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2-DIMENSIONAL PRIMAL DOMAIN DECOMPOSITION THEORY IN DETAIL

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Abstract. We give details of the theory of primal domain decomposition (DD) methods for a 2-dimensional second order elliptic equation with homogeneous Dirichlet boundary conditions and jumping coefficients. The problem is discretized by the finite element method. The computational domain is decomposed into triangular subdomains that align with the coefficients jumps. We prove that the condition number of the vertex-based DD preconditioner is $O((1+\log(H/h))^2)$, independently of the coefficient jumps, where H and h denote the discretization parameters of the coarse and fine triangulations, respectively. Although this preconditioner and its analysis date back to the pioneering work J. H. Bramble, J. E. Pasciak, A. H. Schatz (1986), and it was revisited and extended by many authors including M. Dryja, O. B. Widlund (1990) and A. Toselli, O. B. Widlund (2005), the theory is hard to understand and some details, to our best knowledge, have never been published. In this paper we present all the proofs in detail by means of fundamental calculus.

Keywords: domain decomposition method; finite element method; preconditioning

MSC 2010: 65N55, 65N30, 65F08

1. Introduction

We consider the homogeneous Dirichlet problem for the Poisson equation

$$-\operatorname{div}(\varrho(x)\nabla u(x)) = f(x), \quad x \in \Omega,$$
$$u(x) = 0, \quad x \in \partial\Omega,$$

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where $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain with Lipschitz boundary, $f \in L^2(\Omega)$, and $\varrho \in L^\infty(\Omega)$ is a positive piecewise constant material function. The domain Ω is decomposed into N nonoverlapping open triangular subdomains Ω_i by means of a conforming finite element (FE) discretization $\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega_i}$. This is referred to as the coarse discretization or the domain decomposition (DD). The decomposition aligns with jumps of the material function so that $\varrho(x) = \varrho_i > 0$ for $x \in \Omega_i$. We denote by $\Gamma := \bigcup_{i=1}^{M} \overline{E_i}$ the skeleton of the decomposition, where E_i is the interior of an edge apart from $\partial\Omega$, see Figure 1. We denote the coarse discretization parameter by $H := \max_{i=1,\dots,N} \operatorname{diam}(\Omega_i)$.

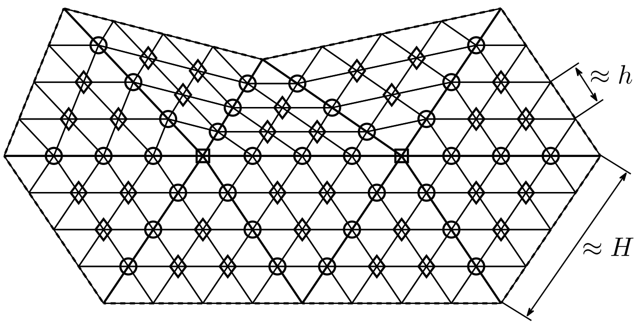

Figure 1. Decomposition of Ω into $N = 10$ subdomains with $n^{\mathrm{V}} = 2$ vertices x_i^{V} (marked by squares); dashed-line depicts $\partial\Omega$; solid-bold-lines denote Γ decomposed into $M = 11$ edges with edge nodes $x_{i,j}^{\mathrm{E}}$ (marked by circles); solid-thin-lines denote the fine triangulation with $n = 65$ nodes; diamonds depict interior nodes $x_{i,j}^{\mathrm{I}}$.

The related weak formulation

$$\text{find } u \in H_0^1(\Omega)\colon\ \underbrace{\sum_{i=1}^{N} \varrho_i \int_{\Omega_i} \nabla u(x) \nabla v(x)\, \mathrm{d}x}_{=:a(u,v)} = \underbrace{\int_{\Omega} f(x) v(x)\, \mathrm{d}x}_{=:b(v)} \quad \forall\, v \in H_0^1(\Omega)$$

is discretized by the conforming finite element (FE) method on a subspace $V := V^h := \langle \varphi_1(x), \dots, \varphi_n(x) \rangle \subset H_0^1(\Omega)$, where $(\varphi_i)_{i=1}^n$ denote the linear Lagrange basis functions related to the nodes depicted in Figure 1. The underlying fine triangulation aligns with the domain decomposition. We arrive at the linear system

$$\mathbf{A}\mathbf{u} = \mathbf{b}, \tag{1.1}$$

where $(\mathbf{A})_{i,j} := a(\varphi_i, \varphi_j)$, $(\mathbf{b})_i := b(\varphi_i)$, and $u^h(x) := \sum_{j=1}^{n} (\mathbf{u})_j \varphi_j(x)$ approximates $u(x)$. By h we denote the fine discretization parameter, which is the maximal fine-triangle diameter.

Primal DD-methods rely on re-sorting the basis functions $(\varphi_i)_{i=1}^n$ into N sets of functions $(\varphi_{i,j}^{\mathrm{I}})_{j=1}^{n_i^{\mathrm{I}}}$, $i = 1, \dots, N$, related to the subdomain interior nodes $x_{i,j}^{\mathrm{I}} \in \Omega_i$, see Figure 1, and a set of functions $(\varphi_k^\Gamma)_{k=1}^{n^\Gamma}$ related to the skeleton nodes $x_k^\Gamma \in \Gamma \backslash \partial\Omega$, each of which either belongs to an edge E_i, $x_k^\Gamma = x_{i,j}^{\mathrm{E}}$, or is a subdomain (coarse) vertex $x_k^\Gamma = x_i^{\mathrm{V}}$, see Figure 1. This translates (1.1) into the saddle-point system

$$
\begin{pmatrix}
\mathbf{A}_1^{\mathrm{I,I}} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{A}_1^{\mathrm{I},\Gamma} \\
\mathbf{0} & \mathbf{A}_2^{\mathrm{I,I}} & \dots & \mathbf{0} & \mathbf{A}_2^{\mathrm{I},\Gamma} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \dots & \mathbf{A}_N^{\mathrm{I,I}} & \mathbf{A}_N^{\mathrm{I},\Gamma} \\
\mathbf{A}_1^{\Gamma,\mathrm{I}} & \mathbf{A}_2^{\Gamma,\mathrm{I}} & \dots & \mathbf{A}_N^{\Gamma,\mathrm{I}} & \mathbf{A}^{\Gamma,\Gamma}
\end{pmatrix}
\begin{pmatrix}
\mathbf{u}_1^{\mathrm{I}} \\ \mathbf{u}_2^{\mathrm{I}} \\ \vdots \\ \mathbf{u}_N^{\mathrm{I}} \\ \mathbf{u}^\Gamma
\end{pmatrix}
=
\begin{pmatrix}
\mathbf{b}_1^{\mathrm{I}} \\ \mathbf{b}_2^{\mathrm{I}} \\ \vdots \\ \mathbf{b}_N^{\mathrm{I}} \\ \mathbf{b}^\Gamma
\end{pmatrix},
\tag{1.2}
$$

where $(\mathbf{A}_k^{\mathrm{I,I}})_{i,j} := a(\varphi_{k,i}^{\mathrm{I}}, \varphi_{k,j}^{\mathrm{I}})$, $(\mathbf{A}_k^{\mathrm{I},\Gamma})_{i,j} = (\mathbf{A}_k^{\Gamma,\mathrm{I}})_{j,i} := a(\varphi_{k,i}^{\mathrm{I}}, \varphi_{k,j}^{\Gamma})$, $(\mathbf{b}_k^{\mathrm{I}})_i := b(\varphi_{k,i}^{\mathrm{I}})$, and $(\mathbf{b}^\Gamma)_i := b(\varphi_i^\Gamma)$. Using a particular-solution approach, (1.2) can be solved in three steps:

1. Solve N independent systems $\mathbf{A}_i^{\mathrm{I,I}}\mathbf{v}_i^{\mathrm{I}} = \mathbf{b}_i^{\mathrm{I}}$, which are FE-counterparts of

$$
\begin{aligned}
-\varrho_i \triangle v_i^{\mathrm{I}}(x) &= f(x), & x &\in \Omega_i, \\
v_i^{\mathrm{I}}(x) &= 0, & x &\in \partial\Omega_i,
\end{aligned}
$$

   on subspaces $V_i := V_i^h := \langle \varphi_{i,1}^{\mathrm{I}}, \dots, \varphi_{i,n_i^{\mathrm{I}}}^{\mathrm{I}} \rangle$.

