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AN ANALYSIS OF THE STABILITY BOUNDARY FOR A LINEAR FRACTIONAL DIFFERENCE SYSTEM

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Abstract. This paper deals with basic stability properties of a two-term linear autonomous fractional difference system involving the Riemann-Liouville difference. In particular, we focus on the case when eigenvalues of the system matrix lie on a boundary curve separating asymptotic stability and unstability regions. This issue was posed as an open problem in the paper J. Čermák, T. Kisela, and L. Nechvátal (2013). Thus, the paper completes the stability analysis of the corresponding fractional difference system.

Keywords: fractional difference system; stability; Laplace transform

MSC 2010: 39A06, 39A30, 39A12

1. Introduction

Fractional calculus is a discipline concerning integrals and derivatives of noninteger orders. In last decades, this originally theoretical concept has found numerous applications in both technology and science and nowadays it belongs to the most developing areas of mathematical analysis. For detailed information on its history and basics we refer, e.g., to monographs [6], [13].

In this paper, we consider a discrete fractional calculus, namely the fractional calculus built on a set of equidistant points $t_n = nh$, $h > 0$, $n \in \mathbb{Z}_0^+$, which is especially important for a numerical analysis of continuous fractional models. The key notion of discrete fractional calculus is the so-called fractional sum introduced as

$$\nabla_h^{-\gamma} f(t_n) = \sum_{k=1}^{n} h \binom{n-k+\gamma-1}{n-k} f(t_k),$$

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where the order γ is a positive real and $f(t_n)$ is a given sequence. The definition of a fractional difference is not unique. We are going to employ one of the two most common approaches, the so-called Riemann-Liouville fractional difference

$$\nabla_h^{\alpha} f(t_n) = \nabla_h^{\lceil\alpha\rceil} \nabla_h^{-(\lceil\alpha\rceil-\alpha)} f(t_n),$$

where the order α is a positive real, $\lceil\alpha\rceil = \min\{m \in \mathbb{Z};\ m \leqslant \alpha\}$ is the ceiling function and $f(t_n)$ is a given sequence. We recall that for a given $m \in \mathbb{Z}^+$ the mth difference is introduced recursively as $\nabla_h^m f(t_n) = \nabla_h \nabla_h^{m-1} f(t_n)$ and $\nabla_h f(t_n) = (f(t_n) - f(t_{n-1}))/h$. For more details on discrete fractional calculus, and in particular its backward version that we discuss, we refer to, e.g., [2], [5], [10].

While the stability analysis in the continuous fractional calculus is a quite established topic (see, e.g., the surveys [7], [12]), the related results in the discrete case are rather rare (see, e.g., [8] or [3], [4]). In [3] the fractional difference system

$$\nabla_h^{\alpha} y(t_n) = Ay(t_n), \quad 0 < \alpha < 1,\ n \in \mathbb{Z}^+, \tag{1.1}$$

$$\nabla_h^{\alpha-1} y(0) = y_0, \quad y_0 \in \mathbb{R}^d, \tag{1.2}$$

where A is a $d \times d$ constant real matrix and $y(t_n)$ is a d-vector, was considered. The authors gave the description of the stability behaviour depending on the position of eigenvalues in the complex plane except for the case when an eigenvalue lies on the boundary of the asymptotic stability region. In this paper, we focus on this missing part and complete the description of the stability behaviour of (1.1).

The paper is organized as follows. In Section 2 we recall the necessary background, such as the basics of the discrete Laplace transform, the used notation and key results of [3]. Section 3 is devoted to the main result, i.e., it involves the theorem completing the known results and the summarizing corollary. Section 4 concludes the paper with some additional comments and comparisons.

2. Used notation and known results

In the sequel, we adopt the notation introduced in [3]. We assume that the matrix A occuring in (1.1) is similar to a Jordan canonical form J. Thus, there exists an invertible matrix P such that $A = PJP^{-1}$, where $J = \operatorname{diag}(J_1, \ldots, J_s)$ and J_l are Jordan blocks of order r_l $(l = 1, \ldots, s)$. In addition, we assume that the matrix A is h^{α}-regressive, i.e. $I - h^{\alpha}A$ is invertible, which ensures the existence and uniqueness of the solution for (1.1), (1.2) (see [3], Proposition 4).

