Jan van Mill
On nowhere first-countable compact spaces with countable \( \pi \)-weight


Persistent URL: [http://dml.cz/dmlcz/144244](http://dml.cz/dmlcz/144244)

**Terms of use:**

© Charles University in Prague, Faculty of Mathematics and Physics, 2015

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* [http://dml.cz](http://dml.cz)
On nowhere first-countable compact spaces with countable $\pi$-weight

JAN VAN MILL

Abstract. The minimum weight of a nowhere first-countable compact space of countable $\pi$-weight is shown to be $\kappa_B$, the least cardinal $\kappa$ for which the real line $\mathbb{R}$ can be covered by $\kappa$ many nowhere dense sets.

Keywords: $\pi$-weight; nowhere first-countable; $\kappa_B$; compact space

Classification: 54D35

1. Introduction

All spaces under discussion are Tychonoff.

In [4], the author showed that there is a (naturally defined) compact space $X$ which is (topologically) homogeneous under $\text{MA} + \neg \text{CH}$ but not under $\text{CH}$. This space has countable $\pi$-weight, character $\omega_1$ and weight $\mathfrak{c}$. It is an open problem whether there can be a compact nowhere first-countable homogeneous space of countable $\pi$-weight and weight less than $\mathfrak{c}$. This cannot be done by a straightforward modification of the method in [4] since from Juhász [2, Theorem 5] it follows that under $\text{MA}$, every compact space of countable $\pi$-weight and weight less than $\mathfrak{c}$ is somewhere first-countable. Hence a homogeneous compactum of countable $\pi$-weight and weight less than $\mathfrak{c}$ is first-countable under $\text{MA}$ ([4, Theorem 1.5]). Let $\lambda$ be the minimum weight of a nowhere first-countable compact space of countable $\pi$-weight. Clearly, $\omega_1 \leq \lambda \leq \mathfrak{c}$. The aim of this note is to show that $\lambda$ is equal to $\kappa_B$, the least cardinal $\kappa$ for which the real line $\mathbb{R}$ can be covered by $\kappa$ many nowhere dense sets. Hence there exists a nowhere first-countable compact space of weight $\kappa_B$ and countable $\pi$-weight. Whether such a space can be homogeneous while $\kappa_B < \mathfrak{c}$ remains an open problem.

2. Preliminaries

Our basic references are Miller [5], Juhász [1] and Kunen [3].

For every space $X$, define $\kappa_B(X)$ to be the least cardinal $\kappa$ such that $X$ can be covered by $\kappa$ many nowhere dense (in $X$) subsets of $X$. In Miller [5, Lemma 1] it is shown that for every crowded Polish space $X$ we have $\kappa_B(X) = \kappa_B$.

DOI 10.14712/1213-7243.2015.121
Let $\text{MA}_\kappa(\text{countable})$ denote the statement that for any countable partial order $\mathbb{P}$ and family $\mathcal{F}$ of dense subsets of $\mathbb{P}$, if $|\mathcal{F}| < \kappa$, then there exists a $\mathbb{P}$-generic filter $G$ over $\mathcal{F}$. It is well-known, see Miller [5, Lemma 2], that $\kappa_B$ is the greatest $\kappa$ for which $\text{MA}_\kappa(\text{countable})$ holds.

The proof of the following result is standard and is included for the sake of completeness.

**Lemma 2.1 (MA$_{\kappa^+}$(countable)).** Let $X$ be a crowded space of weight at most $\kappa$ and of countable $\pi$-weight. Assume that $D$ is a nowhere dense subset of $X$. Then there exist disjoint open sets $U$ and $V$ in $X$ such that $D \subseteq U \cap V$.

**Proof:** Let $\mathcal{U}$ be a countable $\pi$-base for $X$. Put 

$$ \mathbb{P} = \{ (p, q) : (p, q) \in [\mathcal{U}]^{<\omega}) & (\bigcup p \cap \bigcup q = \emptyset) & (\bigcup p \cup \bigcup q \subseteq X \setminus \overline{D}) \}. $$

Order $\mathcal{P}$ in the natural way by $(p_0, q_0) \leq (p_1, q_1)$ iff $\bigcup p_1 \subseteq \bigcup p_0$ and $\bigcup q_1 \subseteq \bigcup q_0$.

Let $\mathcal{V}$ be an open base for $X$ such that $|\mathcal{V}| \leq \kappa$. Let $\mathcal{W} = \{ V \in \mathcal{V} : V \cap D = \emptyset \}$.

