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A NEW APPROACH FOR SOLVING NONLINEAR BVP'S ON THE HALF-LINE FOR SECOND ORDER EQUATIONS AND APPLICATIONS

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Abstract. We present a new approach to solving boundary value problems on noncompact intervals for second order differential equations in case of nonlocal conditions. Then we apply it to some problems in which an initial condition, an asymptotic condition and a global condition is present. The abstract method is based on the solvability of two auxiliary boundary value problems on compact and on noncompact intervals, and uses some continuity arguments and analysis in the phase space. As shown in the applications, Kneser-type properties of solutions on compact intervals and a priori bounds of solutions on noncompact intervals are key ingredients for the solvability of the problems considered, as well as the properties of principal solutions of an associated half-linear equation. The application of this method leads to some new existence results, which complement and extend some previous ones in the literature.

Keywords: global solution; nonlocal boundary value problem; noncompact interval; continuous dependence of solution; fixed point theorem; principal solution

MSC 2010: 34B10, 34B15, 34B18, 34B40

1. Introduction

The purpose of this work is twofold. First, based on the results in [5], [15], we present a new approach that can be applied to solving a wide class of boundary value problems on noncompact intervals for second order equations, also in the singular case or in the presence of a parameter. This method is especially useful in the case

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where the boundary conditions are of nonlocal type; multipoint, integral or functional conditions can also be considered. Then, we show how this method can be applied in some practical cases. In particular, a new existence result, complementing the results in [15], is presented for solutions of a second order nonlinear equation with nonconstant sign weight, satisfying an initial condition, an asymptotic condition and a global condition on the half-line.

In order to present this method, consider the following boundary value problem (BVP) for a second order nonlinear differential equation, possibly depending on a parameter,

$$\begin{cases} \big(a(t)\Phi(x')\big)' = f(t,x,x') + \lambda g(t,x,x'), \quad t \geqslant 0, \\ x \in \mathfrak{B}, \end{cases} \tag{1.1}$$

where $\Phi\colon (-\delta,\delta) \to (-\sigma,\sigma)$ is an increasing homeomorphism, with $0 < \delta$, $\sigma \leqslant \infty$, a is a positive continuous function on $[0,\infty)$, f, g are continuous functions on $[0,\infty)\times\mathbb{R}\times\mathbb{R}$, $\lambda \geqslant 0$ is a real parameter and $\mathfrak{B}$ is a subset of $C[0,\infty)$. The homeomorphism Φ in (1.1) includes, as special cases, the 1-dimensional p-Laplacian operator $\Phi_p(u) = |u|^p \operatorname{sgn} u$, $p > 0$, $u \in \mathbb{R}$, the curvature operator $\Phi_C(u) = u/\sqrt{1+u^2}$, $u \in \mathbb{R}$, and the relativistic operator $\Phi_R(u) = u/\sqrt{1-u^2}$, $|u| < 1$.

Boundary value problems of type (1.1) have received an increasing interest in the last years, due also to their physical applications, since, in some cases, they are related to the existence of radial solutions of some boundary value problems associated to elliptic operators. Many authors have dealt with problems of type (1.1) in various frameworks. Fixed point theorems, continuation principles, upper and lower solutions method and variational approaches have been employed to study the solvability, uniqueness or multiplicity of solutions, see for instance [1], [14], [17], [18] for a good survey of related results.

The method proposed here is particularly significant in the case where the condition $x \in \mathcal{B}$ is *global*, that is, it is nonlocal and can be verified only by examining the behavior of x on the whole interval $[0,\infty)$. Examples of global conditions are $x(t) \geqslant 0$ for all $t \geqslant 0$, $\sup\limits_{t\geqslant 0} x(t) \leqslant R$, where $R > 0$ is a fixed constant, or $\int_0^\infty x(t)\,\mathrm{d}t = 0$. We recall that a condition is called local if it acts on a compact subinterval of $[0,\infty)$, and nonlocal otherwise.

This approach extends to noncompact intervals a method developed by Zanolin et al. in [12], and allows also to treat problems depending on a parameter. It is essentially based on the solvability of two auxiliary boundary value problems (typically one in a compact interval and the other on a half-line) and makes use of some continuity arguments and simple analysis in the phase space.

The abstract method is presented in Section 2, while in Section 3 some applications show how it can be applied in some practical cases. In the applications presented here, the lack of useful lower bounds, due to the fact that the boundary conditions include both the initial condition $x(0) = 0$ and the global condition $x(t) > 0$ for $t > 0$, leads to difficulties in using classical approaches for solving problems of type (1.1), while our method is able to overcome these issues. The theory of principal solutions of half-linear equations, developed in [9] (see also [2], [8]), plays an important role in the study of the behavior of the positive solutions of an associated boundary value problem on a noncompact interval. Combining Kneser-type properties of solutions of a problem on a compact interval, a priori bounds of positive solutions on a noncompact interval and continuity arguments, we prove an existence result which complements and extends some previous ones in [6], [15]. Some comments and related problems complete the paper.

2. The method

In order to study the solvability of (1.1), we consider two auxiliary boundary value problems splitting the half-line $[0, \infty)$ in two parts, and study the properties of solutions of the equations in (1.1) in each of them.

Let $T > 0$ be fixed. If $\mathfrak{B}_0 \subset C[0, T]$ and $\mathfrak{B}_T \subset C[T, \infty)$ are two given subsets, we denote by $\widetilde{\mathfrak{B}}_0$ the subset of $C[0, \infty)$ consisting of functions whose "restriction" to $[0, T]$ belongs to $\mathfrak{B}_0$. Analogously, by $\widetilde{\mathfrak{B}}_T$ we denote the subset of $C[0, \infty)$ consisting of functions whose restriction to $[T, \infty)$ belongs to $\mathfrak{B}_T$, i.e.,

$$\widetilde{\mathfrak{B}}_0 = \bigl\{u \in C[0, \infty)\colon \exists v \in \mathfrak{B}_0 \text{ such that } u(t) = v(t) \text{ in } [0, T]\bigr\},$$
$$\widetilde{\mathfrak{B}}_T = \bigl\{u \in C[0, \infty)\colon \exists v \in \mathfrak{B}_T \text{ such that } u(t) = v(t) \text{ in } [T, \infty)\bigr\}.$$

Let $\mathfrak{B}_0$ and $\mathfrak{B}_T$ be such that

$$\widetilde{\mathfrak{B}}_0 \cap \widetilde{\mathfrak{B}}_T \subseteq \mathfrak{B}. \tag{2.1}$$

First of all, for simplicity, we treat the case where the boundary value problem does not depend on the parameter λ, i.e., we study the solvability of (1.1) with $\lambda = 0$.