The key tool utilized in this paper is the discrete Laplace transform. For a given real sequence $f(t_n)$ it is defined as

$$\mathcal{L}\{f\}(s) = h\sum_{k=1}^{\infty} f(t_k)(1-hs)^{k-1} \tag{2.1}$$

for all points $s \in \mathbb{C}$ at which the series converges. More detailed information on its connection with the standard Laplace transform and other properties can be found, e.g., in [1] or [3].

As shown in [3], applying (2.1) to (1.1), (1.2) yields the Laplace transform of the solution in a form

$$\mathcal{L}\{y\}(s) = (s^\alpha I - A)^{-1} y_0 = P^{-1}(s^\alpha I - J)^{-1} P y_0, \tag{2.2}$$

where $(s^\alpha I - J)^{-1}$ is a diagonal block matrix. The number of blocks corresponding to every eigenvalue λ_i $(i = 1, \ldots, m)$ is equal to its geometric multiplicity p_i and their form is given by the upper triangular matrix

$$\begin{pmatrix} (s^\alpha - \lambda_i(A))^{-1} & (s^\alpha - \lambda_i(A))^{-2} & \ldots & (s^\alpha - \lambda_i(A))^{-r_q} \\ 0 & (s^\alpha - \lambda_i(A))^{-1} & \ldots & (s^\alpha - \lambda_i(A))^{-r_q+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & (s^\alpha - \lambda_i(A))^{-1} \end{pmatrix}, \tag{2.3}$$

where $q = 1, \ldots, p_i$ and r_q is the size of the block.

Now, we recall the notion of stability and asymptotic stability adapted for the linear system (1.1):

Definition 2.1. The fractional difference system (1.1) is said to be

(a) stable if for any $y_0 \in \mathbb{R}^d$ there exists $K > 0$ such that the solution $y(t_n)$ of (1.1), (1.2) satisfies $\|y(t_n)\| \leqslant K$ for all $n = 1, 2, \ldots$,

(b) asymptotically stable if for any $y_0 \in \mathbb{R}^d$ the solution $y(t_n)$ of (1.1), (1.2) satisfies $\|y(t_n)\| \to 0$ as $n \to \infty$.

For the sake of lucidity, we restate and reformulate the main results of [3] concerning the stability of (1.1):

Theorem 2.1. *Let $0 < \alpha < 1$. The fractional difference system* (1.1) *is*

(a) *asymptotically stable, if all eigenvalues of A lie inside the region*

$$\mathcal{S}_{\alpha,h} = \{z \in \mathbb{C}\colon\ z = h^{-\alpha}(1-w)^\alpha,\ w \in \mathbb{C},\ |w| > 1\},$$

(b) *not stable, if there exists an eigenvalue of A lying inside the region*

$$\mathcal{U}_{\alpha,h} = \{z \in \mathbb{C}\colon\ z = h^{-\alpha}(1-w)^{\alpha},\ w \in \mathbb{C},\ |w| < 1\}.$$

R e m a r k 2.1. In [3], the sets $\mathcal{S}_{\alpha,h}$ and $\mathcal{U}_{\alpha,h}$ are described by means of modulus and phase of z. We note that both forms are equivalent as can be verified by calculations in the complex plane.

Theorem 2.1 does not discuss the case when an eigenvalue lies on the boundary curve Ψ separating $\mathcal{S}_{\alpha,h}$ and $\mathcal{U}_{\alpha,h}$, which is given by

$$\Psi = \{z \in \mathbb{C}\colon\ z = h^{-\alpha}(1-w)^{\alpha},\ w \in \mathbb{C},\ |w| = 1\} \tag{2.4}$$

(see Figure 1). The only result of [3] concerning eigenvalues lying on Ψ discusses the case of the zero eigenvalue:

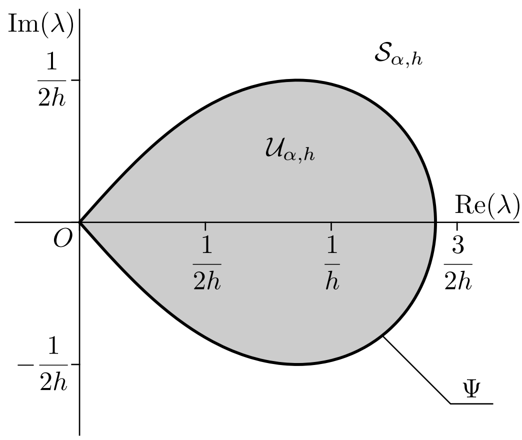

Figure 1. Stability plot for $\alpha = 1/2$.

