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A subclass of strongly clean rings

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Abstract. In this paper, we introduce a subclass of strongly clean rings. Let R be a ring with identity, J be the Jacobson radical of R, and let $J^{\#}$ denote the set of all elements of R which are nilpotent in R/J. An element $a \in R$ is called *very $J^{\#}$-clean* provided that there exists an idempotent $e \in R$ such that $ae = ea$ and $a - e$ or $a + e$ is an element of $J^{\#}$. A ring R is said to be *very $J^{\#}$-clean* in case every element in R is very $J^{\#}$-clean. We prove that every very $J^{\#}$-clean ring is strongly π-rad clean and has stable range one. It is shown that for a commutative local ring R, $A(x) \in M_2\big(R[[x]]\big)$ is very $J^{\#}$-clean if and only if $A(0) \in M_2(R)$ is very $J^{\#}$-clean. Various basic characterizations and properties of these rings are proved. We obtain a partial answer to the open question whether strongly clean rings have stable range one.

This paper is dedicated to Professor Abdullah Harmanci on his 70th birthday

1 Introduction

Throughout this paper, all rings are associative with identity unless otherwise stated. Nicholson in [16] defined clean elements and clean rings, also in [17] Nicholson and Zhou introduced strongly clean rings and Chen continued studying strongly clean rings and introduced strongly J-clean rings in [5]. Other generalizations of clean notion of rings are investigated by many authors ([4], [6], [10], [12]). Let U denote the set of all invertible elements and J be the Jacobson radical of R. In this paper, the set of all elements of R which are nilpotent in R/J will be denoted by $J^{\#}$. Clearly, $J \subseteq J^{\#}$. Let a be an element of R. The element a is called *clean* provided that there exist $e^2 = e \in R$ and $u \in U$ such that $a = e + u$. The element a is *strongly clean* if there exist $e^2 = e \in R$ and $u \in U$ such that $a = e + u$ and $eu = ue$. An element a is called *very clean* if there exists $e^2 = e \in R$ and $u \in U$ such that $a = e + u$ or $a = -e + u$ and $eu = ue$. In general, $a \in R$ is (strongly or very) $\mathcal{T}$-clean if and only if there exists an idempotent $e \in R$ such that ($ae = ea$ and) $a - e$ (or potentially $a + e$ for very cleanness) is in the set related to $\mathcal{T}$. Here,

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$\mathcal{T} \in \{\mathrm{Nil}, J, J^{\#}\}$, and the corresponding sets are $\mathrm{Nil}(R)$ (the set of all nilpotent elements of R), J and $J^{\#}$, respectively. A ring R is said to have *stable range one* if given $a, b \in R$ for which $aR + bR = R$, there exists $y \in R$ such that $a + by \in U$. One of the most important features of stable range one is the cancellation of related modules from direct sums. We know that stable range one in endomorphism rings implies cancellation in direct sums, that is, if A, B, C are modules such that $A \oplus B \cong A \oplus C$, and $\mathrm{End}(A)$ has stable range one, then $B \cong C$ [11, Theorem 2]. Further, if R is directly finite, i.e., any $x, y \in R$ satisfying $xy = 1$ also satisfy $yx = 1$, then so is $M_n(R)$ (for details one can see [6]). But so far it is unknown whether strongly clean rings have stable range one (see [17]). This motivates us to construct a natural subclass of strongly clean rings, namely, very $J^{\#}$-clean rings, which have stable range one.

Clearly, every commutative or Artinian strongly nil clean ring is strongly J-clean. But the converse is not true in general (see [5] or Example 2). Since $\mathrm{Nil}(R) \subseteq J^{\#}$ and $J \subseteq J^{\#}$, we know that strongly J-clean rings and strongly nil-clean rings are strongly $J^{\#}$-clean, and every very nil-clean ring is very $J^{\#}$-clean. Example 2 is a very $J^{\#}$-clean ring, which is not very nil-clean. Every strongly $J^{\#}$-clean ring is very $J^{\#}$-clean but Example 3 is very $J^{\#}$-clean, which is not strongly $J^{\#}$-clean. Any very $J^{\#}$-clean ring is strongly clean (see Theorem 1) but there exists a strongly clean ring which is not very $J^{\#}$-clean (e.g. $\mathbb{Z}_5$). Every strongly clean ring is very clean. Example 4 is a very clean ring, which is not strongly clean. Now we illustrate relations between these classes of rings in the following:

$$\begin{tikzcd}
\text{Strongly } J\text{-clean} \arrow[r] & \text{Strongly } J^{\#}\text{-clean} \arrow[r] & \text{Very } J^{\#}\text{-clean} \arrow[d] \\
\text{Strongly nil-clean} \arrow[r] \arrow[ur] & \text{Very nil-clean} \arrow[ur] & \text{Strongly clean} \arrow[d] \\
 & & \text{Very clean}
\end{tikzcd}$$

None of the implications in the diagram are reversible.

The paper is organized as follows: in Section 2, basic properties of very $J^{\#}$-clean rings are given. We give some examples concerning their relations with clean rings, strongly clean rings, strongly $J^{\#}$-clean rings. Further, we prove that if R is very $J^{\#}$-clean, then R has stable range one. In Section 3, we construct several examples of very $J^{\#}$-clean rings. For instance, if R is an abelian very $J^{\#}$-clean ring, then the ring $R[[x]]$ of power series over R is very $J^{\#}$-clean. In Section 4, we characterize the very $J^{\#}$-cleanness of matrices over commutative local rings. Further, we consider very $J^{\#}$-clean power series rings over such matrix rings.