For every $W \in \mathcal{W}$, put 

$$ W^* = \{ (p, q) \in \mathcal{P} : (\bigcup p \cap W \neq \emptyset) & (\bigcup q \cap W \neq \emptyset) \}. $$

We claim that $W^*$ is dense in $\mathcal{P}$. To prove this, take an arbitrary $(p, q) \in \mathcal{P}$. By assumption, $(\bigcup p \cup \bigcup q) \cap \overline{D} = \emptyset$ and $W \cap D \neq \emptyset$. Since $X$ is crowded, there exist $U, V \in \mathcal{U}$ such that 

$$ \overline{U} \cup \overline{V} \subseteq W \setminus (\overline{D} \cup \overline{p} \cup \overline{q}). $$

Hence $p' = p \cup U$ and $q' = q \cup V$ belong to $\mathcal{P}$ and, clearly, $(p', q') \leq (p, q)$. By our assumptions, there is a filter $F$ in $\mathbb{P}$ such that for every $W \in \mathcal{W}$ we have $W^* \cap F \neq \emptyset$. Put 

$$ U = \bigcup \{ p : (\exists q \in \mathcal{W}^{<\omega})(\langle p, q \rangle \in F) \}, $$

and 

$$ V = \bigcup \{ q : (\exists p \in \mathcal{W}^{<\omega})(\langle p, q \rangle \in F) \}, $$

respectively. Then $U$ and $V$ are clearly as required. \qed

It was shown in Miller [5, Theorem 1] that $\kappa_B$ has uncountable cofinality. (Interestingly, Shelah [6] showed that the measure analogue of this may fail.)

### 3. Proofs

Theorem 5 and Lemma 4 in Juhász [2] imply that if $X$ is countably compact, nowhere first-countable, and has a dense set of points of countable $\pi$-character, then $w(X) \geq \kappa_B$. For completeness sake, we include a simple proof of a weaker result which suffices for our purposes.
Lemma 3.1 (Juhász [2]). Let $\kappa$ be a cardinal for which there exists a compact nowhere first-countable space $X$ with countable $\pi$-weight and weight $\kappa$. Then $\kappa_B \leq \kappa$.

PROOF: Let $\mathcal{B}$ be an open base for $X$ such that $|\mathcal{B}| = \kappa$. Moreover, let $\mathcal{U}$ be a countable $\pi$-base for $X$. For every $B \in \mathcal{B}$, put

$$S(B) = \overline{B} \setminus \bigcup \{U \in \mathcal{U} : U \subseteq B\}.$$ 

Since $\mathcal{U}$ is a $\pi$-base, it is clear that for every $B \in \mathcal{B}$ the set $S(B)$ is a nowhere dense closed subset of $X$.

We claim that $\bigcup_{B \in \mathcal{B}} S(B) = X$. To this end, pick an arbitrary $x \in X$. The collection $\mathcal{V} = \{U \in \mathcal{U} : x \in U\}$ is countable. Since $\chi(x, X) > \omega$, there exists $B \in \mathcal{B}$ which contains no $U \in \mathcal{V}$. Hence for every $U \in \mathcal{U}$ which is contained in $B$ it follows that $x \notin U$, i.e., $x \in S(B)$.

There is an irreducible continuous surjection $f : X \to Y$, where the weight of $Y$ is countable. Hence $Y$ is covered by the collection of nowhere dense closed sets

$$\{f(S(B)) : B \in \mathcal{B}\}.$$ 

Clearly $Y$ is crowded since $X$ is. From this we conclude that $\kappa_B \leq \kappa$, as required.

If $X$ is a compact space and $A$ and $B$ are closed subsets of $X$ such that $A \cup B = X$, then $X(A, B)$ denotes the topological sum $(\{0\} \times A) \cup (\{1\} \times B)$ of $A$ and $B$ and $\pi_{A,B} : X(A, B) \to X$ is defined by

$$\pi_{A,B}(t) = \begin{cases} 
  a & (t = (0, a), a \in A), \\
  b & (t = (1, b), b \in B).
\end{cases}$$

Observe that $t \in A \cap B$ if and only if $|\pi_{A,B}^{-1}(\{t\})| \geq 2$ if and only if $|\pi_{A,B}^{-1}(\{t\})| = 2$.

Lemma 3.2. $\pi_{A,B} : X(A, B) \to X$ is irreducible if and only if $A \setminus B$ is dense in $A$ and $B \setminus A$ is dense in $B$.

PROOF: It will be convenient to denote $\{0\} \times A$ and $\{1\} \times B$ by $A'$ and $B'$, respectively. Assume first that $C \subseteq X(A, B)$ is a proper closed set such that $\pi_{A,B}(C) = X$. We may assume without loss of generality that $U = A' \setminus C$ is nonempty. Put $V = \pi_{A,B}(U)$. Then $V$ is a nonempty relatively open subset of $A$. Moreover, if $x \in V$, then there exists $(1, b) \in B'$ such that $B \ni b = \pi_{A,B}((1, b)) = x$. As a consequence, $V \subseteq B$. There is an open subset $W$ in $X$ such that $W \cap A = V$. Since $V \subseteq B$, obviously $W \subseteq B$. Hence $A \setminus B$ is not dense in $A$.

