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2-FACTORS IN CLAW-FREE GRAPHS WITH LOCALLY DISCONNECTED VERTICES

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Abstract. An edge of G is singular if it does not lie on any triangle of G; otherwise, it is non-singular. A vertex u of a graph G is called locally connected if the induced subgraph $G[N(u)]$ by its neighborhood is connected; otherwise, it is called locally disconnected.

In this paper, we prove that if a connected claw-free graph G of order at least three satisfies the following two conditions: (i) for each locally disconnected vertex v of degree at least 3 in G, there is a nonnegative integer s such that v lies on an induced cycle of length at least 4 with at most s non-singular edges and with at least $s-5$ locally connected vertices; (ii) for each locally disconnected vertex v of degree 2 in G, there is a nonnegative integer s such that v lies on an induced cycle C with at most s non-singular edges and with at least $s-3$ locally connected vertices and such that $G[V(C)\cap V_2(G)]$ is a path or a cycle, then G has a 2-factor, and it is the best possible in some sense. This result generalizes two known results in Faudree, Faudree and Ryjáček (2008) and in Ryjáček, Xiong and Yoshimoto (2010).

Keywords: claw-free graph; 2-factor; closure; locally disconnected vertex; singular edge

MSC 2010: 05C35, 05C38, 05C45

1. Introduction

All graphs considered are simple finite undirected graphs and we refer to [2] for terminology and notation not defined here.

Specifically, C_k denotes the cycle on k vertices and P_k the path on k vertices (i.e. of length $k-1$). We denote the set of all vertices of degree k in G by $V_k(G)$ and denote $V_{\geqslant k}(G) = \bigcup_{i\geqslant k} V_i(G)$. The *distance* in G of two vertices $x, y \in V(G)$ is denoted

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$d_G(x,y)$, and for two subgraphs $F_1, F_2 \subset G$ we denote $d_G(F_1,F_2) = \min\{d_G(x,y) \colon x \in V(F_1), y \in V(F_2)\}$. A *clique* is a (not necessarily maximal) complete subgraph of a graph G, and, for an edge $e \in E(G)$, $\omega_G(e)$ denotes the largest order of a clique containing e. The *line graph* of H, denoted by $L(H)$, is the graph with $E(H)$ as the vertex set, in which two vertices are adjacent if and only if the corresponding edges have a vertex in common. Given a graph F, a graph G is said to be *F-free* if there is no induced subgraph of G that is isomorphic to F. The graph F is generally called a *forbidden subgraph* of G. Specifically, the four-vertex star $K_{1,3}$ will be called the *claw*, and a $K_{1,3}$-free graph will be also said to be *claw-free*. It is a well-known fact that every line graph is claw-free, hence the class of claw-free graphs can be considered a natural generalization of the class of line graphs.

A cycle in G of length $|V(G)|$ is called a *hamiltonian cycle*, and a graph containing a hamiltonian cycle is said to be *hamiltonian*. A *2-factor* of a graph G is a spanning subgraph of G in which every vertex has the same degree 2. Thus, a hamiltonian cycle is a connected 2-factor.

It follows from either [3] or [4] that every claw-free graph G with $\delta(G) \geqslant 4$ has a 2-factor. Yoshimoto [15] showed that a claw-free graph G with $\delta(G) \geqslant 3$ has also a 2-factor if additionally G is 2-connected. Later, Faudree et al. [5] proved the following theorem on forbidden subgraph conditions that imply the existence of 2-factors.

Theorem 1 (Faudree, Faudree, Ryjáček, [5]). *If G is a 2-connected graph of order at least three which is claw-free and C_i-free for all $i \geqslant 6$, then G has a 2-factor.*

Let C_k be a cycle of even length $k \geqslant 4$. Two edges $e_1, e_2 \in E(G)$ are said to be *antipodal* in C_k, if they are at maximum distance in C_k (i.e. $d_{C_k}(e_1,e_2) = k/2-1$). An even cycle C_k in a graph G is said to be *edge-antipodal*, abbreviated EA, if $\min\{\omega_G(e_1), \omega_G(e_2)\} = 2$ for any two antipodal edges $e_1, e_2 \in E(C_k)$. In 2010, Ryjáček et al. introduced a closure for 2-factors, and using this concept, they proved the following theorem.