In what follows, for a positive integer n, $\mathbb{Z}_n$ and $\mathbb{N}$ denote the ring of integers modulo n and the natural numbers, while for a prime integer p, $\mathbb{Z}_{(p)}$ denotes the ring of integers localized at the prime ideal (p), and we write $M_n(R)$ for the rings of all $n \times n$ matrices over a ring R. We write $R[[x]]$ and $\mathrm{Nil}(R)$ for the ring of power series over R and the set of all nilpotent elements of R, respectively. Let $\bar{R}$ denote the quotient ring R/J.

2 Elementary results

Recall that a ring R is called *local* if it has only one maximal left ideal (equivalently, maximal right ideal). It is well known that a ring R is local if and only if $a+b=1$ in R implies that either a or b is invertible if and only if $\bar{R}$ is a division ring. A ring R is said to be *reduced* if it has no non-zero nilpotent elements. Now we begin with the simple result.

Lemma 1. *For a ring R we have that $\bar{R}$ is reduced if and only if $J^{\#}=J$. In particular, $J^{\#}=J$ if R is commutative or local or $\bar{R}$ is the direct sum of division rings.*

It is clear from Lemma 1 that if R is a commutative or local ring, then $a \in R$ is strongly $J^{\#}$-clean if and only if $a \in R$ is strongly J-clean. Recall that a ring R is called *uniquely clean* if every element can be written uniquely as the sum of an idempotent and a unit (see [18]).

Lemma 2. *Let $\bar{R}$ be a direct sum of division rings. Then the following are equivalent.*

(1) R is strongly $J^{\#}$-clean.

(2) $\bar{R}$ is a direct sum of two-element fields.

Proof. Note that if $\bar{R}$ is a direct sum of division rings (every local ring or commutative Artinian ring has this property), then R is (strongly, very) $J^{\#}$-clean if and only if R is (strongly, very, respectively) J-clean, because, by Lemma 1, we have $J^{\#}=J$. Let $\mathbb{F}_n$ denote the field with n elements.

(1) $\Rightarrow$ (2) Since R is strongly J-clean, we have $\bar{R}$ is Boolean, and so $\bar{R} \cong \oplus\mathbb{F}_2$ because $\bar{R}$ is a direct sum of division rings.

(2) $\Rightarrow$ (1) Assume that $\bar{R}$ is a direct sum of two-element fields. Then R is uniquely clean by [18, Corollary 16]. This implies that R is abelian (that is, all idempotents in R are central) and for all $a \in R$ there exists a unique idempotent $e \in R$ such that $e-a \in J$ by [18, Theorem 20]. Thus R is strongly J-clean. $\square$

One may suspect that if $\bar{R}$ is a direct sum of two- or three-element fields, then R is very $J^{\#}$-clean. The following example shows that this is not true in general.

Example 1. Let R denote the ring $\mathbb{Z}_9 \oplus \mathbb{Z}_9$. Then we have $\bar{R}=\mathbb{Z}_3 \oplus \mathbb{Z}_3$ and the only idempotents of the ring R are $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$. Further, note that $J^{\#}=J$. Hence $(2,4) \in R$ is not (strongly) very $J^{\#}$-clean.

(Strongly) Nil-clean elements (rings) are introduced by Diesl in [9], [10]. Clearly, every strongly nil-clean element (ring) is a strongly $J^{\#}$-clean element (ring). But there exists a strongly $J^{\#}$-clean element (ring) which is not strongly nil-clean element (ring) as the following example shows (see [5]).

Example 2. Let $R=\prod_{n=1}^{\infty}\mathbb{Z}_{2^n}$. For each $n \in \mathbb{N}$, $\mathbb{Z}_{2^n}$ is a local ring with the maximal ideal $2\mathbb{Z}_{2^n}$. Then $\mathbb{Z}_{2^n}/2\mathbb{Z}_{2^n} \cong \mathbb{Z}_2$. Hence R is strongly J-clean, and so R

is strongly $J^{\#}$-clean (and very $J^{\#}$-clean). Since the element $r = (0, 2, 2, \dots) \in R$ is not strongly nil-clean (and not very nil-clean), R is not strongly nil-clean (and not very nil-clean).

Every strongly $J^{\#}$-clean (strongly J-clean) ring is very $J^{\#}$-clean (very J-clean) but there exists a very $J^{\#}$-clean (very J-clean) ring which is not strongly $J^{\#}$-clean (strongly J-clean) as the following example shows.

Example 3. The ring $\mathbb{Z}_3$ is very $J^{\#}$-clean which is not strongly $J^{\#}$-clean.

Proof. Let $R = \mathbb{Z}_3$. Note that R is strongly (or very) $J^{\#}$-clean if and only if R is strongly (or very) J-clean because R is commutative, and we have $J = J^{\#} = 0$ by Lemma 1. Since $\bar{R}$ is not Boolean, R is not strongly $J^{\#}$-clean, but R is very $J^{\#}$-clean. $\square$

Very clean elements (rings) are introduced by Chen et al. in [8]. Thus any very $J^{\#}$-clean ring is very clean. But the converse need not be true in general as shown below.

Example 4. $\mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$ is a very clean ring which is not very $J^{\#}$-clean.

Proof. Set $R = \mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$. If R is very $J^{\#}$-clean, then, by Theorem 1, it is strongly clean, but it is not strongly clean by [8, Theorem 3.5] or by [2, Example 17]. $\square$

The next result shows that for an element of a ring, being very $J^{\#}$-clean and strongly $J^{\#}$-clean coincide under some conditions.