2. Solve $\mathbf{S}\mathbf{u}^\Gamma = \mathbf{b}^\Gamma - \sum_{i=1}^{N} \mathbf{A}_i^{\Gamma,\mathrm{I}}\mathbf{v}_i^{\mathrm{I}}$, where

$$
\mathbf{S} := \mathbf{A}^{\Gamma,\Gamma} - \sum_{i=1}^{N} \mathbf{A}_i^{\Gamma,\mathrm{I}}(\mathbf{A}_i^{\mathrm{I,I}})^{-1}\mathbf{A}_i^{\mathrm{I},\Gamma}.
\tag{1.3}
$$

3. Solve N concurrent systems $\mathbf{A}_i^{\mathrm{I,I}}\mathbf{w}_i^{\mathrm{I}} = -\mathbf{A}_i^{\mathrm{I},\Gamma}\mathbf{u}^\Gamma$, which are FE-counterparts of

$$
\begin{aligned}
-\varrho_i \triangle w_i^{\mathrm{I}}(x) &= 0, & x &\in \Omega_i, \\
w_i^{\mathrm{I}}(x) &= u^\Gamma(x), & x &\in \partial\Omega_i \cap \Gamma, \\
w_i^{\mathrm{I}}(x) &= 0, & x &\in \partial\Omega_i \cap \partial\Omega,
\end{aligned}
$$

   and set $\mathbf{u}_i^{\mathrm{I}} := \mathbf{v}_i^{\mathrm{I}} + \mathbf{w}_i^{\mathrm{I}}$.

The method can be also viewed in terms of the block LDL^T-factorization

$$
\mathbf{A} = \begin{pmatrix} \mathbf{I}^{\mathrm{I}} & \mathbf{0} \\ \mathbf{A}^{\Gamma,\mathrm{I}}(\mathbf{A}^{\mathrm{I,I}})^{-1} & \mathbf{I}^\Gamma \end{pmatrix}
\begin{pmatrix} \mathbf{A}^{\mathrm{I,I}} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} \end{pmatrix}
\begin{pmatrix} \mathbf{I}^{\mathrm{I}} & (\mathbf{A}^{\mathrm{I,I}})^{-1}\mathbf{A}^{\mathrm{I},\Gamma} \\ \mathbf{0} & \mathbf{I}^\Gamma \end{pmatrix},
$$

where $\mathbf{I}^{\mathrm{I}}$, $\mathbf{I}^{\Gamma}$ denote the identity matrices, $\mathbf{A}^{\mathrm{I},\mathrm{I}}$ and $\mathbf{A}^{\mathrm{I},\Gamma} = (\mathbf{A}^{\Gamma,\mathrm{I}})^T$ are the upper-block-diagonal and off-diagonal part of $\mathbf{A}$, respectively.

The idea of primal DD-preconditioners is to replace the Schur complement $\mathbf{S}$ in Step 2 by an approximation $\hat{\mathbf{S}}$, which is cheap to invert, the condition number $\kappa(\hat{\mathbf{S}}^{-1}\mathbf{S})$ increases modestly with H/h and is independent of $(\varrho_i)_{i=1}^N$.

The primal DD-methods can be viewed as a block Gauss elimination combined with preconditioned Krylov space methods. The idea of re-ordering the nodes dates back to the nested-dissection sparse direct solver developed by George [5]. The base for the analysis of DD-preconditioners was given in a famous series of papers by Bramble, Pasciak, and Schatz, cf. [1]. Analysis in the Schwarz framework was presented by Dryja, Smith, and Widlund [2]. Let us mention at least two other important DD-methods such as balancing DD proposed and analyzed by Mandel and Brezina [6], or finite element tearing and interconnecting proposed by Farhat and Roux [4] and analyzed by Mandel and Tezaur [7]. We refer to the monograph by Toselli and Widlund [9] for a more comprehensive overview.

The aim of this paper is to present a complete theory for the vertex-based DD-preconditioner in 2 dimensions by means of simple calculus. Although many other DD-preconditioners rely on this theory, to our best knowledge it has never been presented in a single paper or a monograph without external references. Neither have we found a complete proof of the 2-dimensional counterpart of the edge lemma, a brief sketch of which is given in [3]. Moreover, we found and corrected an inaccuracy in the proof [1] of a frequently-used discrete Sobolev inequality. We hope that our effort will be of some help to researchers, at a position similar to ours, who need to get a deeper understanding of the theory in order to develop their novel DD-methods.

The rest of the paper is organized as follows: In Section 2, we give the construction of the preconditioner. In Section 3, we present the analysis of the condition number of the DD-preconditioned algebraic system.

2. Vertex-based preconditioner

In Section 1, we re-ordered the basis functions $(\varphi_i)_{i=1}^n$ into N sets of interior functions and a set of skeleton functions, which arrived at (1.2). Similarly we shall now re-order the set of skeleton basis functions $(\varphi_i^\Gamma)_{i=1}^{n^\Gamma}$ into M, the number of edges, sets of functions $(\varphi_{i,j}^{\mathrm{E}})_{j=1}^{n_i^{\mathrm{E}}}$, $i = 1, \ldots, M$, related to the nodes $x_{i,j}^{\mathrm{E}} \in E_i$, see Figure 1, and into a set of functions $(\varphi_i^{\mathrm{V}})_{i=1}^{n^{\mathrm{V}}}$ related to the subdomain vertices $x_i^{\mathrm{V}} \in \Gamma$. This re-ordering induces a permutation of the Schur complement (1.3), still denoted by $\mathbf{S}$,

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}^{\mathrm{E},\mathrm{E}} & \mathbf{S}^{\mathrm{E},\mathrm{V}} \\ \mathbf{S}^{\mathrm{V},\mathrm{E}} & \mathbf{S}^{\mathrm{V},\mathrm{V}} \end{pmatrix}, \tag{2.1}$$

where the E-blocks of rows or columns are associated with the edge functions $\varphi_{i,j}^{\mathrm{E}}$ and the V-blocks are associated with the vertex functions φ_i^{V}. The matrix $\mathbf{S}^{\mathrm{E,E}}$ admits the block structure

$$\mathbf{S}^{\mathrm{E,E}} = \begin{pmatrix} \mathbf{S}_{1,1}^{\mathrm{E,E}} & \dots & \mathbf{S}_{1,M}^{\mathrm{E,E}} \\ \vdots & \ddots & \vdots \\ \mathbf{S}_{M,1}^{\mathrm{E,E}} & \dots & \mathbf{S}_{M,M}^{\mathrm{E,E}} \end{pmatrix}, \tag{2.2}$$

where $(\mathbf{S}_{i,j}^{\mathrm{E,E}})_{k,l}$ is related to the interaction of the basis functions $\varphi_{i,k}^{\mathrm{E}}$ and $\varphi_{j,l}^{\mathrm{E}}$. From (1.3) we can see that the block structure is sparse, since $\mathbf{S}_{i,j}^{\mathrm{E,E}}$ is zero if there is no subdomain adjacent to both E_i and E_j.

Denote the overall number of interior edge nodes by $n^{\mathrm{E}} := \sum_{i=1}^{M} n_i^{\mathrm{E}}$. We introduce the matrix

$$\mathbf{R}^{\mathrm{E}} = (\mathbf{R}_1^{\mathrm{E}}, \dots, \mathbf{R}_M^{\mathrm{E}}) \in \mathbb{R}^{n^{\mathrm{V}} \times n^{\mathrm{E}}}, \quad \mathbf{R}_i^{\mathrm{E}} \in \mathbb{R}^{n^{\mathrm{V}} \times n_i^{\mathrm{E}}},$$

the transpose of which linearly interpolates the function values from the coarse vertices x_k^{V} into interior nodes $x_{i,j}^{\mathrm{E}}$ of an associated edge E_i. That means the entries of $\mathbf{R}^{\mathrm{E}}$ are given by the values of the coarse-space basis functions

$$(\mathbf{R}_i^{\mathrm{E}})_{k,j} = \varphi_k^H(x_{i,j}^{\mathrm{E}}), \tag{2.3}$$

where $(\varphi_i^H)_{i=1}^{n^{\mathrm{V}}}$ are the FE-functions uniquely defined by the values at the vertices x_i^{V} of the domain decomposition. We change the base $(\varphi_i^{\mathrm{V}})_{i=1}^{n^{\mathrm{V}}}$ to $(\varphi_i^H)_{i=1}^{n^{\mathrm{V}}}$ so that

$$\mathbf{S} = \begin{pmatrix} \mathbf{I}^{\mathrm{E}} & \mathbf{0} \\ -\mathbf{R}^{\mathrm{E}} & \mathbf{I}^{\mathrm{V}} \end{pmatrix} \begin{pmatrix} \mathbf{S}^{\mathrm{E,E}} & \widetilde{\mathbf{S}}^{\mathrm{E,V}} \\ \widetilde{\mathbf{S}}^{\mathrm{V,E}} & \widetilde{\mathbf{S}}^{\mathrm{V,V}} \end{pmatrix} \begin{pmatrix} \mathbf{I}^{\mathrm{E}} & -(\mathbf{R}^{\mathrm{E}})^T \\ \mathbf{0} & \mathbf{I}^{\mathrm{V}} \end{pmatrix}, \tag{2.4}$$

where $\mathbf{I}^{\mathrm{E}}$, $\mathbf{I}^{\mathrm{V}}$ are the identity matrices. Now the block $\mathbf{A}^H := \widetilde{\mathbf{S}}^{\mathrm{V,V}}$ is the FE-discretization of the bilinear form $a(u,v)$ in the coarse base.