Theorem 2 (Ryjáček, Xiong, Yoshimoto, [13]). *Let G be a claw-free graph in which every locally disconnected vertex is in an induced cycle of length 4 or 5, or in an induced EA-C_6. Then G has a 2-factor.*

The neighborhood of a vertex v in G is denoted by $N_G(v)$. A vertex v of G is *locally connected* if $G[N_G(v)]$ is connected; otherwise, it is *locally disconnected*. Let $LC(G)$ denote the set of all locally connected vertices of G. A graph G is called *locally connected* if every vertex of G is locally connected, i.e. $LC(G) = V(G)$. An edge e of G is *singular* if it does not lie on any triangle of G; otherwise, it is *non-singular*. Recently, the last two authors proved the following result, which is a common extension of two known results in [1] and [7], hence also of the results in [8] and [11].

Theorem 3 (Tian, Xiong, [14]). *Let G be a connected claw-free graph of order at least three such that*

(i) *for each locally disconnected vertex v of degree at least 3 in G, there is a nonnegative integer s such that v lies on an induced cycle of length at least 4 with at most s non-singular edges and with at least $s-3$ locally connected vertices;*

(ii) *for each locally disconnected vertex v of degree 2 in G, there is a nonnegative integer s such that v lies on an induced cycle C with at most s non-singular edges and with at least $s-2$ locally connected vertices and such that $G[V(C) \cap V_2(G)]$ is a path or a cycle.*

Then G is Hamiltonian.

Motivated by an extension of Theorems 1 and 2, we use a condition similar to that in Theorem 3 and obtain the following sufficient condition for a claw-free graph to have a 2-factor that is an extension of Theorems 1 and 2.

Theorem 4. *Let G be a connected claw-free graph of order at least three such that*

(i) *for each locally disconnected vertex v of degree at least 3 in G, there is a nonnegative integer s such that v lies on an induced cycle of length at least 4 with at most s non-singular edges and with at least $s-5$ locally connected vertices;*

(ii) *for each locally disconnected vertex v of degree 2 in G, there is a nonnegative integer s such that v lies on an induced cycle C with at most s non-singular edges and with at least $s-3$ locally connected vertices and such that $G[V(C) \cap V_2(G)]$ is a path or a cycle.*

Then G has a 2-factor.

The following corollary is a direct consequence of Theorem 4, because it is the special case of Theorem 4 for $s = 5$ in condition (i) and for $s = 3$ in condition (ii).

Corollary 5. *Let G be a connected claw-free graph of order at least three such that*

(i) *every locally disconnected vertex of degree at least 3 lies on an induced cycle of length at least 4 with at most 5 non-singular edges;*

(ii) *every locally disconnected vertex of degree 2 lies on an induced cycle C with at most 3 non-singular edges such that $G[V(C) \cap V_2(G)]$ is a path or a cycle.*

Then G has a 2-factor.

In Section 2, we shall present Ryjáček's closure concept in claw-free graphs and some auxiliary results, which are then applied to the proof of our main result in Section 3. We prove that Theorem 4 is an extension of Theorems 1 and 2 in Sections 4 and 5, respectively. In the last section, we discuss the sharpness of Theorem 4, point out that there exist many graphs which satisfy the conditions in Corollary 5 but not the ones in Theorems 1 or 2, and propose an open problem.

2. The closure of a claw-free graph

A locally connected vertex v is said to be *eligible* if $G[N_G(v)]$ is not complete. For a vertex x of a graph G, the graph G^*_x with $V(G^*_x) = V(G)$ and $E(G^*_x) = E(G) \cup \{uv\colon\ u, v \in N_G(x)\}$ is called the *local completion* of G at x. For a claw-free graph G, let $G_1 = G$. For $i \geqslant 1$, if G_i is defined and has an eligible vertex x_i, then let $G_{i+1} = (G_i)^*_{x_i}$. If $G_s = (G_{s-1})^*_{x_{s-1}}$ has no eligible vertex, then let $\mathrm{cl}(G) = G_s$ and let us call it the *closure* of G. Ryjáček [10] showed that the closure of G is uniquely determined and G is hamiltonian if and only if $\mathrm{cl}(G)$ is hamiltonian. The latter result was extended to 2-factors as follows.

Theorem 6 (Ryjáček, Saito, Schelp, [12]). *If G is a claw-free graph, then G has a 2-factor if and only if $\mathrm{cl}(G)$ has a 2-factor.*

Ryjáček [10] also established the following relationship between claw-free graphs and triangle-free graphs.