Proposition 1. *Let R be a ring, $2 \in J$, and $a \in R$. Then a is very $J^{\#}$-clean if and only if it is strongly $J^{\#}$-clean.*

Proof. If $a \in R$ is strongly $J^{\#}$-clean, then it is very $J^{\#}$-clean. Conversely, assume that $a \in R$ is very $J^{\#}$-clean. Then there exist an idempotent $e \in R$ and $v \in J^{\#}$ such that $ae = ea$ and $a = e + v$ or $a = -e + v$. If $a = -e + v$, then $a = e + (v - 2e)$. As $2 \in J$, it easy to verify that $v - 2e \in J^{\#}$, hence $a \in R$ is strongly $J^{\#}$-clean. This completes the proof. $\square$

Remark 1. If u is invertible, $v \in J^{\#}$ and $uv = vu$, then we have that $u + v$ and $u - v$ is invertible.

Proof. Since $v \in J^{\#}$ if and only if $-v \in J^{\#}$, we only need to prove one of $u + v \in U$ and $u - v \in U$. We prove that $u - v \in U$. Now, we have $v^n \in J$, thus $1 - u^{-n}v^n \in U$. Now

$$1 - u^{-n}v^n = 1 - (u^{-1}v)^n = (1 - u^{-1}v)\big(1 + u^{-1}v + \cdots + (u^{-1}v)^{n-1}\big).$$

Hence $1 - u^{-1}v$ is invertible, and so $u - v = u(1 - u^{-1}v) \in U$, because $u \in U$. $\square$

By the following result, we determine the set of all invertible elements of a very $J^{\#}$-clean ring.

Proposition 2. *If R is a very $J^{\#}$-clean ring, then*

$$U = \{u \in R \mid u - 1 \in J^{\#} \text{ or } u + 1 \in J^{\#}\}.$$

Proof. Let $u \in U$. Since R is very $J^{\#}$-clean, there exist an idempotent $e \in R$ and $v \in J^{\#}$ such that $ue = eu$ and $u = e + v$ or $u = -e + v$. Assume that $u = e + v$. Then $u - v = e \in U$ implies that $e = 1$ and so $u = v + 1$. Assume that $u = -e + v$. Then $v - u = e \in U$ implies that $e = 1$ and so $u = v - 1$.

On the other hand, suppose that $u = v - 1$ where $v \in J^{\#}$. Then we can find some $n \in \mathbb{N}$ such that $v^n \in J$. Hence $1 - av^n \in U$ for any $a \in R$. If $1 - v^n \in U$, then $1 - v \in U$ because

$$1 - v^n = (1 - v)(1 + v + \cdots + v^{n-1}),$$

and so $u \in U$. Suppose that $u = v + 1$ where $v \in J^{\#}$. Then we can find some $n \in \mathbb{N}$ such that $v^n \in J$. Therefore $1 - av^n \in U$ for any $a \in R$. If $1 + (-1)^{n-1}v^n \in U$, then $1 + v \in U$ because

$$1 + (-1)^{n-1}v^n = (1 + v)(1 - v + \cdots + (-1)^{n-1}v^{n-1}),$$

and so $u \in U$. Hence

$$U = \{u \in R \mid u - 1 \in J^{\#} \text{ or } u + 1 \in J^{\#}\},$$

as required. □

Every very nil-clean ring is very $J^{\#}$-clean, but there exists a very $J^{\#}$-clean ring which is not very nil-clean (see Example 2). Clearly, if J is nil, then $a \in R$ is very $J^{\#}$-clean if and only if $a \in R$ is very nil-clean.

Now we give the relations among strongly cleanness, very nil-cleanness and very $J^{\#}$-cleanness for the rings.

Theorem 1. *Let R be a ring. If R is very $J^{\#}$-clean, then R is strongly clean and $\bar{R}$ is very nil-clean. If R is strongly clean, $\bar{R}$ is very nil-clean and $2 \in J^{\#}$, then R is very $J^{\#}$-clean.*

Proof. Suppose that R is very $J^{\#}$-clean, and let $a \in R$. Then there exist an idempotent $e \in R$ and $v \in J^{\#}$ such that $ae = ea$ and $a = e + v$ or $a = -e + v$. This implies that $a = (1 - e) + (2e - 1 + v)$ or $a = 1 - e + v - 1$. As $ev = ve$ and $(2e - 1)^{-1} = 2e - 1$, we get $2e - 1 + v \in U$ or $v - 1 \in U$ by Remark 1 and Proposition 2. Hence $a \in R$ is strongly clean because $1 - e$ is an idempotent. Thus R is strongly clean. Further, $\bar{a} = \bar{e} + \bar{v}$ or $\bar{a} = -\bar{e} + \bar{v}$ where $\bar{v}^n = \bar{0}$ for some $n \in \mathbb{N}$. Therefore $\bar{R}$ is very nil-clean.

Assume that R is strongly clean, $\bar{R}$ is very nil-clean, $2 \in J^{\#}$ and let $a \in R$. Then there exists an idempotent $e \in R$ such that $a = e + u$ and $ea = ae$ where $u \in U$. As $\bar{R}$ is very nil-clean, we can find an idempotent $\bar{f} \in \bar{R}$ such that $\bar{u}\bar{f} = \bar{f}\bar{u}$ and $\bar{u} = \bar{f} + \bar{w}$ or $\bar{u} = -\bar{f} + \bar{w}$ where $\bar{w} \in \bar{R}$ is nilpotent. Further, $\bar{f} = \bar{u} - \bar{w} \in U(\bar{R})$

or $\bar{f} = \bar{w} - \bar{u} \in U(\bar{R})$, and then $\bar{f} = \bar{1}$. Hence $u = 1 + w + r$ or $u = -1 + w + r$ for some $r \in J$. Therefore

$$a = e + u = e + 1 + w + r = (1 - e) + (2e + w + r)$$

or

$$a = e + u = e - 1 + w + r = -(1 - e) + (w + r).$$

Obviously, $(w + r)^m \in J$ or $(2e + w + r)^m \in J$ for some $m \in \mathbb{N}$. Consequently, R is very $J^{\#}$-clean. □

Recall that an element $a \in R$ is called *strongly π-rad clean* provided that there exists an idempotent $e \in R$ such that $ae = ea$ and $a - e \in U$ and $(eae)^n = ea^n e \in J(eRe)$ for some integer $n \geq 1$. A ring R is said to be *strongly π-rad clean* in case every element in R is strongly π-rad clean (see [9]). For instance, if R is local, then it is strongly π-rad clean. It is well known that $eJe = J(eRe)$ for any $e^2 = e \in R$ (see [13, Theorem 1.3.3]).