Consider the following two auxiliary boundary value problems on the compact interval $[0, T]$ and on the half-line $[T, \infty)$, respectively:

$$\begin{cases} \bigl(a(t)\Phi(x')\bigr)' = f(t, x, x'), \quad t \in [0, T], \\ x \in \mathfrak{B}_0, \end{cases} \tag{2.2}$$

and

$$
(2.3) \qquad \begin{cases} \bigl(a(t)\Phi(x')\bigr)' = f(t,x,x'), & t \geqslant T, \\ x \in \mathfrak{B}_T. \end{cases}
$$

Assume that both the above problems are solvable, and let

$$
(2.4) \qquad \begin{aligned} \Gamma_0 &= \bigl\{\bigl(y(T), y'(T)\bigr)\colon\ y \text{ is a solution of (2.2)}\bigr\} \subset \mathbb{R}^2, \\ \Gamma_T &= \bigl\{\bigl(z(T), z'(T)\bigr)\colon\ z \text{ is a solution of (2.3)}\bigr\} \subset \mathbb{R}^2. \end{aligned}
$$

Roughly speaking, the set Γ_0 consists of all the points in the phase space reached at time $t = T$ by at least one solution of (2.2), while the set Γ_T consists of all the points with the following property: the solution of the Cauchy problem given by the equation in (2.3) with any of these points as initial condition at time $t = T$ is also a solution of (2.3).

Theorem 2.1. *Assume that* (2.1) *holds, and let* Γ_0, Γ_T *be the sets defined in* (2.4). *Assume also that there exist two continuums* S_0, S_T *in* $\mathbb{R}^2$, *with* $S_0 \subseteq \Gamma_0$, $S_T \subseteq \Gamma_T$. *If*

$$
(2.5) \qquad S_0 \cap S_T \neq \emptyset
$$

then (1.1) *with* $\lambda = 0$ *is solvable. Furthermore, if* (c_1, d_1), (c_2, d_2) *are two different points in* $S_0 \cap S_T$, *then the problem has at least two distinct solutions.*

Proof. Let $(c,d) \in S_0 \cap S_T$. Then $(c,d) \in \Gamma_0$ and there exists a solution y of (2.2) such that $y(T) = c$, $y'(T) = d$. Furthermore, $(c,d) \in \Gamma_T$ and there exists a solution z of (2.3) such that $z(T) = c$, $z'(T) = d$. Put

$$
x(t) = \begin{cases} y(t), & t \in [0,T], \\ z(t), & t > T, \end{cases}
$$

then x satisfies the equation in (1.1) with $\lambda = 0$ for all $t \geqslant 0$, and x, $a(t)\Phi(x') \in C^1[0,\infty)$. Next, since $y \in \mathfrak{B}_0$, then $x \in \widetilde{\mathfrak{B}}_0$, and since $z \in \mathfrak{B}_T$, then $x \in \widetilde{\mathfrak{B}}_T$. Condition (2.1) implies that $x \in \mathfrak{B}$ and therefore x is a solution of (1.1) with $\lambda = 0$. The second assertion immediately follows by observing that if x_1 is the solution corresponding to (c_1, d_1) and x_2 is the solution corresponding to (c_2, d_2), then $x_1(T) = c_1$, $x_1'(T) = d_1$, $x_2(T) = c_2$, $x_2'(T) = d_2$. □

Now consider the case where the boundary value problem depends on a real parameter λ, and consider the two auxiliary problems

$$\begin{cases} \big(a(t)\Phi(x')\big)' = f(t,x,x') + \lambda g(t,x,x'), & t \in [0,T], \\ x \in \mathfrak{B}_0, \end{cases} \tag{2.6}$$

and

$$\begin{cases} \big(a(t)\Phi(x')\big)' = f(t,x,x') + \lambda g(t,x,x'), & t \geqslant T, \\ x \in \mathfrak{B}_T, \end{cases} \tag{2.7}$$

where $\mathfrak{B}_0 \subset C[0,T]$, $\mathfrak{B}_T \subset C[T,\infty)$ satisfy (2.1). Let

$$\begin{aligned} \widehat{\Gamma}_0 &= \big\{\big(\Lambda, y(T), y'(T)\big) \colon\ y \text{ is a solution of (2.6) for } \lambda = \Lambda\big\} \subset \mathbb{R}^3, \\ \widehat{\Gamma}_T &= \big\{\big(\Lambda, z(T), z'(T)\big) \colon\ z \text{ is a solution of (2.7) for } \lambda = \Lambda\big\} \subset \mathbb{R}^3. \end{aligned} \tag{2.8}$$

The following existence result holds, which is the extension of Theorem 2.1 to the 3-dimensional case. The proof of this result can be easily deduced from the previous one, and is therefore omitted.