Theorem 7 (Ryjáček, [10]). *If G is a claw-free graph, then there is a triangle-free graph H such that $L(H) = \mathrm{cl}(G)$.*

In a claw-free graph G, the locally disconnected vertices can be partitioned into three classes, depending on the structure of the graphs $G[N(v)]$: Let $LD_0(G)$ denote the class of all vertices v for which $G[N(v)]$ is disconnected with two components of order one, let $LD_1(G)$ denote the class of all vertices v for which $G[N(v)]$ is disconnected with exactly one component of order one, and let $LD_2(G)$ denote the class of all vertices v for which $G[N(v)]$ is disconnected with no component of order one. Note that for a locally disconnected vertex v in a claw-free graph G, $G[N(v)]$ consists of exactly two complete subgraphs of G. Pfender proved the following result.

Lemma 8 (Pfender, [9]). $(LD_0(\mathrm{cl}(G)) \cup LD_1(\mathrm{cl}(G))) \subseteq (LD_0(G) \cup LD_1(G))$ *and* $LD_2(\mathrm{cl}(G)) \subseteq LD_2(G)$ *for every claw-free graph G.*

Recently, Tian and Xiong extended Lemma 8 as follows.

Lemma 9 (Tian, Xiong, [14]). *For $i \in \{0,1,2\}$, $LD_i(\mathrm{cl}(G)) \subseteq LD_i(G)$ for every claw-free graph G.*

For the proof of our main result, we need the following proposition, showing that if a graph G satisfies the assumptions of Theorem 4, then its closure $\mathrm{cl}(G)$ satisfies the assumptions of Corollary 5.

Proposition 10. *Let G be a graph satisfying the assumptions of Theorem 4. Then $\mathrm{cl}(G)$ is a connected claw-free graph of order at least three such that*

(1) *every locally disconnected vertex of degree at least 3 in $\mathrm{cl}(G)$ lies on an induced cycle of length at least 4 with at most 5 non-singular edges;*

(2) *every locally disconnected vertex of degree 2 in $\mathrm{cl}(G)$ lies on an induced cycle C' with at most 3 non-singular edges such that $\mathrm{cl}(G)[V(C') \cap V_2(\mathrm{cl}(G))]$ is a path or a cycle.*

In order to prove Proposition 10, we need the following lemmas. A *branch* in G is a nontrivial path with end vertices that do not lie in $V_2(G)$ and with internal vertices of degree 2 (if such exist). If a branch has length 1, then it has no internal vertices of degree 2. We use $\mathcal{B}(G)$ to denote the set of branches in G.

Lemma 11 (Tian, Xiong, [14]). *Let G be a claw-free graph. If the length of $L \in \mathcal{B}(G)$ is at least 3 in G, then $L \in \mathcal{B}(\mathrm{cl}(G))$.*

Lemma 12. *Let G be a claw-free graph and C an induced cycle in G with at most s non-singular edges and with at least $s-l$ locally connected vertices, where s and l are nonnegative integers and $l \leqslant s$. If $x \in V(C)$ is locally disconnected in $\mathrm{cl}(G)$, then there is an induced cycle C' of length at least 4 in $\mathrm{cl}(G)$ with $x \in V(C') \subseteq V(C)$ and with at most l non-singular edges.*

P r o o f of Lemma 12. Since $x \in V(C)$ is locally disconnected in $\mathrm{cl}(G)$, there is an induced cycle C' in $\mathrm{cl}(G)$ such that $x \in V(C') \subseteq V(C)$ and $|V(C)| \geqslant 4$. It remains to prove that C' has at most l non-singular edges in $\mathrm{cl}(G)$.

Note that every vertex of C' is locally disconnected in $\mathrm{cl}(G)$. By Lemma 9, $V(C') \cap LD_i(\mathrm{cl}(G)) \subseteq V(C) \cap LD_i(G)$ for $i = 0,1,2$. Hence the number of non-singular edges in C' is no more than the number s of non-singular edges in C. If C has no locally connected vertices in G, then $s = l$, hence we are done. Now we consider $s \neq l$.

Suppose $\{u_1, u_2, \ldots, u_{s-l}\} \subseteq V(C) \cap LC(G)$. As mentioned before, $\mathrm{cl}(G)$ is uniquely determined by the graph G, i.e., $\mathrm{cl}(G)$ is independent of the order of eligible vertices during the construction. Note that each u_i is an eligible vertex in G by the hypothesis that C is an induced cycle. Let $G_1 = G^*_{u_1}$ and $N_G(u_1) \cap V(C) = \{v_1, v_2\}$. Then there exists an induced cycle C_1 in G_1 with $V(C_1) = V(C) \setminus \{u_1\}$ and $E(C_1) =$