Theorem 2. *If a ring R is very $J^{\#}$-clean, then it is strongly π-rad clean.*

Proof. Let R be a very $J^{\#}$-clean ring and $a \in R$. Then there exist an idempotent $e \in R$ and $v \in J^{\#}$ such that $ae = ea$ and $a = e + v$ or $a = -e + v$. Assume that $a = e + v$ where $v^n \in J$ for some $n \in \mathbb{N}$. This implies that $a = (1 - e) + (2e - 1 + v)$. As $ev = ve$ and $(2e - 1)^{-1} = 2e - 1$, it is easy to verify that $2e - 1 + v \in U$ by Remark 1. Hence $a(1 - e) = (1 - e)a$ and $a - (1 - e) \in U$ and

$$[(1 - e)a(1 - e)]^n = [(1 - e)v(1 - e)]^n = (1 - e)v^n(1 - e) \in (1 - e)J(1 - e)$$

for some $n \in \mathbb{N}$. Assume that $a = -e + v$ where $v^m \in J$ for some $m \in \mathbb{N}$. This implies that $a = (1 - e) + (v - 1)$. By Proposition 2, $v - 1 \in U$. Thus $a(1 - e) = (1 - e)a$ and $a - (1 - e) \in U$ and

$$[(1 - e)a(1 - e)]^m = [(1 - e)v(1 - e)]^m = (1 - e)v^m(1 - e) \in (1 - e)J(1 - e)$$

for some $m \in \mathbb{N}$. Therefore R is strongly π-rad clean, as asserted. □

The converse of Theorem 2 need not be true as the following example shows.

Example 5. Since $\mathbb{Z}_5$ is a local ring, it is strongly π-rad clean, but not very $J^{\#}$-clean. Because $\bar{2} \in \mathbb{Z}_5$ is not very $J^{\#}$-clean as $J^{\#}(\mathbb{Z}_5) = J(\mathbb{Z}_5) = 0$.

It is an open question that whether strongly clean rings have stable range one (see [17, Question 1]). In the next result, we obtain that very $J^{\#}$-clean rings have this property. So by Theorem 3, we can give a partial answer to the open question. We know from [19] that a ring R has stable range one if and only if $\bar{R}$ has stable range one. Recall that an element a of a ring R is called *strongly π-regular* if there exist a positive integer n and $x \in R$ such that $a^n = a^{n+1}x$. A ring R is said to be *strongly π-regular* if every element of R is strongly π-regular. Ara showed that if R is strongly π-regular, then R has stable range one (see [3, Theorem 4]).

Theorem 3. *Let R be a very $J^{\#}$-clean ring. Then $\bar{R}$ is strongly π-regular, hence R has stable range one.*

Proof. Let R be a very $J^{\#}$-clean ring and $a \in R$. Then there exist an idempotent $e \in R$ and $v \in J^{\#}$ such that $ae = ea$ and $a = e + v$ or $a = -e + v$. Assume that $a = e+v$ where $v^n \in J$ for some $n \in \mathbb{N}$. This implies that $a^n(1-e) = v^n(1-e) \in J$ and $a = (1-e) + (2e-1+v)$. As $ev = ve$ and $(2e-1)^{-1} = 2e-1$, we get $u := 2e-1+v \in U$ by Remark 1. Hence $\bar{a}^n = \bar{a}^n\bar{e} = \bar{u}^n\bar{e}$ and $\bar{a}^{n+1} = \bar{a}^{n+1}\bar{e} = \bar{u}^{n+1}\bar{e}$ in $\bar{R}$. This gives $\bar{a}^n = \bar{a}^{n+1}(\bar{u})^{-1} = (\bar{u})^{-1}\bar{a}^{n+1}$, that is, $\bar{a} \in \bar{R}$ is strongly π-regular. Suppose that $a = -e+v$ where $v^m \in J$ for some $m \in \mathbb{N}$. Write $a = (1-e)+(v-1)$. This implies that $a^m(1-e) = v^m(1-e) \in J$ and

$$a^m e = (ae)^m = \big((v-1)e\big)^m = (v-1)^m e.$$

Since $v^m \in J$, we have $v-1 \in U$. Hence $\bar{a}^m = \bar{a}^m\bar{e} = \overline{v-1}^m\bar{e}$ and

$$\bar{a}^{m+1} = \bar{a}^{m+1}\bar{e} = v \bar{-} 1^{n+1}\bar{e}$$

in $\bar{R}$. This gives

$$\bar{a}^m = \bar{a}^{m+1}(\overline{v-1})^{-1} = (\overline{v-1})^{-1}\bar{a}^{m+1},$$

that is, $\bar{a} \in \bar{R}$ is strongly π-regular, and so $\bar{R}$ is strongly π-regular. Thus $\bar{R}$ has stable range one from [3, Theorem 4]. By the remark above, R has stable range one. $\square$

Let R be a ring and $a \in R$. Set

$$\operatorname{ann}_l(a) = \{r \in R \mid ra = 0\}$$

and

$$\operatorname{ann}_r(a) = \{r \in R \mid ar = 0\}.$$

Then we have the following lemma.