The primal, so-called vertex-based DD-preconditioner is constructed by neglecting $\widetilde{\mathbf{S}}^{\mathrm{E,V}}$, $\widetilde{\mathbf{S}}^{\mathrm{V,E}}$, and by skipping the off-diagonal blocks in (2.2), i.e.

$$\hat{\mathbf{S}} = \begin{pmatrix} \mathbf{I}^{\mathrm{E}} & \mathbf{0} \\ -\mathbf{R}^{\mathrm{E}} & \mathbf{I}^{\mathrm{V}} \end{pmatrix} \begin{pmatrix} \overline{\mathbf{S}}^{\mathrm{E,E}} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^H \end{pmatrix} \begin{pmatrix} \mathbf{I}^{\mathrm{E}} & -(\mathbf{R}^{\mathrm{E}})^T \\ \mathbf{0} & \mathbf{I}^{\mathrm{V}} \end{pmatrix},$$

where $\overline{\mathbf{S}}^{\mathrm{E,E}} := \mathrm{diag}(\mathbf{S}_{1,1}^{\mathrm{E,E}}, \dots, \mathbf{S}_{M,M}^{\mathrm{E,E}})$.

In each iteration of, e.g., the preconditioned conjugate gradient method an action of $\hat{\mathbf{S}}^{-1}$ is required. We have the formula

$$\begin{aligned} \hat{\mathbf{S}}^{-1} &= \begin{pmatrix} \mathbf{I}^{\mathrm{E}} & (\mathbf{R}^{\mathrm{E}})^T \\ \mathbf{0} & \mathbf{I}^{\mathrm{V}} \end{pmatrix} \begin{pmatrix} (\overline{\mathbf{S}}^{\mathrm{E,E}})^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{A}^H)^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{I}^{\mathrm{E}} & \mathbf{0} \\ \mathbf{R}^{\mathrm{E}} & \mathbf{I}^{\mathrm{V}} \end{pmatrix} \\ &= \sum_{i=1}^{M} \begin{pmatrix} \mathbf{I}_i^{\mathrm{E}} \\ \mathbf{0} \end{pmatrix} (\mathbf{S}_{i,i}^{\mathrm{E,E}})^{-1} (\mathbf{I}_i^{\mathrm{E}}, \mathbf{0}) + \begin{pmatrix} (\mathbf{R}^{\mathrm{E}})^T \\ \mathbf{I}^{\mathrm{V}} \end{pmatrix} (\mathbf{A}^{\mathbf{H}})^{-\mathbf{1}} \underbrace{(\mathbf{R}^{\mathrm{E}}, \mathbf{I}^{\mathrm{V}})}_{=:\mathbf{R}^{\mathbf{H}}}. \end{aligned} \tag{2.5}$$

This results in a modification of Step 2 of the three-steps method.

2a. Set

$$\mathbf{c}^{\Gamma} := \begin{pmatrix} \mathbf{c}^{\mathrm{E}} \\ \mathbf{c}^{\mathrm{V}} \end{pmatrix} := \mathbf{b}^{\Gamma} - \mathbf{A}^{\Gamma,\mathrm{I}}\mathbf{v}^{\mathrm{I}}.$$

2b. Solve M independent local systems $\mathbf{S}_{i,i}^{\mathrm{E,E}}\mathbf{w}_i^{\mathrm{E}} = \mathbf{c}_i^{\mathrm{E}}$.

2c. Solve the global coarse system $\mathbf{A}^H\mathbf{w}^H = \mathbf{c}^{\mathrm{V}} + \mathbf{R}^{\mathrm{E}}\,\mathbf{c}^{\mathrm{E}}$.

2d. Set

$$\hat{\mathbf{u}}^{\Gamma} := \begin{pmatrix} \mathbf{w}^{\mathrm{E}} + (\mathbf{R}^{\mathrm{E}})^T\mathbf{w}^H \\ \mathbf{w}^H \end{pmatrix}.$$

The action of $\hat{\mathbf{S}}^{-1}$ comprises the solution to a global system with the coarse matrix $\mathbf{A}^H$ and the solution to M local edge problems with matrices $\mathbf{S}_{i,i}^{\mathrm{E,E}}$, which are local Schur complements related to the systems

$$\begin{pmatrix} \mathbf{A}_j^{\mathrm{I,I}} & \mathbf{0} & \mathbf{A}_{j,i}^{\mathrm{I,E}} \\ \mathbf{0} & \mathbf{A}_k^{\mathrm{I,I}} & \mathbf{A}_{k,i}^{\mathrm{I,E}} \\ \mathbf{A}_{i,j}^{\mathrm{E,I}} & \mathbf{A}_{i,k}^{\mathrm{E,I}} & \mathbf{A}_i^{\mathrm{E,E}} \end{pmatrix} \begin{pmatrix} \mathbf{w}_j^{\mathrm{I}} \\ \mathbf{w}_k^{\mathrm{I}} \\ \mathbf{w}_i^{\mathrm{E}} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{c}_i^{\mathrm{E}} \end{pmatrix},$$

where the relation of i, j, and k is such that the domains Ω_j and Ω_k are connected via the edge E_i. We solve the system for $\mathbf{w}_i^{\mathrm{E}}$. It is an FE-discretization on the space $V_j + V_k + V_i^{\mathrm{E}}$, where $V_i^{\mathrm{E}} := \langle \varphi_{i,l}^{\mathrm{E}} \rangle_{l=1}^{n_i^{\mathrm{E}}}$, of the following problem solved over the patch $\Omega_j \cup \Omega_k$:

$$\begin{aligned} -\varrho_j \triangle w_j^{\mathrm{I}}(x) &= 0, & x &\in \Omega_j, \\ w_j^{\mathrm{I}}(x) &= 0, & x &\in \partial\Omega_j \setminus E_i, \\ -\varrho_k \triangle w_k^{\mathrm{I}}(x) &= 0, & x &\in \Omega_k, \\ w_k^{\mathrm{I}}(x) &= 0, & x &\in \partial\Omega_k \setminus E_i, \\ w_i^{\mathrm{E}}(x) := w_j^{\mathrm{I}}(x) &= w_k^{\mathrm{I}}(x), & x &\in E_i, \\ \varrho_j \frac{dw_j^{\mathrm{I}}}{dn_j}(x) + \varrho_k \frac{dw_k^{\mathrm{I}}}{dn_k}(x) &= c_i^{\mathrm{E}}(x), & x &\in E_i, \end{aligned}$$

where n_j and n_k denote the outward unit normals to Ω_j and Ω_k, respectively.

The resulting preconditioner admits the factorization

$$\hat{\mathbf{A}} = \begin{pmatrix} \mathbf{I}^{\mathrm{I}} & \mathbf{0} \\ \mathbf{A}^{\Gamma,\mathrm{I}}(\mathbf{A}^{\mathrm{I,I}})^{-1} & \mathbf{I}^{\Gamma} \end{pmatrix} \begin{pmatrix} \mathbf{A}^{\mathrm{I,I}} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{S}} \end{pmatrix} \begin{pmatrix} \mathbf{I}^{\mathrm{I}} & (\mathbf{A}^{\mathrm{I,I}})^{-1}\mathbf{A}^{\mathrm{I},\Gamma} \\ \mathbf{0} & \mathbf{I}^{\Gamma} \end{pmatrix}. \tag{2.6}$$

3. Analysis of the condition number

We shall analyze the condition number $\kappa(\hat{\mathbf{A}}^{-1}\mathbf{A})$ by means of finding spectral bounds $\lambda_{\min} > 0$ and $\lambda_{\max} > 0$ such that

$$\forall\, u \in V\colon\ \lambda_{\min}\hat{a}(u,u) \leqslant a(u,u) \leqslant \lambda_{\max}\hat{a}(u,u),$$

where $\hat{a}(u,u)$ is the quadratic form related to $\hat{\mathbf{A}}$. It will turn out that under shape-regularity and quasi-uniformity of both the coarse and fine discretizations the condition number κ is bounded by $C(1+\ln(H/h))^2$ from above. The constant C as well as all the other generic constants that appear in the theory below are independent of H, h, and $(\varrho_i)_{i=1}^N$.

3.1. Orthogonal space splitting. Let us re-visit the algebraic construction of $\hat{\mathbf{A}}$. First we re-sorted the basis functions according to the interior and skeleton nodes. This leads to

$$V = (V_1 \oplus_a \ldots \oplus_a V_N) + V^\Gamma,$$

where $V^\Gamma := \langle \varphi_1^\Gamma, \ldots, \varphi_{n^\Gamma}^\Gamma \rangle$ and where the a-orthogonality of V_i and V_j, for $i \neq j$, follows from $\Omega_i \cap \Omega_j = \emptyset$.