$(E(C)\setminus\{u_1v_1,u_1v_2\})\cup\{v_1v_2\}$. Since u_1v_1,u_1v_2,v_1v_2 are non-singular, C_1 has at most $s-1$ non-singular edges. Notice that C_1 is an induced cycle and that u_i is an eligible vertex in G_1 for $i=2,\dots,s-l$. By recursively performing the local completion on u_i for $i=1,\dots,s-l$, we can obtain an induced cycle C_{s-l} in G_{s-l} such that C_{s-l} has at most $s-(s-l)=l$ non-singular edges and $V(C_{s-l})=V(C)\setminus\{u_1,u_2,\dots,u_{s-l}\}$. By Lemma 9, $V(C')\cap LD_i(\mathrm{cl}(G))\subseteq V(C_{s-l})\cap LD_i(G_{s-l})$ for $i=0,1,2$. Hence the number of non-singular edges in C' is no more than the number l of non-singular edges in C_{s-l}. □

Now we present the proof of Proposition 10.

Proof of Proposition 10. First suppose that x is a locally disconnected vertex of degree at least 3 in $\mathrm{cl}(G)$. Then $x\in LD_1(\mathrm{cl}(G))$ or $x\in LD_2(\mathrm{cl}(G))$. By Lemma 9, $x\in LD_1(G)$ or $x\in LD_2(G)$. This implies that x is a locally disconnected vertex in G and $d_G(x)\geqslant 3$. By assumption (i) of Theorem 4, x lies on an induced cycle of length at least 4 in G with at most s non-singular edges and with at least $s-5$ locally connected vertices. By Lemma 12, x satisfies condition (1) of Proposition 10.

Next suppose that x is a locally disconnected vertex of degree 2 in $\mathrm{cl}(G)$. Then x is a locally disconnected vertex of degree 2 in G. By assumption (ii) of Theorem 4, x lies on an induced cycle C with at most s non-singular edges and with at least $s-3$ locally connected vertices such that $G[V(C)\cap V_2(G)]$ is a path or a cycle. By Lemma 12, x lies on an induced cycle C' with $V(C')\subseteq V(C)$ and with at most 3 non-singular edges.

If $G[V(C)\cap V_2(G)]$ is a cycle, then, since G is connected, G is a cycle. Hence $\mathrm{cl}(G)$ is a cycle and we are done. If $G[V(C)\cap V_2(G)]=\{x\}$, then since $x\in V(C')\subseteq V(C)$ and $x\in V_2(\mathrm{cl}(G))\subseteq V_2(G)$, we have $V(C')\cap V_2(\mathrm{cl}(G))=\{x\}$ and we are also done. Thus, suppose that $|V(C)\cap V_2(G)|\geqslant 2$ and L is the branch such that $(V(C)\cap V_2(G))\supseteq V(L)$. By assumption (ii) of Theorem 4, $L\in\mathcal{B}(G)$ is the unique branch in C. By Lemma 11, $L\in\mathcal{B}(\mathrm{cl}(G))$ is the unique branch in C'. This implies that $\mathrm{cl}(G)[V(C')\cap V_2(\mathrm{cl}(G))]\subseteq V(L)$ is a path. □

3. Proof of Theorem 4

In this section we present the proof of the main result of this paper. An *even* graph is a graph in which every vertex has a positive even degree. A connected even subgraph is called a *circuit*. For $m\geqslant 2$, a star $K_{1,m}$ is a complete bipartite graph with independent sets $A=\{c\}$ and B with $|B|=m$; the vertex c is called the center and the vertices in B are called the leaves of $K_{1,m}$.

Let $\mathscr{S}$ be a set of edge-disjoint circuits and stars with at least three edges in a graph H. We call $\mathscr{S}$ a *system that dominates* H or simply a *dominating system* if every edge of H is either contained in one of the circuits or stars of $\mathscr{S}$ or is adjacent to one of the circuits. Gould and Hynds gave the following characterization of a graph H with $L(H)$ that has a 2-factor.

Theorem 13 (Gould, Hynds, [6])**.** *Let H be a graph. Then $L(H)$ has a 2-factor with c components if and only if there is a system with c elements that dominates H.*

The following result, which is also necessary for our proof, follows immediately from Proposition 10 (2).

Lemma 14. *Let G be a graph satisfying the assumptions of Theorem 4. Then every branch $L \in \mathcal{B}(G)$ of length at least 2 lies on an induced cycle C such that C has at most 3 non-singular edges and L is the unique branch of length at least 2 in C.*

Let M and M' be two sets of edges of a graph G. We use $M\Delta M'$ to denote the symmetric difference of M and M', i.e. $M\Delta M' = (M \cup M') \setminus (M \cap M')$. An edge e is called a *pendant* edge if the degree of the end vertex of e is 1; otherwise, it is *non-pendant*. If G is a line graph, then the graph H for which $L(H) = G$ will be called the *preimage* of G and denoted $H = L^{-1}(G)$. For any subgraph C of a line graph G, we let $L^{-1}(C)$ denote the preimage of C.