Lemma 3. *Let R be a ring and $a = e+v$ or $a = -e+v$ very $J^{\#}$-clean decomposition of a in R. Then $\operatorname{ann}_l(a) \subseteq \operatorname{ann}_l(e)$ and $\operatorname{ann}_r(a) \subseteq \operatorname{ann}_r(e)$.*

Proof. Let $r \in \operatorname{ann}_l(a)$. Then $ra = 0$. Since $ev = ve$, we have $re = rv$ or $re = -rv$, and so $re = rve = rev$ or $re = -rve = -rev$. It follows that $re(1-v) = 0$ or $re(1+v) = 0$, and so $re = 0$ because $1+v,\ 1-v \in U$. That is, $r \in \operatorname{ann}_l(e)$. Therefore $\operatorname{ann}_l(a) \subseteq \operatorname{ann}_l(e)$. Similarly, we can prove that $\operatorname{ann}_r(a) \subseteq \operatorname{ann}_r(e)$. $\square$

Theorem 4. *Let R be a ring and $f \in R$ be an idempotent. Then $a \in fRf$ is very $J^{\#}$-clean in R if and only if a is very $J^{\#}$-clean in fRf.*

Proof. Suppose $a = e + v$, $e^2 = e \in fRf$, $v \in J^{\#}(fRf)$, and $ev = ve$. Obviously, $v \in J^{\#}$ because $v^n \in J(fRf) = fJf \subseteq J$ for some $n \in \mathbb{N}$. Hence $a \in fRf$ is very $J^{\#}$-clean in R. Similarly, one can show that if $a = -e + v$, $e^2 = e \in fRf$, $v \in J^{\#}(fRf)$, and $ev = ve$, then $a \in fRf$ is very $J^{\#}$-clean in R.

Conversely, suppose that $a = -e + v$, $e^2 = e \in R$, $v \in J^\#$, and $ev = ve$. As $a \in fRf$, we see that

$$1 - f \in \mathrm{ann}_l(a) \cap \mathrm{ann}_r(a) \subseteq \mathrm{ann}_l(e) \cap \mathrm{ann}_r(e).$$

Hence $(1 - f)v = 0 = v(1 - f)$ and $fv = vf = v$. We observe that $a = fef + fvf$, $(fef)^2 = fef$, and

$$(fvf)^m = v^m \in fJf = J(fRf) \subseteq J^\#(fRf)$$

for some $m \in \mathbb{N}$. Furthermore,

$$(fef)(fvf) = fevf = fvef = (fvf)(fef).$$

Similarly, one can prove that $a \in fRf$ is very $J^\#$-clean in fRf where $a = e + v$, $e^2 = e \in R$, $v \in J^\#$, and $ev = ve$. Therefore the proof is completed. □

Corollary 1. *A ring R is very $J^\#$-clean if and only if eRe is very $J^\#$-clean for any idempotent $e \in R$.*

Proof. Let $a \in eRe$. Since R is very $J^\#$-clean, we see that $a \in eRe$ is very $J^\#$-clean in R. According to Theorem 4, $a \in eRe$ is very $J^\#$-clean in eRe. The converse is clear by using $e = 1$. □

As is well known, every homomorphic image of a (strongly) clean ring is (strongly) clean (see [12], [16], [17]). Analogously, we can give the following result.

Proposition 3. *Every homomorphic image of very $J^\#$-clean rings is very $J^\#$-clean.*

Proof. Let R be a very $J^\#$-clean ring and $\varphi\colon R \to S$ a surjective ring homomorphism. Then for any $b \in S$, there exists $a \in R$ such that $\varphi(a) = b$. Since R is very $J^\#$-clean, we can find an idempotent $e \in R$ and $v \in J^\#$ such that $ae = ea$ and $a = e + v$ or $a = -e + v$. Assume that $a = -e + v$ and $v^n \in J$ for some $n \in \mathbb{N}$. Then $\varphi(a) = -\varphi(e) + \varphi(v)$ and $\varphi(a)\varphi(e) = \varphi(e)\varphi(a)$. Obviously, $(\varphi(e))^2 = \varphi(e) \in S$. Since $\varphi(J) \subseteq J(S)$, we have $\varphi(v^n) = \varphi(v)^n \in J(S)$ and so $\varphi(v) \in J^\#(S)$. Similarly, one can show that $\varphi(a) = \varphi(e) + \varphi(v) \in S$ is very $J^\#$-clean in S where $a = e + v$ and $v \in J^\#$. □

If I is a left ideal of a ring R, *idempotents lift modulo I* if, given $a \in R$ with $a^2 - a \in I$, there exists $e^2 = e \in R$ such that $a - e \in I$ (see [16]). Note that R is a clean ring if and only if R/J is a clean ring and idempotents lift modulo J (see [12, Proposition 6]). Recall that a ring R is called *abelian* if every idempotent is central.

Theorem 5. *Let I be an ideal of an abelian ring R with $I \subseteq J$. Then R is very $J^\#$-clean if and only if R/I is very $J^\#$-clean and idempotents lift modulo I.*

Proof. Assume that R is very $J^{\#}$-clean. Then R/I is very $J^{\#}$-clean by Proposition 3. Further, by Theorem 1, R is strongly clean, and so idempotents lift modulo I by [12, Proposition 6].

Conversely, suppose that R/I is very $J^{\#}$-clean and idempotents lift modulo I and let $a \in R$. By assumption, for $\bar{a} \in R/I$, there exists an idempotent $\bar{e} \in R/I$ such that $\bar{a}\bar{e} = \bar{e}\bar{a}$ and $\bar{a} - \bar{e}$ or $\bar{a} + \bar{e}$ is an element of $J^{\#}(R/I)$. Assume that $\bar{a} = -\bar{e} + \bar{v}$ where $\bar{v} \in J^{\#}(R/I)$. Then we can find some $t \in \mathbb{N}$ such that $\bar{v}^t \in J(R/I) = J/I$ and so $v \in J^{\#}$. Since idempotents lift modulo I, we may assume that $e^2 = e$. Hence $a + e - v \in I \subseteq J$ and so a is a very $J^{\#}$-clean element because e is central. Similarly, one can prove that if $\bar{a} = \bar{e} + \bar{v}$ and $\bar{v} \in J^{\#}(R/I)$, then a is a very $J^{\#}$-clean element. □

3 Examples

The purpose of this section is to construct several examples for very $J^{\#}$-clean rings.