Theorem 2.2. *Let $0 < \alpha < 1$, let A have the zero eigenvalue $\lambda_1 = 0$ and let all its nonzero eigenvalues belong to $\mathcal{S}_{\alpha,h}$. Denote by $\hat{r} \in \mathbb{Z}^+$ the maximal size of Jordan blocks corresponding to λ_1. The fractional difference system* (1.1) *is*

(a) *asymptotically stable, if $\hat{r} < \alpha^{-1}$,*
(b) *stable, but not asymptotically stable, if $\hat{r} = \alpha^{-1}$,*
(c) *not stable, if $\hat{r} > \alpha^{-1}$.*

3. Main result

Investigating the stability of (1.1) with eigenvalues of A lying on Ψ requires a different approach than that utilized in [3]. In particular, we cannot employ the radius of convergence nor Wiener's theorem. We overcome this inconvenience by the technique suggested in [8].

Theorem 3.1. *Let $0 < \alpha < 1$ and let A be a $d \times d$ matrix. Let $\lambda \in \Psi \setminus \{0\}$ be the eigenvalue of A with algebraic multiplicity d and geometric multiplicity $p \leqslant d$ ($p \in \mathbb{Z}^+$). The fractional difference system* (1.1) *is*

(a) *stable but not asymptotically stable if $p = d$,*

(b) *not stable if $p < d$.*

P r o o f. Let $y(t_n)$ be the solution of (1.1), (1.2) and let λ be the only eigenvalue of A. Considering (2.2) and (2.3), we can see that the Laplace transform of $y(t_n)$ can be written as a linear combination

$$\mathcal{L}\{y\}(s) = \sum_{j=1}^{d} b_j G_j(s) \tag{3.1}$$

where b_j ($j = 1, \ldots, d$) are constant real d-vectors and

$$G_j(s) = \frac{1}{(s^\alpha - \lambda)^j}, \quad j = 1, \ldots, d. \tag{3.2}$$

Obviously, $G_j(s)$ has a pole $\lambda^{1/\alpha}$ of order j, hence we can write it in the form

$$G_j(s) = \frac{F_j(s)}{(s - \lambda^{1/\alpha})^j}, \quad j = 1, \ldots, d, \tag{3.3}$$

where $F_j(s) = \mathcal{L}\{f_j\}(s)$ is a complex function analytic on $\mathbb{C}$ and $f_j(t_n)$ is a suitable auxiliary sequence. Utilizing the binomial formula (see, e.g., [11]), we can find the inverse Laplace transform of the term $1/(s - \lambda^{1/\alpha})^j$. Indeed, we have

$$\begin{aligned}
\frac{1}{(s - \lambda^{1/\alpha})^j} &= \left(\frac{-h}{1 - hs - 1 + h\lambda^{1/\alpha}}\right)^j = \left(\frac{h(1 - h\lambda^{1/\alpha})^{-1}}{1 - (1 - hs)/(1 - h\lambda^{1/\alpha})}\right)^j \\
&= h^j (1 - h\lambda^{1/\alpha})^{-j} \sum_{k=0}^{\infty} \binom{j + k - 1}{k} \left(\frac{1 - hs}{1 - h\lambda^{1/\alpha}}\right)^k \\
&= h \sum_{k=1}^{\infty} \binom{j + k - 2}{k - 1} \frac{h^{j-1}}{(1 - h\lambda^{1/\alpha})^{j+k-1}} (1 - hs)^{k-1} \\
&= \mathcal{L}\left\{\binom{j + n - 2}{n - 1} \frac{h^{j-1}}{(1 - h\lambda^{1/\alpha})^{j+n-1}}\right\}(s).
\end{aligned} \tag{3.4}$$