Theorem 2.2. *Assume that* (2.1) *holds, and let* $\widehat{\Gamma}_0$, $\widehat{\Gamma}_T$ *be the sets defined in* (2.8). *Assume also that there exist two continuums* $\widehat{S}_0$, $\widehat{S}_T$ *in* $\mathbb{R}^3$, *with* $\widehat{S}_0 \subseteq \widehat{\Gamma}_0$, $\widehat{S}_T \subseteq \widehat{\Gamma}_T$. *If*

$$\widehat{S}_0 \cap \widehat{S}_T \neq \emptyset \tag{2.9}$$

then (1.1) *is solvable. Furthermore, if* (λ_1, c_1, d_1), (λ_2, c_2, d_2) *are two different points in* $\widehat{S}_0 \cap \widehat{S}_T$, *then* (1.1) *has at least two distinct solutions.*

The key point in Theorems 2.1, 2.2 is the choice of the sets $\mathfrak{B}_0$ and $\mathfrak{B}_T$. Indeed, the auxiliary boundary value problems (2.2) and (2.3) (or (2.6) and (2.7)) need not only to be solvable, but also to admit a sufficiently large number of solutions so that the condition (2.5) (or (2.9)) is satisfied. Furthermore, the set of solutions of both the problems needs to have some good topological properties, in order to have the existence of the closed, connected subsets $S_0 \subseteq \Gamma_0$, $S_T \subseteq \Gamma_T$ ($\widehat{S}_0 \subseteq \widehat{\Gamma}_0$, $\widehat{S}_T \subseteq \widehat{\Gamma}_T$). This condition is fulfilled if, for instance, the solutions of both auxiliary problems have the property of continuous dependence on the initial data. More specifically, in order to apply Theorem 2.1, we have to choose two auxiliary problems such that each of them has at least a continuum of solutions (i.e., to prove the existence of S_0 and S_T), to study the properties of S_0 and S_T and to prove the existence of at least one intersection point. The same applies to Theorem 2.2.

In the next section, we present some applications of Theorems 2.1 and 2.2, pointing out the construction of the auxiliary problems and the related properties of the sets of solutions. The existence result for the first boundary value problem presented here is new and complements some previous results in [6], [15].

3. Applications

As a first example, consider the boundary value problem

$$\tag{3.1} \begin{cases} \big(r(t)\Phi(x')\big)' = q(t)f(x), \quad t \geqslant 0, \\ x(0)=0,\ x(t)>0, \quad t>0, \quad x'(t)<0, \quad t \text{ large}, \quad 0<\lim\limits_{t\to\infty} x(t) < \infty, \end{cases}$$

where $\Phi(u) = |u|^p \operatorname{sgn} u$ for $u \in \mathbb{R}$ and $p > 0$, r is a positive continuous function on $[0,\infty)$, f is a continuous increasing function on $\mathbb{R}$, satisfying the super-homogeneity condition

$$\tag{3.2} \text{(a)} \quad \lim_{u\to 0^+} \frac{f(u)}{\Phi(u)} = 0, \qquad \text{(b)} \quad \lim_{u\to\infty} \frac{f(u)}{\Phi(u)} = \infty,$$

and q is a continuous function on $[0,\infty)$, with $q(t) \leqslant 0$, $q(t) \not\equiv 0$, for $t \in [0,1]$, and $q(t) \geqslant 0$ for $t > 1$, $q(t) \not\equiv 0$ for large t.

The "crucial" point in the boundary conditions of (3.1) is the global request that the solution has to be positive for all $t > 0$ and zero at the initial point. This causes a lack of useful lower bounds and makes standard approaches difficult to apply.

Let

$$R(t) = \int_1^t r^{-1/p}(s)\,\mathrm{d}s, \quad J = \lim_{T\to\infty} \int_1^T \left(r^{-1}(t)\int_t^T q(s)\,\mathrm{d}s\right)^{1/p} \mathrm{d}t.$$

In [15] it is proved that the equation in (3.1) has at least one solution satisfying $x(0) = 0$, $x(t) > 0$ for $t > 0$, $\lim\limits_{t\to\infty} x(t) = 0$ if and only if $R(\infty) < \infty$ or $R(\infty) = \infty$ and $J = \infty$. We therefore study (3.1) in the remaining case, that is, we assume

$$\tag{3.3} R(\infty) = \infty, \quad J < \infty.$$

We point out that the condition that q has no constant sign on $[0,\infty)$ is necessary for the existence of at least one solution of (3.1). Indeed, if $q(t) \geqslant 0$ for $t \geqslant 0$, we can consider the function $G(t) = r(t)\Phi(x')x$, where x is a solution of (3.1). Since $G'(t) = q(t)f(x)x + r(t)|x'|^{p+1}$, then G is nondecreasing, and, as $G(0) = 0$, we obtain $G(t) \geqslant 0$ for $t > 0$. Thus, the positivity of x yields the existence of a point $t_0 > 0$ such that $G(t) > 0$ for $t \geqslant t_0$ which yields $x'(t) > 0$ for $t \geqslant t_0$, which is a contradiction. If $q(t) \leqslant 0$ for $t \geqslant 0$, for any solution x of (3.1) the quasiderivative $x^{[1]}(t) = r(t)\Phi(x'(t))$ is nonincreasing. Clearly, if $\lim\limits_{t\to\infty} x^{[1]}(t) = k \geqslant 0$, then $x'(t) \geqslant 0$ for all t, which is a contradiction. Then $\lim\limits_{t\to\infty} x^{[1]}(t) = -k < 0$, which implies $x^{[1]}(t) < -k/2$ for large t. Integrating the inequality $x'(t) < -r(t)^{-1/p}(k/2)^{1/p}$ on $[T, t_0]$, with T sufficiently large, we get

$$x(T) - x(t_0) < -(k/2)^{1/p} \int_{t_0}^{T} r^{-1/p}(s)\,\mathrm{d}s,$$

which contradicts as $T \to \infty$ the positivity of x. Therefore, for the solvability of (3.1), the function q needs to change its sign at least once.