Let R be a ring and σ be an endomorphism of R. Let $R[[x,\sigma]]$ be the set of all power series over the ring R. For any $\sum\limits_{i=0}^{\infty} a_i x^i, \sum\limits_{i=0}^{\infty} b_i x^i \in R[[x,\sigma]]$, we define

$$\sum_{i=0}^{\infty} a_i x^i + \sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} (a_i + b_i) x^i,$$

and

$$\Big(\sum_{i=0}^{\infty} a_i x^i\Big)\Big(\sum_{i=0}^{\infty} b_i x^i\Big) = \sum_{i=0}^{\infty} c_i x^i$$

where $c_i = \sum\limits_{k=0}^{i} a_k \sigma^k\big(b_{i-k}\big)$. Then $R[[x,\sigma]]$ is a ring under the preceding addition and multiplication. Clearly, $R[[x,\sigma]]$ is $R[[x]]$ only when σ is the identity morphism. Furthermore, $J\big(R[[x,\sigma]]\big) = J + xR[[x,\sigma]]$ (see [14, Ex. 5.6]).

Lemma 4. *If $R[[x,\sigma]]$ is abelian, then $\sigma(e) = e$ for every idempotent $e \in R$.*

Proof. Since $R[[x,\sigma]]$ is abelian, we have $xe = ex$ for every idempotent $e \in R$. Hence we get $xe = ex = \sigma(e)x$, and so $\sigma(e) = e$, as asserted. □

Proposition 4. *Let $R[[x,\sigma]]$ be an abelian ring. Then the following are equivalent.*

(1) R is very $J^{\#}$-clean.

(2) $R[[x,\sigma]]$ is very $J^{\#}$-clean.

Proof. (1) ⇒ (2) Let $a(x) \in R[[x,\sigma]]$. Then we can find an idempotent $e \in R$ and $v \in J^{\#}$ such that $a(0) = e + v$ or $a(0) = -e + v$. Assume that $a(0) = e + v$. Then $a(x) = e + v(x)$ where $v(x) = a(x) - e = v + a_1 x + a_2 x^2 + \cdots$. Since $\sigma(e) = e$ for any idempotent $e \in R$ by Lemma 4, we see that $ev(x) = v(x)e$. Further, we conclude that $v(x) \in J^{\#}\big(R[[x,\sigma]]\big)$ because $v \in J^{\#}$ and

$$J\big(R[[x,\sigma]]\big) = J + xR[[x,\sigma]].$$

This implies that $a(x) \in R[[x,\sigma]]$ is very $J^{\#}$-clean. Assume that $a(0) = -e + v$. Similarly, we can show that $a(x) \in R[[x,\sigma]]$ is very $J^{\#}$-clean. Thus $R[[x,\sigma]]$ is very $J^{\#}$-clean.

(2) $\Rightarrow$ (1) Let $a \in R$. Then we can find an idempotent $e(x) \in R[[x,\sigma]]$ and $v(x) \in J^{\#}\big(R[[x,\sigma]]\big)$ such that $ae(x) = e(x)a$ and $a = e(x) + v(x)$ or $a = -e(x) + v(x)$. Obviously, $e(0) \in R$ is an idempotent and $v(0) \in J^{\#}$. Since $a = e(0) + v(0)$ or $a = -e(0) + v(0)$ and $ae(0) = e(0)a$, we obtain that $a \in R$ is very $J^{\#}$-clean, and therefore R is very $J^{\#}$-clean. □

Remark 2. As in the proof of [1, Lemma 2.18], we can show that the idempotents of $R[[x,\sigma]]$ belong to R. Hence if R is abelian, then so is $R[[x,\sigma]]$.

The next result is a characterization of being very $J^{\#}$-clean for abelian rings.

Theorem 6. *Let R be an abelian ring. Then the following conditions are equivalent.*

(1) R is very $J^{\#}$-clean.

(2) $R[[x]]/\langle x^n\rangle$ is very $J^{\#}$-clean for all $n \geq 2$.

(3) $R[[x]]/\langle x^2\rangle$ is very $J^{\#}$-clean.

(4) $R[x]/\langle x^2\rangle$ is very $J^{\#}$-clean.

Proof. (1) $\Rightarrow$ (2) If R is very $J^{\#}$-clean, then $R[[x]]$ is very $J^{\#}$-clean by Proposition 4 and so $R[[x]]/\langle x^n\rangle$ is very $J^{\#}$-clean by Proposition 3 for all $n \geq 2$.

(2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (1) Since R is abelian, so is $R[[x]]$ by Remark 2. Note that $J(R[[x]]) = J + xR[[x]]$. Then $\langle x^2\rangle \subseteq J(R[[x]])$, and so R is very $J^{\#}$-clean by Theorem 5.