Lemma 15. *Let G be a graph satisfying the assumptions of Theorem 4 and H a graph such that $\mathrm{cl}(G) = L(H)$. If B is a 2-connected block of H that is not a cycle and $e = uv \in E(B)$, then e lies on a cycle C such that either*
- (3) *C has at most 5 vertices of degree greater than 2 in H and C has no branch of length at least 3 in H; or*
- (4) *C has at most 3 vertices of degree greater than 2 in H and C has exactly one branch of length at least 3 in H.*

P r o o f of Lemma 15. By Proposition 10, every locally disconnected vertex in $\mathrm{cl}(G)$ satisfies condition (1) or (2) of Proposition 10.

Claim 1. Every branch $L \in \mathcal{B}(H)$ of length at least 3 lies on a cycle C such that C has at most 3 vertices of degree greater than 2 in H and L is the unique branch of length at least 3 in C.

P r o o f of Claim 1. Let $L' \in \mathcal{B}(\mathrm{cl}(G))$ be a branch corresponding to $L \in \mathcal{B}(H)$. Note that $|E(L')| = |E(L)| - 1 \geqslant 2$. By Lemma 14, there exists an induced cycle C' such that C' has at most 3 non-singular edges and L' is the unique branch of length at least 2 in C'.

By the fact that $\mathrm{cl}(G) = L(H)$, $L^{-1}(C')$ is a cycle in H such that $L^{-1}(C')$ has at most 3 vertices of degree greater than 2 in H and L is the unique branch of length at least 3 in H. $\square$

If e lies on a branch $L \in \mathcal{B}(H)$ of length at least 3, then e lies on a cycle satisfying (4) by Claim 1. Now suppose that e lies on a branch $L \in \mathcal{B}(H)$ of length 1 or 2. Let $v_e \in V(\mathrm{cl}(G))$ be the vertex corresponding to the edge e in $E(H)$. Then $d_{\mathrm{cl}(G)}(v_e) \geqslant 3$. Since H is triangle-free, $N_H(u) \cap N_H(v) = \emptyset$. By the fact that B is 2-connected, $N_H(u) \neq \emptyset$ and $N_H(v) \neq \emptyset$. Further, since $\mathrm{cl}(G)$ is claw-free, $\mathrm{cl}(G)[N_{\mathrm{cl}(G)}(v_e)]$ consists of two vertex-disjoint cliques, i.e., v_e is locally disconnected in $\mathrm{cl}(G)$.

By Proposition 10, v_e lies on an induced cycle C_e of length at least 4 in $\mathrm{cl}(G)$ with at most 5 non-singular edges. By the fact that $\mathrm{cl}(G) = L(H)$, $L^{-1}(C_e)$ is a cycle in H such that $e \in E(L^{-1}(C_e))$ and $|V(L^{-1}(C_e)) \cap V_{\geqslant 3}(H)| \leqslant 5$. Since B is not a cycle, $|V(L^{-1}(C_e)) \cap V_{\geqslant 3}(H)| \geqslant 2$. Note that $L^{-1}(C_e)$ has a branch of length 1 or 2. Therefore, if $|V(L^{-1}(C_e)) \cap V_{\geqslant 3}(H)| = 2$, then $L^{-1}(C_e)$ has at most one branch of length at least 3, which implies that $L^{-1}(C_e)$ satisfies (3) or (4); if $|V(L^{-1}(C_e)) \cap V_{\geqslant 3}(H)| = 3$, then $L^{-1}(C_e)$ has at most t ($t \leqslant 2$) branches of length at least 3. Suppose that L_1 and L_2 are the two possible branches of length at least 3 in $L^{-1}(C_e)$ ($L_1 = L_2$ if $t = 1$). By Claim 1, L_i lies on a cycle C_i satisfying (4) for $i = 1, 2$. Thus, either $C = H[E(L^{-1}(C_e))\Delta E(C_1)\Delta E(C_2)]$ (if $t = 2$), or $C = H[E(L^{-1}(C_e))\Delta E(C_1)]$ (if $t = 1$), or $C = C_e$ (if $t = 0$) is a cycle such that $e \in E(C)$ and such that C satisfies (3).