In order to solve (3.1), we can consider the following two auxiliary boundary value problems on $[0,1]$ and on $[1,\infty)$, respectively:

$$\begin{cases} \bigl(r(t)\Phi(x')\bigr)' = q(t)f(x), & t \in [0,1], \\ x(0) = 0, \quad x(t) > 0, & t \in (0,1), \end{cases} \tag{3.4}$$

and

$$\begin{cases} \bigl(r(t)\Phi(x')\bigr)' = q(t)f(x), & t \geqslant 1, \\ x(t) > 0, \quad x'(t) < 0, & t \geqslant 1. \end{cases} \tag{3.5}$$

Notice that $q(t) \leqslant 0$ in $[0,1]$, while $q(t) \geqslant 0$ in $[1,\infty)$.

Step 1. Existence of the continuum S_0 and its properties. By a classical result by Erbe and Wang [10], [19], see also [15], problem (3.4) under conditions (3.2) has at least two solutions y, w satisfying the additional conditions $y(1) = 0$, $w'(1) = 0$, respectively. If we consider the Cauchy problem

$$\begin{cases} \bigl(r(t)\Phi(x')\bigr)' = q(t)f(x_+), & t \in [0,1], \\ x(0) = 0, \quad x'(0) = A > 0, \end{cases} \tag{3.6}$$

where $x_+ = \max\{x, 0\}$, clearly every solution of (3.6) which is positive in $(0,1)$ is also a solution of (3.4), and vice versa. Indeed, since $r(t)\Phi(x')$ is nonincreasing, assuming for contradiction $x'(0) = 0$, it follows that $x'(t) \leqslant 0$ for $t \in [0,1]$, which, together with the condition $x(0) = 0$, contradicts the positivity of x in $(0,1)$.

The following properties for the solutions of (3.6) hold. The proof can be found in [15] and here is omitted.

Lemma 3.1. *Every solution of* (3.6) *is defined on the whole interval* $[0,1]$. *Furthermore, if x is a solution of* (3.6), *with* $x(t_0) \leqslant 0$ *for* $0 < t_0 \leqslant 1$, *then* $x'(t_0) < 0$.

In particular, since all the solutions of (3.6) are persistent, the following generalization of Kneser's theorem (see for instance [4]) can be applied to solutions of (3.6).

Proposition 3.1. *Consider the system*

$$z' = F(t,z), \quad (t,z) \in [a,b] \times \mathbb{R}^n,$$

where F is continuous, and let K_0 be a continuum (i.e., compact and connected) subset of $\{(t,z)\colon\ t = a\}$ *and* $\mathcal{Z}(K_0)$ *the family of all the solutions emanating from* K_0. *If any solution* $z \in \mathcal{Z}(K_0)$ *is defined on the interval* $[a,b]$, *then the cross-section* $\mathcal{Z}(b;K_0) = \{z(b)\colon\ z \in \mathcal{Z}(K_0)\}$ *is a continuum in* $\mathbb{R}^n$.

Now, let $A_y = y'(0) > 0$, $A_w = w'(0) > 0$, where y, w are the solutions of (3.4) satisfying the additional conditions $y(1) = 0$, $w'(1) = 0$, respectively. Then y, w are also nonnegative solutions of (3.6) for $A = A_y$ and $A = A_w$, respectively. Assume, without restriction, $A_y < A_w$. Then Proposition 3.1 ensures that the set

$$T = \bigl\{\bigl(x(1), x'(1)\bigr)\colon\ x \text{ is a solution of (3.6) such that } x'(0) = A \in [A_y, A_w]\bigr\}$$

is a continuum in $\mathbb{R}^2$, containing the points $(w(1),0)$ and $(0,y'(1))$. Notice that Lemma 3.1 gives $w(1) > 0$ and $y'(1) < 0$. Furthermore, T does not contain any point $(0,c)$ with $c \geqslant 0$. It follows that a continuum $S_0 \subseteq T$ exists, such that S_0 is contained in $\overline{\pi} = \{(u,v)\colon\ u \geqslant 0,\ v \leqslant 0\}$, $(0,0) \notin S_0$, and there exist $R, M > 0$ such that $(R,0) \in S_0$, $(0,-M) \in S_0$. Furthermore,

$$S_0 \subseteq \Gamma_0 = \bigl\{\bigl(x(1), x'(1)\bigr)\colon\ x \text{ is a solution of (3.4)}\bigr\},$$

since every point of S_0 corresponds to a solution of (3.6) which is positive on $(0,1)$.

Step 2. Existence of the continuum S_1 and its properties. Now consider problem (3.5). In order to derive the properties of the solutions of (3.5), it is useful to compare them with the principal solutions of an associated half-linear equation. We recall that the notion of principal solutions, introduced for second order nonoscillatory linear equations by Leighton and Morse (see, e.g. [13], Chapter 11), has been extended to the half-linear equation

$$\bigl(r(t)\Phi(x')\bigr)' = p(t)\Phi(x), \quad t \geqslant 1 \tag{3.7}$$

in [9] by using the Riccati equation approach. (See also [8] for a good survey on the theory of principal solutions.) The set of principal solutions of (3.7) is always nonempty and for any $\mu \neq 0$ there exists a unique principal solution z such that $z(1) = \mu$, i.e., principal solutions are determined up to a constant factor. The characteristic properties of principal solutions for (3.7), when p is positive for $t \geqslant 1$, are investigated in [2]. In particular, it is shown that, roughly speaking, principal solutions of (3.7) are the smallest solutions in a neighborhood of infinity and, under the assumptions

$$\int_1^\infty r^{-1/p}(s)\,\mathrm{d}s = \infty, \qquad \lim_{T\to\infty}\int_1^T \left(r^{-1}(t)\int_t^T p(s)\,\mathrm{d}s\right)^{1/p}\mathrm{d}t < \infty,$$

any principal solution z satisfies $z(t)z'(t) < 0$ for every $t \geqslant 1$, and $\lim\limits_{t\to\infty} z(t) = l_z \neq 0$, see [2], Corollary 1. We point out that these properties hold also when $p(t) \geqslant 0$ for $t > 1$, $p(t) \not\equiv 0$ for large t.