Now we take into account the transformation of the base determined by the right factor of (2.6). It transforms the basis functions φ_i^Γ to their discrete harmonic extensions $\widetilde{\varphi}_i^\Gamma := \mathcal{H}(\varphi_i^\Gamma)$. Recall that the discrete harmonic extension $\widetilde{u}^\Gamma$ of $u^\Gamma \in V^\Gamma$ is the solution to the problem

$$\text{find } \widetilde{u}^\Gamma \in V\colon\ \widetilde{u}^\Gamma(x) = u^\Gamma(x) \text{ on } \Gamma \quad \text{and} \quad \forall\, j \quad \forall\, v \in V_j\colon\ a(\widetilde{u}^\Gamma, v) = 0.$$

Note that $\widetilde{u}^\Gamma|_{\Omega_j}$, $j = 1, \ldots, N$, is an FE-counterpart of

$$\begin{aligned} -\triangle \widetilde{u}^\Gamma(x) &= 0, & x &\in \Omega_j, \\ \widetilde{u}^\Gamma(x) &= u^\Gamma(x), & x &\in \Gamma \cap \partial\Omega_j, \\ \widetilde{u}^\Gamma(x) &= 0, & x &\in \partial\Omega \cap \partial\Omega_j. \end{aligned}$$

Denoting $\widetilde{V}^\Gamma := \mathcal{H}(V^\Gamma)$ we arrive at the a-orthogonal decomposition

$$V = V_1 \oplus_a \ldots \oplus_a V_N \oplus_a \widetilde{V}^\Gamma.$$

The Schur complement $\mathbf{S}$ is the FE-discretization of the bilinear form

$$s(u^\Gamma, v^\Gamma) := a(\mathcal{H}(u^\Gamma), \mathcal{H}(v^\Gamma)), \quad u^\Gamma, v^\Gamma \in V^\Gamma,$$

in the base $(\varphi_i^\Gamma)_{i=1}^{n^\Gamma}$. The latter can be deduced from

$$\mathbf{S} = \begin{pmatrix} -\mathbf{A}^{\Gamma,\mathrm{I}}(\mathbf{A}^{\mathrm{I},\mathrm{I}})^{-1} & \mathbf{I}^\Gamma \end{pmatrix} \begin{pmatrix} \mathbf{A}^{\mathrm{I},\mathrm{I}} & \mathbf{A}^{\mathrm{I},\Gamma} \\ \mathbf{A}^{\Gamma,\mathrm{I}} & \mathbf{A}^{\Gamma,\Gamma} \end{pmatrix} \begin{pmatrix} -(\mathbf{A}^{\mathrm{I},\mathrm{I}})^{-1}\mathbf{A}^{\mathbf{I},\Gamma} \\ \mathbf{I}^\Gamma \end{pmatrix},$$

where the transformation factors consist of the nodal coordinates of $(\widetilde{\varphi}_i^\Gamma)_{i=1}^{n^\Gamma}$.

Finally, we take a closer look at the last transformation determined by the factor $\mathbf{R}^H$ in (2.5). It transforms functions to the linear interpolation from its vertex values along all the skeleton edges. We denote this interpolation operator by $I^H\colon\ C_0(\Omega) \to C_0(\Omega)$, $I^H(v)(x) := \sum_{i=1}^{n^{\mathrm{V}}} v(x_i^{\mathrm{V}})\varphi_i^H(x)$. In particular, $I^H(\varphi_i^{\mathrm{V}}) = \varphi_i^H$, see (2.3). Since the latter are discrete harmonics, we end up with the decomposition

$$V = \underbrace{V_1 \oplus_a \ldots \oplus_a V_N}_{=:V^{\mathrm{I}}} \oplus_a \underbrace{(\widetilde{V}^{\mathrm{E}} + V^H)}_{=\widetilde{V}^\Gamma}, \tag{3.1}$$

where $V^H := I^H(V)$, $\widetilde{V}^E := \mathcal{H}(V - V^H) = \mathcal{H}\Big(\sum_{i=1}^{M} V_i^{\mathrm{E}}\Big)$. Therefore, every $u = u^{\mathrm{I}} + u^{\mathrm{E}} + u^{\mathrm{V}} \in V$ admits the unique decomposition

$$u = \widetilde{u}^{\mathrm{I}} \oplus_a (\widetilde{u}^{\mathrm{E}} + u^H),$$

where $u^H := I^H(u)$, $\widetilde{u}^{\mathrm{E}} = \mathcal{H}(u - u^H)$, and $\widetilde{u}^{\mathrm{I}} := u - \widetilde{u}^{\mathrm{E}} - u^H$. The quadratic forms now read as follows:

$$a(u,u) = \underbrace{\sum_{i=1}^{N} a(\widetilde{u}_i^{\mathrm{I}}, \widetilde{u}_i^{\mathrm{I}})}_{=a(\widetilde{u}^{\mathrm{I}},\widetilde{u}^{\mathrm{I}})} + \underbrace{\sum_{i,j=1}^{M} a(\widetilde{u}_i^{\mathrm{E}}, \widetilde{u}_j^{\mathrm{E}})}_{=a(\widetilde{u}^{\mathrm{E}},\widetilde{u}^{\mathrm{E}})} + 2a(\widetilde{u}^{\mathrm{E}}, u^H) + a(u^H, u^H), \tag{3.2}$$

$$\hat{a}(u,u) = \sum_{i=1}^{N} a(\widetilde{u}_i^{\mathrm{I}}, \widetilde{u}_i^{\mathrm{I}}) + \sum_{i=1}^{M} a(\widetilde{u}_i^{\mathrm{E}}, \widetilde{u}_i^{\mathrm{E}}) + a(u^H, u^H) \tag{3.3}$$

with $\widetilde{u}^{\mathrm{I}} = \sum_{i=1}^{N} \widetilde{u}_i^{\mathrm{I}}$, $\widetilde{u}_i^{\mathrm{I}} \in V_i$ and $\widetilde{u}^{\mathrm{E}} = \sum_{i=1}^{M} \widetilde{u}_i^{\mathrm{E}}$, $\widetilde{u}_i^{\mathrm{E}} \in \mathcal{H}(V_i^{\mathrm{E}})$.

3.2. Upper bound

Theorem 3.1. *For all $u \in V$ we have*

$$a(u,u) \leqslant 10\hat{a}(u,u),$$

i.e., $\lambda_{\max} := 10$.

Proof. Let us take an arbitrary $u \in V$ and its unique splitting $u = u^{\mathrm{I}} \oplus_a (\widetilde{u}^{\mathrm{E}} + u^H)$. For each skeleton edge E_i we define its edge-neighbourhood

$$N_i := \{j \in \{1, \dots, M\} \colon\ i \neq j \text{ and } \exists k \in \{1, \dots, N\} \colon\ E_i, E_j \subset \partial\Omega_k\}.$$

Since $|N_i| \leqslant 4$, as each skeleton edge E_i is associated with at most four other edges via two subdomains, and $j \in N_i \Leftrightarrow i \in N_j$, using $2a(v,w) \leqslant a(v,v) + a(w,w)$,

$$\begin{aligned} a(\widetilde{u}^{\mathrm{E}}, \widetilde{u}^{\mathrm{E}}) &= \sum_{i=1}^{M}\sum_{j=1}^{M} a(\widetilde{u}_i^{\mathrm{E}}, \widetilde{u}_j^{\mathrm{E}}) = \sum_{i=1}^{M}\Big\{a(\widetilde{u}_i^{\mathrm{E}}, \widetilde{u}_i^{\mathrm{E}}) + \sum_{j\in N_i} a(\widetilde{u}_i^{\mathrm{E}}, \widetilde{u}_j^{\mathrm{E}})\Big\} \\ &\leqslant \sum_{i=1}^{M}\Big\{a(\widetilde{u}_i^{\mathrm{E}}, \widetilde{u}_i^{\mathrm{E}}) + \sum_{j\in N_i} \frac{1}{2}[a(\widetilde{u}_i^{\mathrm{E}}, \widetilde{u}_i^{\mathrm{E}}) + a(\widetilde{u}_j^{\mathrm{E}}, \widetilde{u}_j^{\mathrm{E}})]\Big\} \\ &\leqslant \Big(1 + \frac{4}{2}\Big)\sum_{i=1}^{M} a(\widetilde{u}_i^{\mathrm{E}}, \widetilde{u}_i^{\mathrm{E}}) + \frac{4}{2}\sum_{j=1}^{M} a(\widetilde{u}_j^{\mathrm{E}}, \widetilde{u}_j^{\mathrm{E}}) = 5\sum_{i=1}^{M} a(\widetilde{u}_i^{\mathrm{E}}, \widetilde{u}_i^{\mathrm{E}}). \end{aligned}$$

Using the latter estimate, the mixed term is estimated as follows:

$$a(\widetilde{u}^{\mathrm{E}}, u^H) \leqslant \frac{1}{2}[a(\widetilde{u}^{\mathrm{E}}, \widetilde{u}^{\mathrm{E}}) + a(u^H, u^H)] \leqslant \frac{5}{2}\sum_{i=1}^{M} a(\widetilde{u}_i^{\mathrm{E}}, \widetilde{u}_i^{\mathrm{E}}) + \frac{1}{2}a(u^H, u^H).$$