If $|V(L^{-1}(C)) \cap V_{\geqslant 3}(H)| = 4$ or 5, we can use the same method as above and finally obtain a cycle C that satisfies (3). $\square$

Now we present the proof of our main result.

P r o o f of Theorem 4. We choose an even subgraph X of H such that

(a) X contains a maximum number of branches of length at least 3;

(b) subject to condition (a), X contains a maximum number of vertices of degree greater than 2 in H;

(c) subject to conditions (a) and (b), X contains a maximum number of edges of H.

Let $F = H - X$. Then we have

Claim 2. Each of the following conditions holds:

(5) X contains all branches of length at least 3 in H;

(6) every component of F has at most one vertex of degree greater than 2 in H;

(7) F is a forest.

P r o o f of Claim 2. (5) Suppose, on the contrary, that there exists a branch B of H with length at least 3 such that B does not lie on X. Then by Lemma 15, B lies on a cycle C satisfying (4). Let X' be the graph with $E(X') = E(X)\Delta E(C)$ (and with the corresponding set of vertices). Then X' contains more branches of length at least 3 in H than X, which contradicts (a).

(6) Suppose, on the contrary, that there exists a component F' of F such that F' has at least two vertices of degree greater than 2 in H. Since F' is connected, there exists a path connecting any two of these vertices. We choose two of the vertices x_1, x_2 such that $x_1, x_2 \in V(F') \cap V_{\geqslant 3}(H)$ and $d_{F'}(x_1, x_2)$ is as small as possible. Let P be a path connecting x_1 and x_2. We claim that $|E(P)| \leqslant 2$: For otherwise, suppose that $|E(P)| \geqslant 3$. By our choice of x_1 and x_2, all of the inner vertices of P are of degree 2 in H. Thus P is a branch of length at least 3 in H. This contradicts (5).

If $|E(P)| = 1$, then $N_{F'}(x_1) \cap N_{F'}(x_2) = \emptyset$. Since the edge x_1x_2 is not a pendant edge, by Lemma 15 (3), the edge x_1x_2 lies on an induced cycle C of length at least 4 with at most 5 vertices of degree greater than 2 in H.

If $|E(P)| = 2$, then let $N_{F'}(x_1) \cap N_{F'}(x_2) = \{x\}$. Since the edge x_1x is not a pendant edge, by Lemma 15 (3), the edge x_1x lies on an induced cycle C of length at least 4 with at most 5 vertices of degree greater than 2 in H. Since $d_{F'}(x) = 2$, the edge xx_2 also lies on C.

Thus, in any case, x_1 and x_2 lie on a common induced cycle C of length at least 4 with at most 5 vertices of degree greater than 2 in H.

We first suppose that $|V(X) \cap V(C) \cap V_{\geqslant 3}(H)| = 1$. Let X' be the graph with $E(X') = E(X) \cup E(C)$ (and with the corresponding set of vertices). Then X' is an even subgraph of H satisfying (a), but X' has more vertices of degree greater than 2 in H in comparison with X, contradicting (b).

Now we suppose that $|V(X) \cap V(C) \cap V_{\geqslant 3}(H)| \geqslant 2$. Let X' be the graph with $E(X') = E(X) \Delta E(C)$ (and with the corresponding set of vertices). Then it is easy to see that X' satisfies (a), but $|V(X') \cap V_{\geqslant 3}(H)| > |V(X) \cap V_{\geqslant 3}(H)|$, contradicting (b).

(7) Suppose, otherwise, that there exists a cycle C' in F. Let X' be the graph with $E(X') = E(X) \cup E(C')$ (and with the corresponding set of vertices). Then X' is an even subgraph of H satisfying (a) and (b), but X' has more edges than X, contradicting (c). □

Claim 3. Every component of F has exactly one vertex of degree greater than 2 in H.

Proof of Claim 3. By (6), we only need to prove that every component of F has at least one vertex of degree greater than 2 in H.

Suppose, on the contrary, that there exists a component F_0 of F such that F_0 has no vertex of degree greater than 2 in H. Since F_0 is connected, there exists a path P in F_0. By (5), $|E(P)| \leqslant 2$.