We are not able to determine the inverse Laplace transform of $F_j(s)$, i.e. $f_j(t_n)$, nevertheless for the stability analysis it suffices to show that a particular value $|F_j(\lambda^{1/\alpha})| = |\mathcal{L}\{f_j\}(\lambda^{1/\alpha})|$ is a finite positive real number. Combining (3.2) and (3.3), we obtain

$$F_j(s) = \frac{(s-\lambda^{1/\alpha})^j}{(s^\alpha - \lambda)^j} \quad (= \mathcal{L}\{f_j\}(s)), \quad j = 1, \dots, d,$$

which implies

$$\begin{aligned}(3.5) \qquad |\mathcal{L}\{f_j\}(\lambda^{1/\alpha})| &= \left| \lim_{s\to\lambda^{1/\alpha}} \left(\frac{s-\lambda^{1/\alpha}}{s^\alpha-\lambda}\right)^j \right| = \left| \left(\lim_{s\to\lambda^{1/\alpha}} \frac{s-\lambda^{1/\alpha}}{s^\alpha-\lambda}\right)^j \right| \\ &= \left| \left(\lim_{s\to\lambda^{1/\alpha}} \frac{1}{\alpha s^{\alpha-1}}\right)^j \right| = \frac{|\lambda^{1/\alpha-1}|^j}{\alpha^j} > 0.\end{aligned}$$

a) Let $d = p$, i.e., let algebraic and geometric multiplicities of λ be equal. In this case, we have $b_j = 0$ for $j = 2, \dots, d$ in (3.1), hence

$$\begin{aligned}\lim_{n\to\infty} \|y(t_n)\| &= \lim_{n\to\infty} \|b_1 \mathcal{L}^{-1}\{G_1\}(t_n)\| = \|b_1\| \lim_{n\to\infty} |\mathcal{L}^{-1}\{G_1\}(t_n)| \\ &= \|b_1\| \lim_{n\to\infty} \left| \mathcal{L}^{-1}\left\{ F_1(s) \frac{1}{s-\lambda^{1/\alpha}} \right\}(t_n) \right| = \|b_1\| \lim_{n\to\infty} \left| f_1(t_n) \frac{1}{(1-h\lambda^{1/\alpha})^n} \right| \\ &= \|b_1\| \lim_{n\to\infty} h \left| \sum_{k=1}^n f_1(t_k) \frac{1}{(1-h\lambda^{1/\alpha})^{n-k+1}} \right| \\ &= \|b_1\| \lim_{n\to\infty} \frac{1}{|1-h\lambda^{1/\alpha}|^n} \left| h \sum_{k=1}^n f_1(t_k)(1-h\lambda^{1/\alpha})^{k-1} \right| \\ &= \|b_1\| |\mathcal{L}\{f_1\}(\lambda^{1/\alpha})| \lim_{n\to\infty} \frac{1}{|1-h\lambda^{1/\alpha}|^n} \\ &= \|b_1\| |\mathcal{L}\{f_1\}(\lambda^{1/\alpha})| = \|b_1\| \frac{|\lambda^{1/\alpha}|^{1-\alpha}}{\alpha} > 0,\end{aligned}$$

where we utilize the convolution theorem for discrete Laplace transform (see, e.g., [3], Lemma 14), (3.4), (2.1), (3.5) and the fact that $\lambda \in \Psi$ implies $|1 - h\lambda^{1/\alpha}| = 1$ (see (2.4)). It follows that for $d = p$ the system (1.1) is stable but not asymptotically stable.

b) Now let $d > p$, therefore there exists an integer u ($2 \leqslant u \leqslant d$) such that $b_j \neq 0$ for $j = 1, \dots, u$ (for nonzero initial vector y_0). Thus we get

$$\lim_{n\to\infty} \|y(t_n)\| = \lim_{n\to\infty} \left\| \sum_{j=1}^u b_j \mathcal{L}^{-1}\{G_j\}(t_n) \right\|.$$