Combining the estimates completes the proof with $\lambda_{\max} := 10$,

$$a(u,u) \leqslant a(\widetilde{u}^{\mathrm{I}}, \widetilde{u}^{\mathrm{I}}) + 10\sum_{i=1}^{M} a(\widetilde{u}_i^{\mathrm{E}}, \widetilde{u}_i^{\mathrm{E}}) + 2a(u^H, u^H) \leqslant 10\hat{a}(u,u).$$

□

3.3. Shape-regular quasi-uniform triangulations

Assumption 3.1. Let us assume that the fine triangulation is from a family of shape-regular discretizations by which we mean that there exists $\alpha_{\min} \in (0, \pi/3\rangle$ independent of h such that every angle in the FE-triangulation, thus also in the domain decomposition, is bounded by $\alpha_{\min}$ from below. Shape-regularity guarantees the angles to be bounded from above by $\alpha_{\max} := \pi - 2\alpha_{\min}$. From the law of sines we have a uniform upper bound on the ratio between the largest and shortest edge of a triangle T_i or a subdomain Ω_i, i.e.,

$$\frac{h_{\max}^i}{h_{\min}^i}, \frac{H_{\max}^i}{H_{\min}^i} \in \Big\langle 1, \frac{1}{\sin\alpha_{\min}}\Big\rangle. \tag{3.4}$$

For the sake of simplicity we assume that to each x_k^{V} being a corner of Ω_i there is exactly one adjacent triangle T such that $T \subset \Omega_i$.

Assumption 3.2. Let us further assume that both the fine and coarse triangulations are from families of quasi-uniform discretizations by which we mean that there exists a common constant $C_{\rm A2} \in (0,1\rangle$ independent of h and H such that for every triangle T_i and every subdomain Ω_i the diameters $h^i_{\max}$ and $H^i_{\max}$, respectively, are bounded by

$$h^i_{\max} \geqslant C_{\rm A2} h, \quad H^i_{\max} \geqslant C_{\rm A2} H. \tag{3.5}$$

For the sake of simplicity we assume that $H \geqslant 2h$.

We will need a discrete Sobolev inequality for the FE-functions.

Lemma 3.1. *Given a linear function v on a triangle with vertices A, B, C and an angle α at A, we have*

$$\|\nabla v\|^2 \leqslant \frac{2[(v(B)-v(A))^2 + (v(C)-v(A))^2]}{\min\{\|B-A\|^2, \|C-A\|^2\} \sin^2 \alpha}.$$

P r o o f. We introduce the coordinate system such that A is at the origin and the line segment $\overline{AB}$ is the x_1-axis; then

$$\frac{\partial v}{\partial x_1} = \frac{v(B)-v(A)}{\|B-A\|}, \quad \cos\alpha \frac{\partial v}{\partial x_1} + \sin\alpha \frac{\partial v}{\partial x_2} = \frac{v(C)-v(A)}{\|C-A\|} =: \frac{\mathrm{d}v}{\mathrm{d}s}.$$

The assertion follows from the following manipulations:

$$\begin{aligned} \|\nabla v\|^2 &= \Big(\frac{\partial v}{\partial x_1}\Big)^2 + \frac{1}{\sin^2\alpha}\Big(\frac{\mathrm{d}v}{\mathrm{d}s} - \cos\alpha \frac{\partial v}{\partial x_1}\Big)^2 \\ &\leqslant \frac{1}{\sin^2\alpha}\Big[\sin^2\alpha\Big(\frac{\partial v}{\partial x_1}\Big)^2 + 2\Big(\frac{\mathrm{d}v}{\mathrm{d}s}\Big)^2 + 2\cos^2\alpha\Big(\frac{\partial v}{\partial x_1}\Big)^2\Big] \\ &\leqslant \frac{2}{\sin^2\alpha}\Big[\Big(\frac{\partial v}{\partial x_1}\Big)^2 + \Big(\frac{\mathrm{d}v}{\mathrm{d}s}\Big)^2\Big]. \end{aligned}$$

□

Corollary 3.1. *Under Assumptions* 3.1 *and* 3.2 *there exists* $C_{\rm C1} > 0$ *such that*

$$\forall\, i \in \{1,\dots,N\} \quad \forall\, u \in V\colon\ h\|\nabla u\|_{L^\infty(\Omega_i)} \leqslant C_{\rm C1}\|u\|_{L^\infty(\Omega_i)}. \tag{3.6}$$

P r o o f. For $x \in T_j \subset \Omega_i$ with vertices A, B, C, Assumption 3.1 and Lemma 3.1 yield

$$\|\nabla u(x)\| \leqslant \frac{\sqrt{2}}{h^j_{\min}\sin\alpha_{\min}}\sqrt{4u(A)^2 + 2u(B)^2 + 2u(C)^2} \leqslant \frac{4\|u\|_{L^\infty(\Omega_i)}}{h^j_{\min}\sin\alpha_{\min}}.$$

The assertion follows from (3.4) and (3.5):

$$\|\nabla u(x)\| \leqslant \frac{4\|u\|_{L^\infty(\Omega_i)}}{h^j_{\max}\sin^2\alpha_{\min}} \leqslant \frac{1}{h}\underbrace{\frac{4}{C_{\mathrm{A2}}\sin^2\alpha_{\min}}}_{=:C_{\mathrm{C1}}}\|u\|_{L^\infty(\Omega_i)}.$$

□

3.4. Stability of the coarse space. The next lemma is crucial for the stability of the coarse space in the energy norm. We are inspired by the proof of Bramble, Pasciak, and Schatz in [1], L.3.3.

Lemma 3.2. *Under Assumptions* 3.1 *and* 3.2 *there exists* $C_{\mathrm{L2}} > 0$ *such that for all* $i \in \{1, \dots, N\}$

$$\forall\, u \in V\colon\ \|u\|^2_{L^\infty(\Omega_i)} \leqslant C_{\mathrm{L2}}\Big(1 + \ln\frac{H}{h}\Big)\Big(\frac{1}{H^2}\|u\|^2_{L^2(\Omega_i)} + |u|^2_{H^1(\Omega_i)}\Big).$$

P r o o f. Without loss of generality, assume that $\|u\|_{L^\infty(\Omega_i)} = |u(0)|$. We shall find an open cone $\Lambda_{0,KH,\gamma} \subset \Omega_i$ with the vertex at the origin 0, the radius KH and the angle $\gamma := \alpha_{\min}$ with K independent of H. For the construction of $\Lambda_{0,KH,\gamma}$ we refer to Figure 2 and the following description. Denote by d_a, d_b, and d_c the distances of the origin to the prolongations of the sides of Ω_i with lengths a, b, and c, respectively, and assume that d_a is the largest distance. We choose $\widetilde{K}H := d_a$. We take the open cone $\Lambda_{A,\widetilde{K}H,\alpha} \subset \Omega_i$ at the vertex A of Ω_i that is opposite to the side a, where α denotes the angle at A. By moving $\Lambda_{A,\widetilde{K}H,\alpha}$ to the origin, we get the cone $\Lambda_{0,\widetilde{K}H,\alpha} \subset \Omega_i$. It remains to find $K > 0$ independent of H such that $K \leqslant \widetilde{K}$. The area $|\Omega_i|$ can be estimated as

$$|\Omega_i| = \frac{ad_a + bd_b + cd_c}{2} \leqslant \frac{a+b+c}{2}\widetilde{K}H.$$

By (3.5) and (3.4) we have an H-independent estimate for the constant $\widetilde{K}$:

$$\widetilde{K} \geqslant 2\frac{\frac{1}{2}H^i_{\max}H^i_{\min}\sin\alpha_{\min}}{3H^i_{\max}H} \geqslant \frac{C_{\mathrm{A2}}}{3}\frac{H^i_{\min}\sin\alpha_{\min}}{H^i_{\max}} \geqslant \frac{C_{\mathrm{A2}}}{3}\sin^2\alpha_{\min} =: K.$$

The construction of $\Lambda_{0,KH,\gamma}$ is completed by shortening the radius and diminishing the angle of $\Lambda_{0,\widetilde{K}H,\alpha}$.

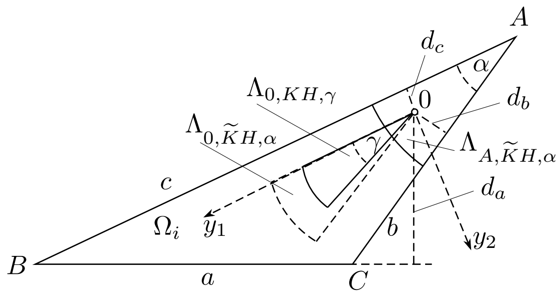

Figure 2. Construction of $\Lambda_{0,KH,\gamma}$.