Let $c > 0$ be a fixed constant. The following lemma compares the solutions of (3.5) and the principal solutions of (3.7), with $p(t) = Mq(t)$, where

$$M = \max_{u\in[0,c]} \frac{f(u)}{\Phi(u)}.$$

Notice that (3.2 a) ensures that M is well defined.

Lemma 3.2. *Assume* (3.2 a) *and* (3.3), *and let z_γ be the principal solution of the half-linear equation*

$$\big(r(t)\Phi(z')\big)' = Mq(t)\Phi(z)$$

with $z_\gamma(1) = \gamma$, $0 < \gamma \leqslant c$. Then for any solution x of (3.5), *with $x(1) = c$, we have*

$$x(t) \geqslant z_\gamma(t), \quad t \geqslant 1, \tag{3.8}$$

$$x'(1) \geqslant \frac{c}{\gamma}\, z'_\gamma(1), \tag{3.9}$$

$$0 < \lim_{t\to\infty} x(t) < \infty. \tag{3.10}$$

P r o o f. The argument is similar to the one in [15], Lemma 2.3. Here we only briefly indicate the main differences.

First of all notice that the assumption $R(\infty) = \infty$ implies that $\lim\limits_{t\to\infty} x^{[1]}(t) = \lim\limits_{t\to\infty} z_\gamma^{[1]}(t) = 0$, where we recall that $x^{[1]}(t) = r(t)\Phi(x'(t))$, $z_\gamma^{[1]}(t) = r(t)\Phi(z'_\gamma(t))$. Indeed, since $x^{[1]}(t)$ is negative nondecreasing, it has a finite nonpositive limit. If $\lim\limits_{t\to\infty} x^{[1]}(t) = -l < 0$, integrating the inequality $x^{[1]}(t) \leqslant -l$ on $[1,t]$, $t > 1$, we get

$$x(t) \leqslant x(1) - l\int_1^t r^{-1/p}(s)\,\mathrm{d}s$$

and letting $t \to \infty$ we obtain a contradiction with the positivity of x. The same holds for z_γ.

Put $g(t) = x(t) - z_\gamma(t)$, since x and z_γ are both positive decreasing, the limit $\lim\limits_{t\to\infty} g(t) = g(\infty)$ exists and it is finite. If $g(\infty) < 0$, there exists $T \geqslant 1$ such that $x(t) < z_\gamma(t)$ for every $t > T$, i.e., $\Phi(x(t)) < \Phi(z_\gamma(t))$ for every $t > T$. Furthermore, since $g(1) = c - \gamma \geqslant 0$, we can assume that $g(T) = 0$. Then, taking into account that $\lim\limits_{t\to\infty} x^{[1]}(t) = \lim\limits_{t\to\infty} z_\gamma^{[1]}(t) = 0$, it results for $t \geqslant T$ that

$$\begin{aligned}
-r(t)\big[\Phi(x') - \Phi(z_\gamma')\big] &= \int_t^\infty q(s)\big[f(x(s)) - M\Phi(z_\gamma(s))\big]\,\mathrm{d}s \\
&\leqslant M\int_t^\infty q(s)\big[\Phi(x(s)) - \Phi(z_\gamma(s))\big]\,\mathrm{d}s < 0,
\end{aligned}$$

i.e., $\Phi(x') > \Phi(z_\gamma')$, which implies $g'(t) > 0$ for $t \geqslant T$. This is a contradiction, as $g(T) = 0$ and $g(t) < 0$ for $t > T$. Thus $g(\infty) \geqslant 0$. Since $g(1) \geqslant 0$, in order to prove that g is nonnegative on $[1,\infty)$, it is sufficient to prove that g has no negative minimum. Now the proof goes on as in [15], Lemma 2.3, and (3.8), (3.9) are proved. The positivity of the limit of x immediately follows from (3.8) since z_γ has a positive limit at infinity. $\square$

The existence of solutions for (3.5) now follows as an easy consequence of a result by Chanturiya [3].

Theorem 3.1. *Assume* (3.2 a) *and* (3.3). *Then for every $c > 0$ fixed,* (3.5) *has a unique solution satisfying $x(1) = c$.*

P r o o f. The existence of a solution x of the equation in (3.5), satisfying $x(1) = c$, $x(t) \geqslant 0$, $x'(t) \leqslant 0$ for all $t \geqslant 1$ follows by [3], Theorem 1. In order to prove that x satisfies also the conditions in (3.5), it is sufficient to show that $x'(t) < 0$ for all $t \geqslant 1$. First of all we remark that, under assumption (3.2 a), any nontrivial solution of the equation in (3.5), defined for all $t \geqslant 1$, is not eventually zero (see, e.g., [16], Theorem 1.2 and Remark 1.1). Now assume, for contradiction, that $t_0 \geqslant 1$ exists, such that $x'(t_0) = 0$. Let $G(t) = r(t)\Phi(x')x$. Since $G'(t) = q(t)f(x)x + r(t)|x'|^{p+1} \geqslant 0$, then G is nondecreasing, with $G(t_0) = 0$. If $G(t) \equiv 0$ for every $t \geqslant t_0$, then the positivity of r yields $x' \equiv 0$ on $[t_0,\infty)$, i.e., x is eventually constant and positive. This is impossible since $f(x) > 0$ for $x > 0$. Then $t_1 > t_0$ exists, such that $G(t) > 0$ for every $t > t_1$. Thus, $x'(t) > 0$ for every $t > t_1$, which is again a contradiction.

The uniqueness of the solution follows from a classical result by Mambriani, in which the same property is proved for a generalized Thomas-Fermi equation. Indeed, if f is increasing, then two positive solutions of the equation in (3.5), both defined for

$t \geqslant 1$, can cross at most in one point, including $t = \infty$. Furthermore, there cannot exist two positive solutions starting from the same point and having a finite limit at infinity. $\square$

For every $c > 0$ let $x(t; c)$ be the unique solution of (3.5) such that $x(1; c) = c$, and let $G\colon (0, \infty) \to (-\infty, 0)$ be the function that associates to every $c > 0$ the value of the derivative of x at $t = 1$:

$$G(c) = x'(1; c).$$

Let S_1 be the graph of G, i.e., $S_1 = \{(c, G(c)),\ c > 0\} \subset \mathbb{R}^2$. Then the following holds.