Let $P = x_1x_2x_i$, $2 \leqslant i \leqslant 3$. Without loss of generality, we suppose that $d_H(x_1) \geqslant d_H(x_i)$. By (6) and our hypothesis, $2 \geqslant d_H(x_1) \geqslant d_H(x_i)$. Then we claim that $d_H(x_1) = 2$ (otherwise, if $d_H(x_1) = 1$, then $d_H(x_i) = 1$, $i = 2$ or 3, hence $|E(H)| \leqslant 2$, contradicting the fact that $\mathrm{cl}(G) = L(H)$ has at least three vertices), $i = 2$ (if

$i = 3$, then, since $d_H(x_1) = 2$, F_0 has a branch of length at least three in H, contradicting (5)), and $d_H(x_2) = 1$ (if $d_H(x_2) = 2$, then F_0 has a branch of length at least three in H, contradicting (5)). Thus, let $x_1' \in N_H(x_1)$. Since $x_1'x_1 \in E(H)$ is not a pendant edge, by Lemma 15, $x_1'x_1$ lies on a cycle C of H. Since $d_H(x_1) = 2$, x_1x_2 also lies on C, but this is impossible since $d_H(x_2) = 1$. $\square$

By Claim 3, every component F_i of F is a star S_i with at least three edges. So $X \cup \left(\bigcup_{i=1} S_i\right)$ is a dominating system of H. Thus $L(H)$ has a 2-factor by Theorem 13. Therefore, by Theorems 7 and 6, G has a 2-factor. $\square$

4. Theorem 4 is an extension of Theorem 1.

Proof. In order to prove that Theorem 4 is an extension of Theorem 1, it is sufficient to prove that Corollary 5 is an extension of Theorem 1. Thus we only need to prove that a graph satisfying the conditions of Theorem 1, must also satisfy condition (i) and (ii) in Corollary 5.

Let G be a graph satisfying the conditions of Theorem 1, and v any locally disconnected vertex in G. Since G is 2-connected, v is not a cut vertex. Then v lies on an induced cycle C. By the assumption of v, the length of C is at least 4. Since G is C_i-free ($i \geqslant 6$), the length of C is at most 5.

First suppose that $d_G(v) \geqslant 3$. Then C has at most 5 non-singular edges. So v satisfies condition (i) in Corollary 5. Thus G satisfies condition (i) in Corollary 5.

Next suppose that $d_G(v) = 2$. Then C has at most 3 non-singular edges. If C has only one non-singular edge, then $G[V(C) \cap V_2(G)] = P_3$; if C has two non-singular edges which are adjacent in G, then $G[V(C) \cap V_2(G)] = P_2$; if C has two non-singular edges which are non-adjacent in G, or C has three non-singular edges, then $G[V(C) \cap V_2(G)]$ is an isolated vertex. In all these cases, v satisfies condition (ii) in Corollary 5. Thus G satisfies condition (ii) in Corollary 5. $\square$

5. Theorem 4 is an extension of Theorem 2

Proof. In order to prove that Theorem 4 is an extension of Theorem 2, it is sufficient to prove that Corollary 5 is an extension of Theorem 2. Thus we only need to prove that a graph satisfying the conditions of Theorem 2, must also satisfy condition (i) and (ii) in Corollary 5.

Let G be a graph satisfying the conditions of Theorem 2, and v any locally disconnected vertex in G. Then v is in an induced cycle C with $|E(C)| = 4$ or 5, or in an induced EA-C_6.

First suppose that v is in an induced cycle C with $|E(C)| = 4$ or 5. By an argument similar to the proof in Section 4, we could prove that v satisfies condition (i) or (ii) in Corollary 5. Thus, G satisfies condition (i) or (ii) in Corollary 5.

Next suppose that v is in an induced EA-C_6. If $d_G(v) \geqslant 3$, then EA-C_6 has at most 5 non-singular edges (if EA-C_6 has 6 non-singular edges, then there exist two antipodal edges $e_1, e_2 \in E(EA\text{-}C_6)$, such that $\min\{\omega_G(e_1), \omega_G(e_2)\} \geqslant 3$; this contradicts the definition of EA-C_6). So v satisfies condition (i) in Corollary 5. Thus G satisfies condition (i) in Corollary 5.

If $d_G(v) = 2$, then EA-C_6 has at most 3 non-singular edges (if EA-C_6 has 4 non-singular edges, then there exist two antipodal edges $e_1, e_2 \in E(EA\text{-}C_6)$, such that $\min\{\omega_G(e_1), \omega_G(e_2)\} \geqslant 3$; this contradicts the definition of EA-C_6). It is straightforward to check that $G[V(EA\text{-}C_6) \cap V_2(G)]$ is a path. So v satisfies condition (ii) in Corollary 5. Thus G satisfies condition (ii) in Corollary 5. □

6. Concluding remarks

6.1. Sharpness. In this subsection, we discuss the sharpness and show that all the conditions of Theorem 4 are the best possible in some sense.