We consider the coordinate system according to a side of $\Lambda_{0,KH,\gamma}$. For $y(\varrho,\vartheta) = \varrho(\cos\vartheta, \sin\vartheta) \in \Lambda_{0,KH,\gamma}$ the fundamental theorem of calculus gives

$$u(0) = u(y(\varrho,\vartheta)) - \int_0^{\varrho} \underbrace{\nabla u(y(t,\vartheta))(\cos\vartheta, \sin\vartheta)}_{=:u'_t(t,\vartheta)} \, \mathrm{d}t.$$

Integrating ϑ from 0 to γ and applying the triangle inequality, we get

$$\gamma |u(0)| \leqslant \left| \int_0^{\gamma} u(y(\varrho,\vartheta)) \, \mathrm{d}\vartheta \right| + \left| \int_0^{\gamma}\int_0^{\varrho} u'_t(t,\vartheta) \, \mathrm{d}t \, \mathrm{d}\vartheta \right|. \tag{3.7}$$

Choose, independently of h and H, $\delta := \min\{(\sqrt{2}-1)/(\sqrt{2}C_{\mathrm{C1}}), K\}$, where C_{C1} is the constant in (3.6). We shall consider two cases. First, if $\delta h < \varrho$, the Cauchy-Schwarz and triangle inequalities yield

$$\begin{aligned} \gamma |u(0)| \leqslant \sqrt{\gamma} \sqrt{\int_0^{\gamma} u^2(y(\varrho,\vartheta)) \, \mathrm{d}\vartheta} &+ \left| \int_0^{\gamma}\int_0^{\delta h} u'_t(t,\vartheta) \, \mathrm{d}t \, \mathrm{d}\vartheta \right| \\ &+ \left| \int_0^{\gamma}\int_{\delta h}^{\varrho} u'_t(t,\vartheta) \, \mathrm{d}t \, \mathrm{d}\vartheta \right|. \end{aligned} \tag{3.8}$$

The second and third terms in (3.8) can be estimated as follows:

$$\begin{aligned} \left| \int_0^{\gamma}\int_0^{\delta h} u'_t(t,\vartheta) \, \mathrm{d}t \, \mathrm{d}\vartheta \right| &\leqslant \gamma \delta h \|\nabla u\|_{L^\infty(\Omega_i)} \leqslant \gamma \delta C_{\mathrm{C1}} |u(0)|, \\ \left| \int_0^{\gamma}\int_{\delta h}^{\varrho} u'_t(t,\vartheta) \, \mathrm{d}t \, \mathrm{d}\vartheta \right| &= \left| \int_{\Lambda_{0,\varrho,\gamma} \setminus \Lambda_{0,\delta h,\gamma}} \nabla u(y) \frac{y}{\|y\|^2} \, \mathrm{d}y \right| \\ &\leqslant \|\nabla u\|_{L^2(\Omega_i)} \sqrt{\gamma} \sqrt{\ln \frac{\varrho}{\delta h}}. \end{aligned}$$

Using the estimates, moving the second term from the right-hand side of (3.8) to the left, squaring the inequality and dividing by γ^2, we have

$$\frac{1}{2}|u(0)|^2 \leqslant (1-\delta C_{\mathrm{C1}})^2 |u(0)|^2 \leqslant \frac{2}{\gamma} \left\{ \int_0^{\gamma} u^2(y(\varrho,\vartheta)) \, \mathrm{d}\vartheta + \ln \frac{\varrho}{\delta h} \|\nabla u\|^2_{L^2(\Omega_i)} \right\}. \tag{3.9}$$

In the second case, $\delta h \geqslant \varrho$, we estimate the first term on the right-hand side of (3.7) by the Cauchy-Schwarz inequality and the second term by $\gamma\delta C_{\mathrm{C1}}|u(0)|$, which leads to

$$\frac{1}{2}|u(0)|^2 \leqslant (1-\delta C_{\mathrm{C1}})^2|u(0)|^2 \leqslant \frac{2}{\gamma}\int_0^\gamma u^2(y(\varrho,\vartheta))\,\mathrm{d}\vartheta. \tag{3.10}$$

Multiplying (3.9) and (3.10) by 2ϱ, integrating ϱ from δh to KH and from 0 to δh, respectively, and summing up the resulting inequalities yields

$$\begin{aligned}\frac{(KH)^2}{2}|u(0)|^2 \leqslant \frac{4}{\gamma}\Big\{&\int_0^{KH}\int_0^\gamma \varrho u^2(y(\varrho,\vartheta))\,\mathrm{d}\vartheta\,\mathrm{d}\varrho\\ &+\|\nabla u\|^2_{L^2(\Omega_i)}\int_{\delta h}^{KH}\varrho\ln\frac{\varrho}{\delta h}\,\mathrm{d}\varrho\Big\}.\end{aligned}$$

By estimating the second term on the right-hand side we complete the proof

$$\begin{aligned}|u(0)|^2 &\leqslant \frac{4}{\gamma}\Big\{\frac{2}{(KH)^2}\|u\|^2_{L^2(\Omega_i)} + \Big(\ln\frac{K}{\delta}+\ln\frac{H}{h}\Big)|u|^2_{H^1(\Omega_i)}\Big\}\\ &\leqslant \underbrace{\frac{4}{\gamma}\max\Big\{1,\frac{2}{K^2},\ln\frac{K}{\delta}\Big\}}_{=:C_{\mathrm{L2}}}\Big(1+\ln\frac{H}{h}\Big)\Big\{\frac{1}{H^2}\|u\|^2_{L^2(\Omega_i)}+|u|^2_{H^1(\Omega_i)}\Big\}.\end{aligned}$$

□

Corollary 3.2. *Under Assumptions* 3.1 *and* 3.2 *there exists* $C_{\mathrm{C2}} > 0$ *such that for all* $i \in \{1,\dots,N\}$

$$\forall\, u \in V\colon\ \|u-\overline{u}_i\|^2_{L^\infty(\Omega_i)} \leqslant C_{\mathrm{C2}}\Big(1+\ln\frac{H}{h}\Big)|u|^2_{H^1(\Omega_i)},$$

where $\overline{u}_i := |\Omega_i|^{-1}\int_{\Omega_i} u(x)\,\mathrm{d}x$ *with* $|\Omega_i|$ *being the area of* Ω_i.

P r o o f. Combining the previous lemma and the Poincaré inequality [8], we obtain

$$\|u-\overline{u}_i\|^2_{L^2(\Omega_i)} \leqslant C_{\mathrm{P}}H^2|u|^2_{H^1(\Omega_i)},$$

where $C_{\mathrm{P}} := 1/\pi^2$, and the assertion follows with $C_{\mathrm{C2}} := C_{\mathrm{L2}}(1+C_{\mathrm{P}})$. □

The next lemma gives stability of the coarse space. It can be found in [9], L.4.12.

Lemma 3.3. *Under Assumptions* 3.1 *and* 3.2 *there exists* $C_{\mathrm{L3}} > 0$ *such that for all* $i \in \{1, \ldots, N\}$

$$\forall\, u \in V \colon\ |I^H(u)|^2_{H^1(\Omega_i)} \leqslant C_{\mathrm{L3}}\Bigl(1 + \ln \frac{H}{h}\Bigr)|u|^2_{H^1(\Omega_i)},$$

as a consequence of which

$$\forall\, u \in V \colon\ a(u^H, u^H) \leqslant C_{\mathrm{L3}}\Bigl(1 + \ln \frac{H}{h}\Bigr)a(u, u).$$

Proof. Denote by P_1, P_2, and P_3 the vertices of a subdomain Ω_i. We have

$$|I^H(u)|^2_{H^1(\Omega_i)} = |I^H(u) - \overline{u}_i|^2_{H^1(\Omega_i)} = \Bigl|\sum_{j=1}^{3}(u(P_j) - \overline{u}_i)\varphi_j^H\Bigr|^2_{H^1(\Omega_i)}. \tag{3.11}$$

For $j \in \{1, 2, 3\}$ and the remaining indices k and l we employ Lemma 3.1 with $A := P_k$, $\alpha := \alpha_k$ the angle at P_k, $B := P_j$, and $C := P_l$. Using (3.4) we conclude

$$\begin{aligned}|\varphi_j^H|^2_{H^1(\Omega_i)} = \|\nabla\varphi_j^H\|^2|\Omega_i| &\leqslant \frac{2 \cdot \frac{1}{2}\|P_j - P_k\|\|P_l - P_k\| \sin\alpha_k}{\min\{\|P_j - P_k\|^2, \|P_l - P_k\|^2\} \sin^2\alpha_k} \\ &\leqslant \frac{H^i_{\max}}{H^i_{\min} \sin\alpha_k} \leqslant \frac{1}{\sin^2\alpha_{\min}} =: \widetilde{c},\end{aligned}$$

where $|\Omega_i|$ denotes the area of Ω_i. By (3.11) and Corollary 3.2 we have

$$|I^H(u)|^2_{H^1(\Omega_i)} \leqslant 3\sum_{j=1}^{3}|u(P_j) - \overline{u}_i|^2|\varphi_j^H|^2_{H^1(\Omega_i)} \leqslant 9\widetilde{c}C_{\mathrm{C2}}\Bigl(1 + \ln \frac{H}{h}\Bigr)|u|^2_{H^1(\Omega_i)},$$

which completes the proof with $C_{\mathrm{L3}} := 9\widetilde{c}C_{\mathrm{C2}}$. □

3.5. Stability of the edge space. To find $\lambda_{\min}$ it remains to estimate the edge-term in (3.3) by (3.2) from above. Being inspired by [9], L.4.23 we introduce a system of edge-based functions $(\theta_i(x))_{i=1}^M \subset C(\widetilde{\Omega})$, where $\widetilde{\Omega} := \overline{\Omega} \setminus \{x_j^{\mathrm{V}} \colon j = 1, \ldots, n^{\mathrm{V}}\}$. For the construction we refer to Figure 3 and the following paragraph.