Theorem 3.2. *Under the assumptions of Theorem* 3.1, S_1 *is an unbounded continuum contained in* $\pi = \{(u, v)\colon\ u > 0,\ v < 0\}$ *and* $\lim\limits_{c \to 0^+} G(c) = 0$.

Proof. First we show that G is a continuous function on $(0, \infty)$. Fixed $\overline{c} > 0$, let $\{c_n\}$ be a positive sequence converging to $\overline{c}$. Denote $x_n(t) = x(t; c_n)$ and $\overline{x}(t) = x(t; \overline{c})$, and choose n large enough so that $\overline{c}/2 \leqslant c_n \leqslant 3\overline{c}/2$. Let $\overline{M} > 0$ such that $f(u) \leqslant \overline{M}\Phi(u)$ for every $u \in [0, 3\overline{c}/2]$, and let $\overline{z}$ be the principal solution of the half-linear equation $(r(t)\Phi(z'))' = \overline{M}q(t)\Phi(z)$ such that $\overline{z}(1) = \overline{c}/2$. Since $x_n(t) \leqslant c_n \leqslant 3\overline{c}/2$ and, taking into account Lemma 3.2, $x_n(t) \geqslant \overline{z}(t) \geqslant \lim\limits_{t \to \infty} \overline{z}(t) = l_z > 0$, the sequence $\{x_n\}$ is equibounded on $[1, \infty)$. Since

$$x_n^{[1]}(1) = -\int_1^\infty q(r) f(x_n(r))\,\mathrm{d}r,$$

using the Lebesgue dominated convergence theorem we get

$$\lim_n G(c_n) = \lim_n x_n'(1) = -\left(\frac{1}{r(1)}\int_1^\infty q(s) f(\overline{x}(s))\,\mathrm{d}s\right)^{1/p} = \overline{x}'(1) = G(\overline{c}).$$

Thus, G is a continuous map and so S_1 is a continuum.

It remains to show that S_1 has $(0, 0)$ as a cluster point. To this aim, let $\{c_n\}$ be a positive sequence such that $\lim_n c_n = 0$, and let $x_n(t) = x(t; c_n)$. Assume without restriction that $c_n \leqslant 1$ for every n, and let $\overline{M} > 0$ be such that $f(u) \leqslant \overline{M}\Phi(u)$ for every $u \in [0, 1]$. Considering the half-linear equation $(r(t)\Phi(z'))' = \overline{M}q(t)\Phi(z)$, let $z_n(t)$ and $\overline{z}(t)$ be the principal solutions satisfying $z_n(1) = c_n$ and $\overline{z}(1) = 1$, respectively. By the uniqueness up to a constant factor clearly we have $z_n(t) = c_n\overline{z}(t)$, $t \geqslant 1$. Then, from (3.9), it results that

$$0 > x_n'(1) \geqslant z_n'(1) = c_n\overline{z}'(1),$$

and letting $n \to \infty$ we have the assertion. $\square$

Step 3. Application of Theorem 2.1. We are now in a position to apply Theorem 2.1.

Theorem 3.3. *Assume that* (3.2) *and* (3.3) *hold. Then* (3.1) *has at least one solution.*

P r o o f. First of all notice that from (3.10) in Lemma 3.2 the problem (3.5) is equivalent to the following one:

$$\begin{cases} \big(r(t)\Phi(x')\big)' = q(t)f(x), \quad t \geqslant 1, \\ x(t) > 0,\ x'(t) < 0, \quad t \geqslant 1, \quad 0 < \lim\limits_{t\to\infty} x(t) < \infty. \end{cases} \tag{3.11}$$

If we put

$$\begin{aligned} \mathfrak{B} &= \big\{x \in C[0,\infty)\colon\ x(0) = 0, \quad x(t) > 0, \quad t > 0, \\ &\qquad\qquad x'(t) < 0, \quad t \text{ large}, \quad 0 < \lim_{t\to\infty} x(t) < \infty\big\}, \\ \mathfrak{B}_0 &= \big\{x \in C[0,1]\colon\ x(0) = 0, \quad x(t) > 0, \quad t \in (0,1]\big\}, \\ \mathfrak{B}_1 &= \big\{x \in C[1,\infty)\colon\ x(t) > 0, \quad x'(t) < 0, \quad t \geqslant 1, \quad 0 < \lim_{t\to\infty} x(t) < \infty\big\}, \end{aligned}$$

then clearly $\widetilde{\mathfrak{B}}_0 \cap \widetilde{\mathfrak{B}}_1 \subset \mathfrak{B}$. We have seen that

$$S_0 \subseteq \Gamma_0 = \big\{\big(x(1), x'(1)\big)\colon\ x \text{ is a solution of } (3.4)\big\}$$

is a continuum in $\mathbb{R}^2$, and also

$$S_1 = \big\{\big(x(1), x'(1)\big)\colon\ x \text{ is a solution of } (3.11)\big\}$$

is a continuum in $\mathbb{R}^2$, emanating from the origin and unbounded, see Figure 1 below.