▷ The condition "$s - 5$ locally connected vertices" in Theorem 4 (i) cannot be replaced by "$s - 6$ locally connected vertices". The graph $L(H_1)$, where H_1 is illustrated in Figure 1, is a graph such that every locally disconnected vertex of degree at least 3 lies on an induced cycle of length 6 with 6 non-singular edges and without a locally connected vertex. However, the line graph $L(H_1)$ has no 2-factor: otherwise, by Theorem 13, H_1 has a dominating system, a contradiction.

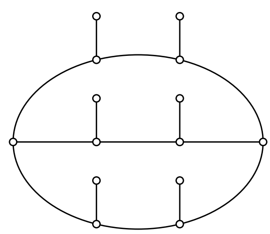

Figure 1. H_1.

▷ The condition "$s - 3$ locally connected vertices" in Theorem 4 (ii) cannot be replaced by "$s - 4$ locally connected vertices". The graph $L(H_2)$, where H_2 is illustrated in Figure 2, is a graph such that every locally disconnected vertex of degree 2 lies on an induced cycle C with 4 non-singular edges and without a locally connected vertex such that $G[V(C) \cap V_2(G)]$ is a path or a cycle.

However, the line graph $L(H_2)$ has no 2-factor: otherwise, by Theorem 13, H_2 has a dominating system, a contradiction.

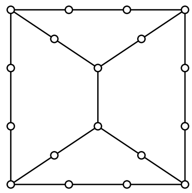

Figure 2. H_2.

▷ The condition "$G[V(C) \cap V_2(G)]$ is a path or a cycle" in Theorem 4 (ii) is necessary. The graph H_3 in Figure 3 is a graph satisfying condition (i) but not (ii) since the three locally disconnected vertices of degree two do not satisfy Condition (ii) of Theorem 4 (although any two of them lie on an induced cycle with only two non-singular edges). It is straightforward to check that H_3 has no 2-factor. One can obtain many such graphs of arbitrarily large order by joining a clique of arbitrary order to any nontrivial maximal clique of H_3.

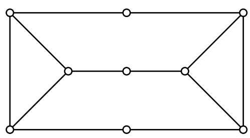

Figure 3. H_3.

6.2. There exist many graphs which satisfy the conditions in Corollary 5 but not the ones in Theorems 1 and 2. In fact, let $k \geqslant 7$ be an integer and G a connected graph obtained from a cycle C_k and some isolated vertices, by joining the isolated vertices with the vertices on the cycle C_k so that there are at most 3 non-singular edges on C_k and $G[V(C_k) \cap V_2(G)]$ is a path or a cycle. Then G satisfies condition (ii) in Corollary 5, but does not satisfy the conditions of Theorems 1 and 2, respectively.

6.3. Open problem. At the end of this section, we propose the following problem for further study.

Problem 16. Does every connected claw-free graph G of order $n \geqslant 3$ with conditions (i) and (ii) of Theorem 4 have a 2-factor with at most $n/9+1$ components?

If Problem 16 has a positive solution, then the upper bound is sharp in the following sense. We use $K'_{2,3}$ to denote the graph obtained from $K_{2,3}$ by attaching one pendant edge to every vertex of degree two. Let s be an integer and $F_1(x_1, y_1)$,

$F_2(x_2, y_2), \ldots, F_s(x_s, y_s)$ ($s \geqslant 3$) s copies of $K'_{2,3}$, where x_i, y_i are two noncutvertices of degree 3 of $F_i(x_i, y_i)$, $1 \leqslant i \leqslant s$. Then the graph H_4 is obtained from these s $F_i(x_i, y_i)$ by identifying y_i and x_{i+1} for all $i \in \{1, 2, \ldots, s-1\}$. For an example in the case when $s = 5$, see Figure 4. Therefore $|E(H_4)| = n = 9s$ and $n/9 + 1 = s + 1$. Because three cutvertices of degree 3 of $F_i(x_i, y_i)$ do not lie on a common circuit in any possible dominating system, H_4 has a dominating system with $n/9 + 1$ components. By Theorem 13, $L(H_4)$ has a 2-factor with exactly $n/9 + 1$ components.

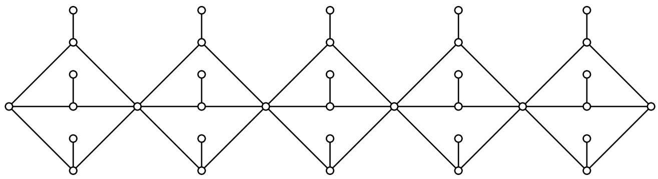

Figure 4. H_4.

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