It follows from the series of equalities in a) that $|\mathcal{L}^{-1}\{G_1\}(t_n)|$ is bounded, so we turn our attention to the stability behaviour of functions $\mathcal{L}^{-1}\{G_j\}(t_n)$ for $j > 1$. Employing a technique similar to that used in the case $j = 1$ and the asymptotic expansion of a binomial coefficient (see, e.g., [11]), we can write

$$\begin{aligned}
\lim_{n\to\infty} |\mathcal{L}^{-1}\{G_j\}(t_n)| &= \lim_{n\to\infty} \left| f_j(t_n) \binom{j+n-2}{n-1} \frac{h^{j-1}}{(1-h\lambda^{1/\alpha})^{j+n-1}} \right| \\
&= \lim_{n\to\infty} h \left| \sum_{k=1}^{n} f_j(t_k) \binom{j+n-k-1}{n-k} \frac{h^{j-1}}{(1-h\lambda^{1/\alpha})^{j+n-k}} \right| \\
&= \lim_{n\to\infty} \left(\frac{h^{j-1}}{|1-h\lambda^{1/\alpha}|^{j+n-1}} \left| h \sum_{k=1}^{n} \binom{j+n-k-1}{n-k} f_j(t_k)(1-h\lambda^{1/\alpha})^{k-1} \right| \right) \\
&\geqslant \lim_{n\to\infty} \left| h^j \binom{j+n-2}{n-1} f_j(t_1) - h^j f_j(t_1) + h^j \sum_{k=1}^{n} f_j(t_k)(1-h\lambda^{1/\alpha})^{k-1} \right| \\
&= \left| h^j \lim_{n\to\infty} \binom{j+n-2}{n-1} f_j(t_1) - h^j f_j(t_1) + h^{j-1}\mathcal{L}\{f_j\}(\lambda^{1/\alpha}) \right| \\
&= \left| h^j f_j(t_1) \lim_{n\to\infty} \frac{(n-1)^{j-1}}{(j-1)!} - h^j f_j(t_1) + h^{j-1}\mathcal{L}\{f_j\}(\lambda^{1/\alpha}) \right| = \infty.
\end{aligned}$$

Since the vectors b_j $(j = 1, \dots, u)$ depend on the choice of y_0, we conclude that $\|y(t_n)\|$ tends to infinity for some y_0 and therefore (1.1) is not stable. □

Theorem 3.1 supplements Theorems 2.1 and 2.2 and completes the stability analysis of (1.1). Considering (2.2), (2.3) and a basic matrix calculus, it enables us to summarize the results:

Corollary 3.1. *Let $0 < \alpha < 1$ and $\lambda_i \in \mathbb{C}$ $(i = 1, \dots, m)$ be distinct eigenvalues of a $d \times d$ matrix A. Further, denote $\Lambda = \{\lambda_i,\ i = 1, \dots, m\}$, $\Lambda_S = \{\lambda_i \colon\ \lambda_i \in \mathcal{S}_{\alpha,h}\}$ and $\Lambda_\Psi = \{\lambda_i \colon\ \lambda_i \in \Psi\}$. The fractional difference system* (1.1) *is*

(a) *asymptotically stable, if and only if $\Lambda = \Lambda_S \cup (\Lambda_\Psi \cap \{0\})$ and the maximal size of the Jordan blocks corresponding to the zero eigenvalue is less than α^{-1},*
(b) *stable, if and only if $\Lambda = \Lambda_S \cup \Lambda_\Psi$ and all nonzero elements of Λ_Ψ have the same algebraic and geometric multiplicities and the maximal size of the Jordan blocks corresponding to the zero eigenvalue is less than or equal to α^{-1}.*

4. Concluding remarks

We have completed the stability analysis of the fractional difference system (1.1) by investigation of the case of eigenvalues lying on the boundary curve Ψ. This result enables us to write the stability assertions for (1.1) in the form of equivalences such as Corollary 3.1. We can rephrase this corollary for the scalar case as follows: The equation $\nabla_h^\alpha y(t_n) = \lambda y(t_n)$ $(0 < \alpha < 1)$ is stable if and only if $\lambda \in \mathcal{S}_{\alpha,h} \cup \Psi$ and asymptotically stable if and only if $\lambda \in \mathcal{S}_{\alpha,h} \cup \{0\}$.

The discussed stability behaviour corresponds with the results achieved for the continuous analogue of (1.1) in [14], where, among other similarities, the zero eigenvalue also represents the only point of the boundary curve that allows asymptotic stability.

In [9] the author performed the stability analysis of the continuous fractional differential system involving the Caputo fractional derivative. We note that the results and techniques presented in this paper and in [3] can also be utilized for stability analysis of the discrete counterpart of this problem with corresponding results.

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