Since, clearly, $S_0 \cap S_1 \neq \emptyset$, then (3.1) has at least one solution. $\square$

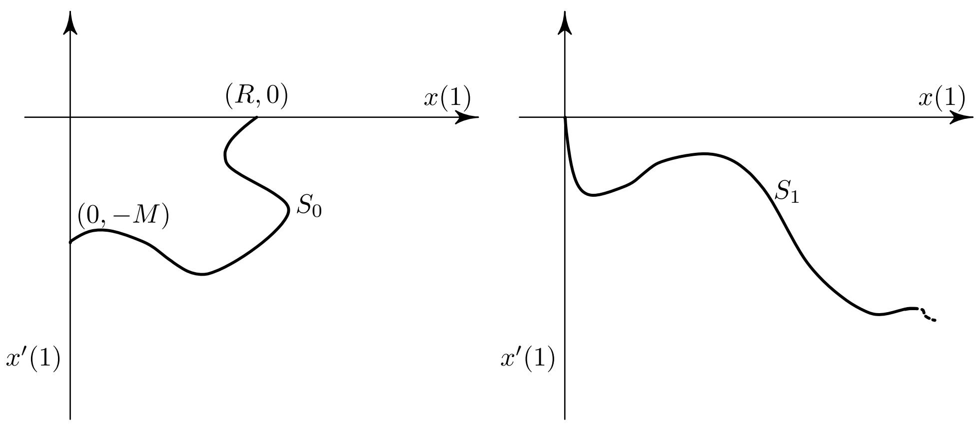

Figure 1. The connected sets S_0 and S_1.

Theorem 3.3 complements the results in [15], since it deals with complementary assumptions on the coefficients, and the results in [7] since the zero initial condition is assumed.

Other applications of Theorems 2.1 and 2.2 can be found in [5], [15]. More precisely, in [15] the boundary value problem

$$\begin{cases} \big(r(t)\Phi(x')\big)' = q(t)f(x), \quad t \geqslant 0, \\ x(0) = 0, \ x(t) > 0, \quad t > 0, \quad \lim\limits_{t\to\infty} x(t) = 0, \end{cases} \tag{3.12}$$

is considered, where, as in the previous example, $\Phi(u) = |u|^p \operatorname{sgn} u$ for $u \in \mathbb{R}$ and $p > 0$, r is a positive continuous function on $[0,\infty)$, f is a continuous increasing function on $\mathbb{R}$, satisfying the super-homogeneity condition (3.2) and q is a continuous function on $[0,\infty)$, with $q(t) \leqslant 0$, $q(t) \not\equiv 0$, for $t \in [0,1]$, and $q(t) \geqslant 0$ for $t > 1$, $q(t) \not\equiv 0$ for large t.

It is also possible to apply Theorem 2.1 to this problem in order to prove the existence of at least one solution. The reasoning is very similar to that seen in the first example, and the details can be found in [15].

An application of Theorem 2.2 can be found in [5]. We limit ourself to describe here the main ideas. The problem is the following:

$$\begin{cases} \big(r(t)\Phi(x')\big)' + f(t,x) = \lambda q(t)F(x), \quad t \geqslant 0, \\ x(0) = 0, \quad x'(1) \leqslant 0, \quad x(t) > 0, \quad t > 0, \quad 0 < \lim\limits_{t\to\infty} x(t) < \infty, \end{cases} \tag{3.13}$$

where $\Phi(u) = |u|^p \operatorname{sgn} u$ for $u \in \mathbb{R}$ and $0 < p \leqslant 1$, r is a positive continuous function on $[0,\infty)$ and f is a continuous function on $[0,\infty) \times [0,\infty)$ such that $f(t,u) \geqslant 0$ on $[0,1] \times [0,\infty)$, $f(t,u) \equiv 0$ on $[1,\infty) \times [0,\infty)$, and there exists $[T_1,T_2] \subseteq [0,1]$ such that $\min\limits_{t\in[T_1,T_2]} f(t,u) > 0$ for any $u > 0$. Furthermore, $\lambda > 0$ is a real parameter, q is a continuous function on $[0,\infty)$ such that $q(t) \equiv 0$ on $[0,1]$, and F is a positive continuous function on $[0,\infty)$ and locally Lipschitz on $(0,\infty)$. No assumption is made on the sign of q.

Put, similarly to the previous examples,

$$R(t) = \int_1^t r^{-1/p}(s)\,\mathrm{d}s, \quad J = \lim_{T\to\infty} \int_1^T \left(r^{-1}(t)\int_t^T |q(s)|\,\mathrm{d}s\right)^{1/p} \mathrm{d}t.$$

The existence of solutions of (3.13) is studied under the assumptions

$$R(\infty) = \infty, \quad J < \infty, \tag{3.14}$$

and the super-homogeneity or sub-homogeneity condition on f, i.e., in the case where f satisfies

$$\lim_{u\to 0+}\Big(\max_{t\in[0,1]}\frac{f(t,u)}{\Phi(u)}\Big)=0,\quad \lim_{u\to\infty}\Big(\min_{t\in[T_1,T_2]}\frac{f(t,u)}{\Phi(u)}\Big)=\infty, \tag{3.15}$$

or

$$\lim_{u\to 0+}\Big(\min_{t\in[T_1,T_2]}\frac{f(t,u)}{\Phi(u)}\Big)=\infty,\quad \lim_{u\to\infty}\Big(\max_{t\in[0,1]}\frac{f(t,u)}{\Phi(u)}\Big)=0. \tag{3.16}$$

Since $q\equiv 0$ in $[0,1]$ and $f\equiv 0$ for $t\geqslant 1$, the auxiliary boundary value problems considered are

$$\begin{cases} \big(r(t)\Phi(x')\big)'+f(t,x)=0, & t\in[0,1],\\ x(0)=0,\quad x(t)>0, & t\in(0,1), \end{cases} \tag{3.17}$$

and

$$\begin{cases} \big(r(t)\Phi(x')\big)'=\lambda q(t)F(x), \quad t\geqslant 1,\\ x'(1)\leqslant 0,\quad x(t)>0,\quad t\geqslant 1,\quad 0<\lim\limits_{t\to\infty}x(t)<\infty. \end{cases} \tag{3.18}$$

Step 1. Existence of the continuum $\widehat{S}_0$ and its properties. Notice that problem (3.17) does not depend on the parameter λ. Since f satisfies the assumption (3.15) or (3.16), the proof that the set of all solutions of (3.17) contains a continuum $S_0\subset\mathbb{R}^2$ follows with minor changes from the analogous result seen in the first application. Also in this case S_0 is contained in $\overline{\pi}=\{(u,v)\colon\ u\geqslant 0,\ v\leqslant 0\}$, $(0,0)\notin S_0$, and there exist $R,M>0$ such that $(R,0)\in S_0$, $(0,-M)\in S_0$, see Figure 1. Thus

$$\widehat{S}_0=\{(\lambda,c,d)\colon\ \lambda\in\mathbb{R},\ (c,d)\in S_0\}$$

is a continuum in $\mathbb{R}^3$. Roughly speaking, it is a "cylindric" surface, orthogonal to the plane $\lambda=0$, whose projection on $\lambda=0$ is the continuum S_0.