We decompose each subdomain Ω_j, $j \in \{1, 2, \ldots, N\}$, with all three edges being parts of the skeleton, i.e., $E_{j_1}, E_{j_2}, E_{j_3} \subset \Gamma$, into six triangles ω_k. Without loss of generality we take $x \in \omega_1 \setminus \{P_1\}$ and introduce local coordinates $x = (x_1, x_2)$. We denote the angle at P_1 by α_1 and define the related edge functions θ_{j_1}, θ_{j_2}, and θ_{j_3} in ω_1 by

$$\theta_{j_1}(x) := 1 - \frac{2}{3\tan(\frac{1}{2}\alpha_1)}\frac{x_1}{x_2}, \quad \theta_{j_2}(x) = \theta_{j_3}(x) := \frac{1}{3\tan(\frac{1}{2}\alpha_1)}\frac{x_1}{x_2}. \tag{3.12}$$

The edge functions are analogously defined in $\omega_2, \ldots, \omega_6$. For a subdomain Ω_j with only one or two edges assigned to the skeleton the construction of the related edge functions is similar. Note that the system completed by edge-functions assigned to $\partial\Omega$ forms a partition of unity on $\widetilde{\Omega}$.

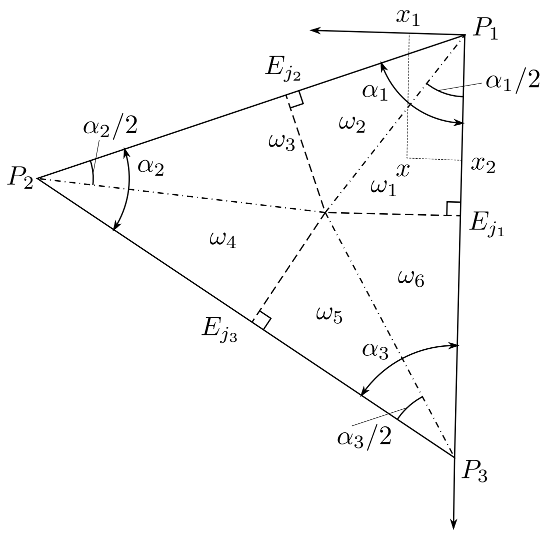

Figure 3. Decomposition of Ω_j used for the construction of θ_{j_1}, θ_{j_2}, and θ_{j_3}.

Lemma 3.4. *Under Assumption* 3.1 *there exists* $C_{\rm L4} > 0$ *such that for all* $i \in \{1, \ldots, M\}$

$$\|\nabla\theta_i(x)\| \leqslant C_{\rm L4}/r^H(x) \quad \text{almost everywhere in } \widetilde{\Omega},$$

where, for $x \in \Omega_j$ *with the vertices* P_1, P_2, *and* P_3, $r^H(x) := \min\limits_{k=1,2,3} \|x - P_k\|$.

Proof. The assertion follows from the construction above. For $x \in \omega_1$ we have

$$\begin{aligned} \|\nabla\theta_{j_1}(x)\| &= \frac{2}{3\tan(\frac{1}{2}\alpha_1)} \frac{\sqrt{(x_1)^2 + (x_2)^2}}{(x_2)^2} \\ &\leqslant \underbrace{\frac{2}{3\tan(\frac{1}{2}\alpha_{\min})\cos^2(\frac{1}{2}\alpha_{\max})}}_{=:C_{\rm L4}} \underbrace{\frac{1}{\sqrt{(x_1)^2 + (x_2)^2}}}_{\leqslant 1/r^H(x)} \end{aligned}$$

by Assumption 3.1. The estimate holds true for $\|\nabla\theta_{j_2}(x)\|$ and $\|\nabla\theta_{j_3}(x)\|$. The other cases, $x \in \omega_k$, are analogous. □

Similarly to replacing the FE-projection by interpolation when estimating the FE-approximation error, we will estimate the energy of the FE-interpolation of

$\theta_i(u-u^H)$, rather than the energy of $\widetilde{u}_i^{\mathrm{E}}$. We need the so-called edge lemma, the proof of which is sketched in [3].

Lemma 3.5. *Under Assumptions* 3.1 *and* 3.2 *there exists* $C_{\mathrm{L5}}>0$ *such that for all edges* E_i, $i\in\{1,\dots,M\}$ *and both the adjacent domains* Ω_j, $E_i\subset\partial\Omega_j$, *we have*

$$\forall\, u\in V\colon\ |I^h(\theta_i w)|^2_{H^1(\Omega_j)}\leqslant C_{\mathrm{L5}}\Big\{\Big(1+\ln\frac{H}{h}\Big)\|w\|^2_{L^\infty(\Omega_j)}+|w|^2_{H^1(\Omega_j)}\Big\},\tag{3.13}$$

where $w:=u-u^H$ *and* $I^h\colon C_0(\Omega)\to V$ *is the FE-interpolation operator, i.e.,* $I^h(v)(x):=\sum\limits_{i=1}^{n}v(x_i)\varphi_i(x)$, *where* x_i *is the node related to* φ_i.

P r o o f. Let us take an edge E_i and an adjacent domain Ω_j. By Assumption 3.1 with each coarse vertex P_k, $k=1,2,3$, of Ω_j an exactly one fine triangle T with vertices $A=P_k$, B, and C is associated. In the case that none of B and C lies on E_i, $I^h(\theta_i w)$ vanishes on T. We are left to analyze the other two triangles, for both of which we can consider $C\in E_i$. The contribution of such a triangle to $|I^h(\theta_i w)|^2_{H^1(\Omega_j)}$ is, due to (3.4), as follows:

$$\begin{aligned}|I^h(\theta_i w)|^2_{H^1(T)}&=\frac{w^2(C)}{\|C-A\|^2\sin^2\alpha}\,\frac{\|B-A\|\|C-A\|\sin\alpha}{2}\\&\leqslant\frac{h^T_{\max}}{2h^T_{\min}\sin\alpha_{\min}}w^2(C)\leqslant\underbrace{\frac{1}{2\sin^2\alpha_{\min}}}_{=:\widetilde{k}_1}\|w\|^2_{L^\infty(\Omega_j)}.\end{aligned}\tag{3.14}$$

In case of a triangle $T\subset\Omega_j$ such that none of its vertices A, B, and C is a vertex of Ω_j, Lemma 3.1 yields

$$\|\nabla I^h(\theta_i w)\|^2\leqslant\frac{2\{[(\theta_i w)(B)-(\theta_i w)(A)]^2+[(\theta_i w)(C)-(\theta_i w)(A)]^2\}}{\min\{\|B-A\|^2,\|C-A\|^2\}\sin^2\alpha},$$

where α denotes the angle at A. Since $\theta_i w$ is piecewise differentiable along the line segments $\overline{AB}$ and $\overline{AC}$, we can adopt the Lagrange mean value theorem. The latter combined with (3.4), the construction (3.12), and Lemma 3.4 yield

$$\begin{aligned}\|\nabla I^h(\theta_i w)\|^2&\leqslant\frac{2\|\nabla(\theta_i w)\|^2_{L^\infty(T)}}{\sin^2\alpha}\underbrace{\frac{\|B-A\|^2+\|C-A\|^2}{\min\{\|B-A\|^2,\|C-A\|^2\}}}_{\leqslant 1+(h^T_{\max}/h^T_{\min})^2}\\&\leqslant\underbrace{\frac{4\big(1+1/\sin^2\alpha_{\min}\big)}{\sin^2\alpha_{\min}}}_{=:\widetilde{k}_2}\{\|\nabla\theta_i\|^2_{L^\infty(T)}\|w\|^2_{L^\infty(T)}+\|\theta_i\|^2_{L^\infty(T)}\|\nabla w\|^2_{L^\infty(T)}\}\\&\leqslant\widetilde{k}_2\{(C_{\mathrm{L4}}/r^{H,h}(x))^2\|w\|^2_{L^\infty(T)}+\|\nabla w\|^2_{L^\infty(T)}\},\end{aligned}$$

where $\|\nabla f\|_{L^\infty(T)} := \operatorname*{ess\,sup}_{x\in T} \|\nabla f(x)\|$ and $r^{H,h}(x) := \operatorname{dist}(T(x), \{P_1, P_2, P_3\})$, where $T(x)$ is the open triangle containing x, which is a well-defined function up to the interfaces between fine triangles. Denote by $\widetilde{\Omega}_j$ the union of such non-corner triangles. They contribute to $|I^h(\theta_i w)|^2_{H^1(\Omega_j)}$ as follows:

$$(3.15)\quad |I^h(\theta_i w)|^2_{H^1(\widetilde{\Omega}_j)} \leqslant \widetilde{k}_2\Big\{\|w\|^2_{L^\infty(\Omega_j)}(C_{\mathrm{L4}})^2\int_{\widetilde{\Omega}_j}(1/r^{H,h}(x))^2\,\mathrm{d}x + |w|^2_{H^1(\widetilde{\Omega}_j)}\Big\}.$$

