X$ and satisfies that $f$ is a homeomorphism on the pre-image of the points with character at most $\omega_1$.

Clearly an elementarily reconstructible space that has character at most $\omega_1$ will be reconstructible. A reader of Tall’s paper [10] will realize that in the forcing extension mentioned above, if there is a Lindelöf space $X$ with points $G_\delta$ and character at most $\omega_1$ which has cardinality greater than $\omega_1$ then this will imply the consistency of there being regular Lindelöf spaces that are elementarily reconstructible. It may possibly be true that $X$ itself will be elementarily reconstructible, but we do not know\(^1\) if a supercompact cardinal is sufficient for this claim. However, we can prove, sketched below in Proposition 4.6, that a 2-huge cardinal (see [7, p. 331]) is sufficient.

On the other hand, not only does the poset $2^{<\omega_1}$ render our space to be non-Lindelöf, it also creates a subspace which cannot be embedded into a Lindelöf space with points $G_\delta$.

**Proposition 4.4.** If $Y \subset 2^{<\omega_1}$ is a $w^{\Diamond^*}$-sequence, then in the forcing extension by $2^{<\omega_1}$, there is a $\psi \in 2^{\omega_1}$ such that $T_\psi(Y) = \{\alpha : \psi \upharpoonright \alpha \in Y\}$ is stationary.

Since $\{\psi \upharpoonright \alpha : \alpha \in T_\psi(Y)\}$, as a subspace of $2^{<\omega_1}$, is homeomorphic to $T_\psi(Y)$ as a subspace of the ordinal $\omega_1$, this next proposition shows that our space $X$ is not reconstructible.

**Proposition 4.5.** If $S$ is a stationary subset of $\omega_1$, then $S$ cannot be embedded in a Lindelöf space with points $G_\delta$.

**Proof:** Assume that $Z$ is a Lindelöf space with $S$ as a subspace. Since $S$ cannot equal a union of non-stationary sets, and $Z$ is Lindelöf, there is a point $z$ of $Z$

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\(^1\)the excellent referee noted the difficulty and suggested huge cardinals
with the property that every neighborhood of \( z \) meets \( S \) in a non-stationary set. Let us show that \( z \) is not a \( G_\delta \)-point. Let \( \{ U_n : n \in \omega \} \) be a family of open subsets of \( Z \), each meeting \( S \) in a non-stationary set. Since \( S \) is a subspace, \( S \setminus U_n \) is a closed subset of \( S \) that misses the stationary set \( U_n \). Of course this implies that \( S \setminus U_n \) is countable. This shows that each \( G_\delta \) of \( Z \) that contains \( z \) will also contain many points of \( S \). \( \square \)

Following Kunen [9, VII.3.1], let \( L_\nu'(\kappa) \) denote the standard Silver variant of the Levy collapse of a strongly inaccessible cardinal \( \kappa \) to \( \omega_2 \) with countable conditions. If \( \kappa \) is strongly inaccessible, then \( L_\nu'(\kappa) \) has cardinality \( \kappa \) and satisfies the \( \kappa \)-chain condition. We will need that if \( \lambda < \kappa \) is also strongly inaccessible, then \( L_\nu'(\kappa) \) is isomorphic to the iteration \( L_\nu'(\lambda) \ast L_\nu'(\kappa) \) (see [9, VII.3.5]). A cardinal \( \kappa \) is 2-huge if there is an elementary embedding \( j \) from \( V \) into a submodel \( M \) such that \( \kappa \) is the critical point of \( j \) and \( M \) has the property that every subset of \( M \) with cardinality at most \( j(j(\kappa)) \) is also a member of \( M \). Let us note that \( j(\kappa) \) is a measurable cardinal (see [7, p. 331]). We recall that Arhangelskii [1] showed that every Lindelöf space with points \( G_\delta \) has cardinality less than the first measurable cardinal.

**Lemma 4.6.** Suppose that \( \kappa \) is a 2-huge cardinal and let \( G \) be \( L_\nu'(\kappa) \)-generic. In the forcing extension \( V[G] \), every Lindelöf, points \( G_\delta \), regular space of cardinality greater than \( \aleph_1 \) is reconstructibly Lindelöf.

**Proof:** We work with forcing terminology rather than in the extension \( V[G] \). Suppose that \( \lambda \geq \kappa \) is a cardinal and that there is a \( L_\nu'(\kappa) \)-name \( \check{\tau} \) of a topology on \( \lambda \) that is forced to be Lindelöf, regular, and with points \( G_\delta \). By Arhangelskii’s result and the fact that \( j(\kappa) \) is measurable in \( V[G] \), we have that \( \lambda \) is smaller than \( j(\kappa) \). Now we apply the elementary embedding \( j \) and work briefly in the model \( M \). We have that \( j(\check{\tau}) \) is a \( L_\nu'(j(\kappa)) \)-name of a Lindelöf, points \( G_\delta \) topology on the set \( j(\lambda) \). Following Tall [10], it can be shown that it is forced (in \( M \)) that the closure, \( Y \), of the set \( Z = j[\lambda] = \{ j(\alpha) : \alpha \in \lambda \} \) in the space \( (j(\lambda), j(\check{\tau})) \) is Lindelöf and that \( j^{-1} \) maps \( Z \) continuously onto the space \( (\lambda, \check{\tau}) \) as per the requirements of Definition 4.3. Finally, since \( \lambda < j(\kappa) \), we have that \( j(\lambda) \) is less than the strongly inaccessible cardinal \( j(j(\kappa)) \), and so it follows that the \( L_\nu'(j(\kappa)) \)-name \( j(\check{\tau}) \) is forced to be Lindelöf even in the model \( V \). Finally, from the point of view of the forcing extension by \( L_\nu'(\kappa) \), and the fact that \( L_\nu'(j(\kappa)) \) is isomorphic to \( L_\nu'(\kappa) \ast L_\nu'(j(\kappa)) \), we have that \( X = (\lambda, \check{\tau}) \) is forced by \( L_\nu'(\kappa) \) to be reconstructibly Lindelöf. \( \square \)

We close with the obvious question.

**Question 2.** Does CH imply there is a regular Lindelöf space with points \( G_\delta \) that is elementarily reconstructible?

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