Step 2. Existence of the continuum $\widehat{S}_1$ and its properties. For every $c>0$ fixed, let $I_c=[c/2,3c/2]$ and assume that

$$M_{c,F}=\inf_{c>0}\frac{\min_{I_c}F(x)}{\max_{I_c}F(x)}>0. \tag{3.19}$$

The class of continuous functions satisfying the above conditions is sufficiently wide. For instance if F is a sum of positive powers then it satisfies (3.19). See [5] for a complete discussion on this assumption. The following existence result for (3.18), based on the Banach contraction theorem, holds.

Theorem 3.4. *Assume* (3.14) *and*

$$\frac{\int_1^\infty q_-(t)\,\mathrm{d}t}{\int_1^\infty q_+(t)\,\mathrm{d}t} \leqslant M_{c,F}, \tag{3.20}$$

for every $c > 0$, where q_-, q_+ denote the negative and the positive part of q, respectively. Then for any $c > 0$ there exists $\lambda_c > 0$, depending continuously on c, such that for any positive $\lambda \leqslant \lambda_c$ the problem (3.18) *has a unique solution x satisfying $x(1) = c$, $c/2 \leqslant x(t) \leqslant 3c/2$, for all $t \geqslant 1$.*

Under the assumptions of Theorem 3.4, let $D = \{(\lambda, c)\colon\ 0 < \lambda \leqslant \lambda_c,\ c > 0\}$ and denote by $x(t;\lambda,c)$, $(\lambda,c) \in D$, the unique solution of (3.18) satisfying $x(1;\lambda,c) = c$ and laying in the strip I_c. Let $G\colon\ D \to (-\infty, 0]$ be the function that associates to every $(\lambda, c) \in D$ the value of the derivative of x at $t = 1$:

$$G(\lambda, c) = x'(1;\lambda,c),$$

and let $\widehat{S}_1$ be the graph of G, i.e. $\widehat{S}_1 = \{(\lambda, c, G(\lambda,c)),\ (\lambda,c) \in D\} \subset \mathbb{R}^3$. The properties of the set $\widehat{S}_1$ are described by the following theorem, based on the proof of the continuity of the map G.

Theorem 3.5. *Under the assumptions of Theorem* 3.4*, $\widehat{S}_1$ is an unbounded continuum and $\lim\limits_{c\to 0^+} G(\lambda,c) = 0$ in D, $\lim\limits_{\lambda\to 0^+} G(\lambda,c) = 0$ for every $c > 0$.*

Step 3. Application of Theorem 2.2. Taking into account Theorems 3.4 and 3.5, the following holds.

Theorem 3.6. *Assume that* (3.15) *or* (3.16) *holds and let* (3.14) *and* (3.20) *be satisfied. Then the boundary value problem* (3.13) *has at least one solution for every $\lambda > 0$ sufficiently small.*

Proof. We have seen that $\widehat{S}_0$, $\widehat{S}_1$ are continuums in $\mathbb{R}^3$, both contained in $\{(u,v,w)\colon\ u \geqslant 0,\ v \geqslant 0,\ w \leqslant 0\}$. In particular, $\widehat{S}_0 = \{(\lambda,c,d)\colon\ \lambda > 0,\ (c,d) \in S_0\}$ while $\widehat{S}_1$ is the unbounded graph of a continuous function G of (λ, c). Then clearly $\widehat{S}_0$ and $\widehat{S}_1$ have nonempty intersection, and there exists a continuous curve whose image γ lays in the intersection

$$\widehat{S}_0 \cap \widehat{S}_1 \supseteq \gamma,$$

and whose projection on the λ-axis is an interval $(0, \widehat{\lambda})$ for $\widehat{\lambda} > 0$ sufficiently small. $\square$

Several open problems are related to the results presented in this section.

First of all, in the boundary value problems (3.1) and (3.12), the function q is assumed to have only one change of sign, and in particular $q(t) \geqslant 0$ for $t \geqslant 1$. Thus the auxiliary problem on $[1, \infty)$ has nonnegative weight, and this allows us to use the principal solutions associated to a suitable half-linear equation as positive lower bounds for the solutions of the problem in $[1, \infty)$. If q has no constant sign on $[1, \infty)$, then the principal solutions of the half-linear equation still exist, but, in general, they are no more positive on the whole half-line. How to deal with this case is still an open problem for (3.12), in which the limit of the solution at infinity is required to be zero.

The abstract method proposed in Section 2 allows also to prove the multiplicity of solutions of (1.1), provided we are able to prove that the sets S_0 and S_1 (or $\widehat{S}_0$ and $\widehat{S}_1$) have more than one point of intersection. Several results in the literature deal with multiplicity of solutions for problems in compact intervals, see for instance [11] and the literature therein, and can be applied to (2.2). It is an open problem how to use these results to prove the existence of more than one solution for (1.1).

In (3.13) we have assumed that $p \leqslant 1$. This condition is useful to obtain the uniqueness of the solutions of the BVP (3.18) which take values on I_c. The uniqueness property allows us to define the function G and to consider its graph, that is, the surface $\widehat{S}_1$. Under the more general condition $p > 0$, it is still possible to prove, for instance using the Shauder-Tychonoff fixed point theorem, that (3.18) has at least one solution. Then G comes out to be, in general, a multimap with real values, and it is an open problem whether it has a continuous selection, whose graph would define the set $\widehat{S}_1$.

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