For the converse implication, assume without loss of generality that $A \setminus B$ is not dense in $A$. Then $(\{0\} \times A \setminus \overline{B}) \cup (\{1\} \times B)$ is a proper closed subset of $X_{A,B}$ which is mapped onto $X$ by $\pi_{A,B}$.
Lemma 3.3. There is a nowhere first-countable compact space of weight $\kappa_B$ and countable $\pi$-weight.

Proof: Let $\tau: \kappa_B \to \kappa_B$ be a surjection every fiber of which has size $\kappa_B$. Moreover, let $\{D_\alpha : \alpha < \kappa_B\}$ be a family of closed and nowhere dense subsets of $2^\omega$ covering $2^\omega$. Our space will be the inverse limit $X_{\kappa_B}$ of a continuous inverse system $\{X_\alpha, \beta \leq \alpha < \kappa_B, f^\alpha_\beta\}$ such that $X_0 = 2^\omega$ and for every $\alpha < \kappa_B$ and $\beta \leq \alpha$,

1. $X_\alpha$ is a compact space of weight at most $|\alpha| \cdot \omega$,
2. $f^\alpha_\beta: X_\alpha \to X_\beta$ is a continuous, irreducible surjection,
3. there are closed sets $A_\alpha$ and $B_\alpha$ in $X_\alpha$ such that
   a. $A_\alpha \cup B_\alpha = X_\alpha$,
   b. $A_\alpha \cap B_\alpha \supseteq (f^\alpha_0)^{-1}(D_{\tau(\alpha)})$,
   c. $A_\alpha \setminus B_\alpha$ and $B_\alpha \setminus A_\alpha$ are dense in $A_\alpha$ respectively $B_\alpha$,
   d. $X_{\alpha+1} = X_\alpha(A_\alpha, B_\alpha)$ and $f_\alpha^{\alpha+1} = \pi_{A_\alpha, B_\alpha}$.

The construction of this inverse sequence is a triviality by a repeated application of Lemmas 2.1 and 3.2. The only thing left to verify is that $X_{\kappa_B}$ has weight $\kappa_B$ and is nowhere first-countable.

Striving for a contradiction, assume that $X_{\kappa_B}$ is first-countable at $t$. Since $\kappa_B$ has uncountable cofinality (see §2), there exists $\beta < \kappa_B$ such that

(†) $$(f^\kappa_B)^{-1}(\{f^{\kappa_B}_\beta(t)\}) = \{t\}. $$

Let $\xi < \kappa_B$ be such that $f^\kappa_B(\xi) \in D_\xi$. Pick $\alpha > \beta$ so large that $\tau(\alpha) = \xi$. Then clearly

$$|(f^{\alpha+1}_\alpha)^{-1}(\{f^{\kappa_B}_\alpha(t)\})| = 2,$$

which contradicts (†).

That the weight of $X_{\kappa_B}$ is at most $\kappa_B$ follows by construction. And that it has weight at least $\kappa_B$ is a consequence of Lemma 3.1 and the fact that it is nowhere first-countable. Observe that $X_0$ has countable weight, and that $X_{\kappa_B}$ admits a continuous, irreducible map onto $X_0$. Hence $X_{\kappa_B}$ has countable $\pi$-weight. □

4. Questions

1. Is there in ZFC a homogeneous nowhere first-countable compact space of countable $\pi$-weight and weight $\kappa_B$?
2. What are the cardinals of the form $w(X)$, where $X$ is a nowhere first-countable compactum of countable $\pi$-weight?
   (Let $\Pi$ denote this set of cardinals. We showed that $\kappa_B \in \Pi$. Moreover, $\mathfrak{c} \in \Pi$. To check this, let $X$ be the absolute of the unit interval. Then $X$ has countable $\pi$-weight, is nowhere first-countable, and has weight $\mathfrak{c}$ (since it contains a copy of $\beta \omega$). We do not know whether there can be a cardinal $\kappa \in \Pi \setminus \{\kappa_B, \mathfrak{c}\}$.)
A natural question is whether there can be a $\kappa$ in $\Pi$ of countable cofinality. This question may have a very simple answer. Indeed, assume that there is a sequence

$$\kappa_0 < \kappa_1 < \cdots < \kappa_n < \cdots$$

in $\Pi$. For every $n$ let $X_n$ be a witness of the fact that $\kappa_n \in \Pi$. Then $X = \prod_{n<\omega} X_n$ is a witness that $\kappa = \sup_{n<\omega} \kappa_n \in \Pi$.

REFERENCES


KdV Institute for Mathematics, University of Amsterdam, Science Park 904, P.O. Box 94248, 1090 GE Amsterdam, The Netherlands

E-mail: j.vanMill@uva.nl

URL: http://staff.fnwi.uva.nl/j.vanmill/

(Received January 27, 2015)