It remains to estimate the integral. We have

$$(3.16)\qquad \int_{\widetilde{\Omega}_j}\frac{1}{(r^{H,h}(x))^2}\,\mathrm{d}x \leqslant \sum_{k=1}^{3}\int_{\widetilde{\Omega}_j}\frac{1}{\inf\limits_{y\in T(x)}\|y-P_k\|^2}\,\mathrm{d}x.$$

Let us introduce three systems of local polar coordinates each of which has its origin at a coarse vertex P_k, its x_1-axis coincides with an edge of Ω_j, and Ω_j lies in the upper half-space. We denote by $v^T_{\min}$ the smallest height of a triangle T. The law of sines, Assumptions 3.1 and 3.2 yield

$$v^T_{\min} \geqslant h^T_{\min}\sin\alpha_{\min} \geqslant h^T_{\max}\sin^2\alpha_{\min} \geqslant C_{\mathrm{A2}}h\sin^2\alpha_{\min}.$$

Thus, by choosing $c := C_{\mathrm{A2}}\sin^2\alpha_{\min}$ the domain

$$\Lambda := \{x=(x_1,x_2)=\varrho(\cos\alpha,\sin\alpha)\in\mathbb{R}^2\colon\ ch\leqslant\varrho\leqslant H \text{ and } 0\leqslant\alpha\leqslant\alpha_{\max}\}$$

covers $\widetilde{\Omega}_j$ with respect to each of the coordinate systems. Let us denote the respective counterparts of Λ associated with P_1, P_2, and P_3 by Λ_1, Λ_2, and Λ_3. Let us adopt the k-th local polar coordinates $x(\varrho,\alpha)$ and note that

$$\inf_{y\in T(x(\varrho,\alpha))}\|y-P_k\| \geqslant \max\{ch, \varrho-h\}.$$

We have the estimate

$$\begin{aligned}\int_{\widetilde{\Omega}_j}\frac{1}{\inf\limits_{y\in T(x)}\|y-P_k\|^2}\,\mathrm{d}x &\leqslant \int_0^{\alpha_{\max}}\int_{ch}^{H}\frac{\varrho}{(\max\{ch,(\varrho-h)\})^2}\,\mathrm{d}\varrho\,\mathrm{d}\alpha\\ &= \alpha_{\max}\Big(\frac{2c+1}{2c^2}+\ln\frac{H-h}{ch}+\frac{h}{ch}-\frac{h}{H-h}\Big) \leqslant \alpha_{\max}\Big(\underbrace{\frac{4c+1}{2c^2}}_{=:\widetilde{c}}+\ln\frac{H}{ch}\Big),\end{aligned}$$

where we used $H\geqslant 2h$ from Assumption 3.2. Using the latter and (3.16), (3.15) is estimated by

$$|I^h(\theta_i w)|^2_{H^1(\widetilde{\Omega}_j)} \leqslant \widetilde{k}_2\Big\{\|w\|^2_{L^\infty(\Omega_j)}(C_{\mathrm{L4}})^2 3\alpha_{\max}\Big[\widetilde{c}+\ln\frac{H}{ch}\Big] + |w|^2_{H^1(\widetilde{\Omega}_j)}\Big\}.$$

After adding the two contributions (3.14), the assertion follows with

$$C_{\mathrm{L5}} := \max\{3\widetilde{k}_2(C_{\mathrm{L4}})^2\alpha_{\max}(\widetilde{c} - \ln c) + 2\widetilde{k}_1, 3\widetilde{k}_2(C_{\mathrm{L4}})^2\alpha_{\max}, \widetilde{k}_2\}.$$

□

Now we can analyze the stability of the edge space. The following lemma is proved in [9].

Lemma 3.6. *Under Assumptions* 3.1 *and* 3.2 *there exists* $C_{\mathrm{L6}} > 0$ *such that*

$$\forall\, u \in V\colon \sum_{i=1}^{M} a(\widetilde{u}_i^{\mathrm{E}}, \widetilde{u}_i^{\mathrm{E}}) \leqslant C_{\mathrm{L6}}\Bigl(1 + \ln\frac{H}{h}\Bigr)^2 a(u,u).$$

Proof. Denote by Ω_{i_1} and Ω_{i_2} the domains adjacent to E_i and by $\{E_{j_i}\}_{i=1}^{M_j}$, $M_j \leqslant 3$, the edges adjacent to Ω_j. Recall that $w := u - I^H(u)$ and $\widetilde{u}_i^{\mathrm{E}} := \mathcal{H}(w_i^{\mathrm{E}})$, where $w_i^{\mathrm{E}} := w$ on E_i and $w_i^{\mathrm{E}} := 0$ elsewhere on $\Gamma \cup \partial\Omega$. The discrete harmonicity of $\widetilde{u}_i^{\mathrm{E}}$ and Lemma 3.5 yield

$$\begin{aligned}
\sum_{i=1}^{M} a(\widetilde{u}_i^{\mathrm{E}}, \widetilde{u}_i^{\mathrm{E}}) &= \sum_{i=1}^{M}\sum_{j=1}^{2} \varrho_{i_j} |\widetilde{u}_i^{\mathrm{E}}|^2_{H^1(\Omega_{i_j})} \leqslant \sum_{i=1}^{M}\sum_{j=1}^{2} \varrho_{i_j} |I^h(\theta_i w)|^2_{H^1(\Omega_{i_j})} \\
&= \sum_{j=1}^{N} \varrho_j \sum_{i=1}^{M_j} |I^h(\theta_{j_i} w)|^2_{H^1(\Omega_j)} \\
&\leqslant \sum_{j=1}^{N} \varrho_j 3C_{L5}\Bigl\{\Bigl(1 + \ln\frac{H}{h}\Bigr)\|w\|^2_{L^\infty(\Omega_j)} + |w|^2_{H^1(\Omega_j)}\Bigr\}.
\end{aligned}$$

Now $(\alpha+\beta)^2 \leqslant 2(\alpha^2+\beta^2)$ and Corollary 3.2 give

$$\begin{aligned}
\|w\|^2_{L^\infty(\Omega_j)} &= \|(u - \overline{u}_j) - (I^H(u) - \overline{u}_j)\|^2_{L^\infty(\Omega_j)} \\
&\leqslant 2(\|u - \overline{u}_j\|^2_{L^\infty(\Omega_j)} + \underbrace{\|I^H(u) - \overline{u}_j\|^2_{L^\infty(\Omega_j)}}_{\leqslant \|u - \overline{u}_j\|^2_{L^\infty(\Omega_j)}}) \leqslant 4C_{\mathrm{C2}}\Bigl(1 + \ln\frac{H}{h}\Bigr)|u|^2_{H^1(\Omega_j)}.
\end{aligned}$$

Similarly, Lemma 3.3 gives

$$\begin{aligned}
|w|^2_{H^1(\Omega_j)} = |u - I^H(u)|^2_{H^1(\Omega_j)} &\leqslant 2(|u|^2_{H^1(\Omega_j)} + |I^H(u)|^2_{H^1(\Omega_j)}) \\
&\leqslant 2(1 + C_{\mathrm{L3}})\Bigl(1 + \ln\frac{H}{h}\Bigr)|u|^2_{H^1(\Omega_j)}.
\end{aligned}$$

Combining the estimates yields $C_{\mathrm{L6}} := 3C_{\mathrm{L5}}[4C_{\mathrm{C2}} + 2(1 + C_{\mathrm{L3}})]$. □

3.6. Lower bound

Theorem 3.2. *Under Assumptions* 3.1 *and* 3.2 *there exists* $C > 0$ *such that*

$$\forall\, u \in V\colon\ \hat{a}(u,u) \leqslant \underbrace{C\Big(1 + \ln \frac{H}{h}\Big)^2}_{=1/\lambda_{\min}(H,h)} a(u,u).$$

P r o o f. Comparing (3.2) and (3.3), the assertion is a consequence of Lemma 3.3 and 3.6 with $C := 1 + C_{\mathrm{L3}} + C_{\mathrm{L6}}$. □

We conclude with an estimate of the condition number:

$$\kappa(\hat{\mathbf{A}}^{-1}\mathbf{A}) \leqslant 10C\Big(1 + \ln \frac{H}{h}\Big)^2$$

with $C > 0$ independent of H, h, and $(\varrho_i)_{i=1}^N$ in a family of shape-regular quasi-uniform triangulations.

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