(3) $\Leftrightarrow$ (4) Since $R[x]/\langle x^2\rangle \cong R[[x]]/\langle x^2\rangle$, there is nothing to show. □

Let R be a ring and $\sigma\colon R \to R$ be an endomorphism. Set

$$D_2(R,\sigma) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \;\middle|\; a, b \in R \right\},$$

addition and multiplication are defined as follows:

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} + \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ 0 & a+c \end{pmatrix};$$
$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = \begin{pmatrix} ac & ad + b\sigma(c) \\ 0 & ac \end{pmatrix}.$$

Then $D_2(R,\sigma)$ is a ring with the identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Denote $D_2(R, 1_R)$ by $D_2(R)$, where $1_R\colon R \to R$, $r \mapsto r$. Further, it can be verified that

$$J\big(D_2(R,\sigma)\big) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \;\middle|\; a \in J,\, b \in R \right\}.$$

Proposition 5. *Let R be an abelian ring and $\sigma\colon R \to R$ be an endomorphism. Then the following are equivalent.*

(1) R is very $J^{\#}$-clean.

(2) $D_2(R,\sigma)$ is very $J^{\#}$-clean.

Proof. Note that since R is abelian, $\sigma(e) = e$ for every idempotent $e \in R$ by Lemma 4 and Remark 2.

(1) $\Rightarrow$ (2) Let $A := \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in D_2(R,\sigma)$. Then there exists an idempotent $e \in R$ such that $ae = ea$ and $v := a - e \in J^{\#}$ or $v := a + e \in J^{\#}$. Assume that $v := a - e \in J^{\#}$ and $v^n \in J$ for some $n \in \mathbb{N}$. Since $V^n = \begin{pmatrix} v^n & * \\ 0 & v^n \end{pmatrix} \in J\big(D_2(R,\sigma)\big)$ where $V = \begin{pmatrix} v & b \\ 0 & v \end{pmatrix}$,

$$A - \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} = V \in J^{\#}\big(D_2(R,\sigma)\big).$$

As R is abelian and $\sigma(e) = e$, we see that $EA = AE$ where $E^2 = E = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$ (because $EA = AE$ if and only if $eb = b\sigma(e) = be$). Therefore $A \in D_2(R,\sigma)$ is very $J^{\#}$-clean. Assume that $v := a + e \in J^{\#}$. Similar to the preceding discussion, it can be shown that $A \in D_2(R,\sigma)$ is very $J^{\#}$-clean, as required.

(2) $\Rightarrow$ (1) Let $a \in R$. Then $A := \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in D_2(R,\sigma)$. By hypothesis, there exists an idempotent $E := \begin{pmatrix} e & b \\ 0 & e \end{pmatrix} \in D_2(R,\sigma)$ such that $AE = EA$ and

$$A + E \in J^{\#}\big(D_2(R,\sigma)\big)$$

or $A - E \in J^{\#}\big(D_2(R,\sigma)\big)$. As E is an idempotent, we have $e = e^2$. Further, we get $ea = ae$, and that $a - e \in J^{\#}$ or $a + e \in J^{\#}$. Therefore R is very $J^{\#}$-clean. $\square$

Let R be a ring and V an R-R-bimodule which is a general ring (possibly with no unity) in which $(vw)r = v(wr)$, $(vr)w = v(rw)$ and $(rv)w = r(vw)$ hold for all $v, w \in V$ and $r \in R$. Then *ideal-extension* (it is also called *Dorroh extension*) $I(R;V)$ of R by V is defined to be the additive abelian group $I(R;V) = R \oplus V$ with multiplication $(r,v)(s,w) = (rs, rw + vs + vw)$.

Proposition 6. *An ideal-extension $S = I(R;V)$ is very $J^{\#}$-clean if the following conditions are satisfied.*

(1) R is very $J^{\#}$-clean;

(2) If $e^2 = e \in R$, then $ev = ve$ for all $v \in V$;

(3) If $v \in V$, then $v + w + vw = 0$ for some $w \in V$.

Furthermore, if S is very $J^{\#}$-clean, then R is very $J^{\#}$-clean.

Proof. Suppose that (1), (2) and (3) are satisfied. Let $s = (r, w) \in S$ and (by (1)) write $r = e+v$ or $r = -e+v$, $e^2 = e$, $v \in J^\#$ and $re = er$. Assume that $r = -e+v$ and $v^n \in J$ for some $n \in \mathbb{N}$. Then $s = -(e, 0) + (v, w)$ and $(e, 0)^2 = (e, 0) \in S$. Note that $(0, V) \subseteq J(S)$ if and only if (3) holds (see [15]). Since $(v, w)^n = (v^n, *)$, it suffices to show that $(v^n, 0) \in J(S)$. For any $(p, q) \in S$,

$$(1, 0) - (v^n, 0)(p, q) = (1 - v^n p, -v^n q) \in U(S)$$

because

$$(1 - v^n p, -v^n q) = (1 - v^n p, 0)(1, (1 - v^n p)^{-1}(-v^n q))$$

and

$$(1, (1 - v^n p)^{-1}(-v^n q)) = (1, 0) + (0, (1 - v^n p)^{-1}(-v^n q)) \in U(S)$$

by $(0, V) \subseteq J(S)$. Thus $(v^n, 0) \in J(S)$ and so $(v, w) \in J^\#(S)$. By (2), $(r, w)(e, 0) = (e, 0)(r, w)$. The case where $r = e + v$ can be similarly handled.

On the other hand, suppose that S is very $J^\#$-clean and let $a \in R$. Then $(a, 0) = (e, t) + (v, w)$ or $(a, 0) = -(e, t) + (v, w)$, $(e, t)^2 = (e, t)$, $(v, w) \in J^\#(S)$ and $(a, 0)(e, t) = (e, t)(a, 0)$. Assume that $(a, 0) = (e, t)+(v, w)$ and $(v, w)^m \in J(S)$ for some $m \in \mathbb{N}$. Since $(v, w)^m \in J(S)$, $(e, t)^2 = (e, t)$ and $(a, 0)(e, t) = (e, t)(a, 0)$, we get $a = e + v$, $v^m \in J$, $e^2 = e \in R$, and $ae = ea$. Hence a is strongly $J^\#$-clean. Suppose $(a, 0) = -(e, t) + (v, w)$ and $(v, w)^n \in J(S)$ for some $n \in \mathbb{N}$. Similarly, it can be shown that $-a$ is strongly $J^\#$-clean and so R is very $J^\#$-clean. □

Example 6. Let R be an abelian very $J^\#$-clean ring, n a positive integer and

$$S = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \middle| \; a, a_{ij} \in R(i < j) \right\}.$$

Then S is very $J^\#$-clean and noncommutative if $n \geq 3$.

Proof. Let

$$V = \left\{ \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \middle| \; a_{ij} \in R(i < j) \right\}.$$

Then $S \cong I(R; V)$. By applying Proposition 6, (1) is clear; (2) holds because R is abelian and (3) follows because of $V \subseteq J(S)$. □

4 Very $J^\#$-clean 2×2 matrices

Let $f, g \in R[x]$ be polynomials over a commutative ring R and let (f, g) denote the ideal generated by f, g. A polynomial $f(x) \in R[x]$ is a monic polynomial of degree n if $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where $a_{n-1}, \ldots, a_1, a_0 \in R$. If $\varphi \in M_n(R)$, we use $\chi(\varphi)$ to stand for the characteristic polynomial $\det(xI_n - \varphi)$.

The aim of this section is to characterize a single very $J^{\#}$-clean 2×2 matrix over a commutative local ring by means of the factorization of its characteristic polynomial.

We begin with the following result from [7], and give the proof of it for the sake of completeness.

Lemma 5. *Let R be a commutative ring and $\varphi \in M_n(R)$. Then the following are equivalent.*

(1) $\varphi \in J^{\#}\big(M_n(R)\big)$.

(2) $\chi(\varphi) \equiv x^n \pmod{J}$.

(3) There exists a monic polynomial $h \in R[x]$ such that $h \equiv x^{\deg h} \pmod{J}$ for which $h(\varphi) = 0$.

Proof. Note that $J\big(M_n(R)\big) = M_n\big(J\big)$ and $M_n(R)/J\big(M_n(R)\big) = M_n(\bar{R})$. Furthermore, since R is commutative, we have that $\mathrm{Nil}(R) \subseteq J$.

(1) $\Rightarrow$ (2) If $\varphi \in J^{\#}\big(M_n(R)\big)$, then $\bar{\varphi}$ is nilpotent in $M_n(\bar{R})$. According to [4, Proposition 3.5.4], we get $\chi(\varphi) \equiv x^n \pmod{\mathrm{Nil}(R)}$. So $\chi(\varphi) \equiv x^n \pmod{J}$ because $\mathrm{Nil}(R) \subseteq J$.

(2) $\Rightarrow$ (3) Set $h = \chi(\varphi)$. Then $h \equiv x^{\deg h} \pmod{J}$. By Cayley-Hamilton Theorem, $h(\varphi) = 0$.

(3) $\Rightarrow$ (1) Assume that $h = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ where $a_i \in J$ for $0 \leq i \leq n-1$. Then $\bar{h} \equiv x^n \pmod{\mathrm{Nil}(\bar{R})}$ and $\bar{h}(\bar{\varphi}) = 0$. Again, by [4, Proposition 3.5.4], $\bar{\varphi}$ is nilpotent in $M_n(\bar{R})$. This gives $\varphi \in J^{\#}\big(M_n(R)\big)$. $\square$

Definition 1. [7, Definition 2.4] For $r \in R$, define

$$\mathbb{J}_r = \{f \in R[x] \mid f \text{ is monic, and } f \equiv (x-r)^{\deg f} \pmod{J^{\#}}\}.$$

Remark 3. If R is commutative, then $J^{\#}$ is simply the Jacobson radical. So we get

$$\mathbb{J}_r = \{f \in R[x] \mid f \text{ is monic, and } f \equiv (x-r)^{\deg f} \pmod{J}\}.$$

By $f \equiv (x-r)^{\deg f} \pmod{J}$, we mean $f - (x-r)^{\deg f} \in J[x]$. Furthermore, it is well known that

$$\chi(\varphi) = x^2 - \mathrm{tr}(\varphi)x + \det(\varphi) \quad \text{and} \quad \chi(-\varphi) = x^2 + \mathrm{tr}(\varphi)x + \det(\varphi)$$

because $tr(-\varphi) = -\mathrm{tr}(\varphi)$ and $\det(\varphi) = \det(-\varphi)$ for $\varphi \in M_2(R)$. In general, note that

$$\chi(-\varphi)(x) = \det\big(xI_n - (-\varphi)\big) = (-1)^n \det\big((-x)I_n + \varphi\big) = (-1)^n\chi(\varphi)(-x)$$

and $\det(-\varphi) = (-1)^n\det(\varphi)$ for $\varphi \in M_n(R)$.

For an easy reference, we mention the following lemmas without proofs. Recall that a commutative ring R is called *projective-free* if every finitely generated projective R-module is free. Any commutative local ring is projective-free.

Lemma 6. *[7, Lemma 2.5] Let R be a projective-free ring and $h \in R[x]$ a monic polynomial of degree n, let $\varphi \in M_n(R)$. If $h(\varphi) = 0$ and there exists a factorization $h = h_0h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$, then φ is strongly $J^{\#}$-clean.*

Lemma 7. *[7, Theorem 2.6] Let R be a projective-free ring and $h \in R[x]$ a monic polynomial of degree n. Then the following are equivalent.*

(1) Every $\varphi \in M_n(R)$ with $\chi(\varphi) = h$ is strongly $J^{\#}$-clean.

(2) There exists a factorization $h = h_0h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$.

In the proof of Lemma 8 and Theorem 7, we refer to Lemma 6 and Lemma 7.

Lemma 8. *Let R be a commutative local ring and $h \in R[x]$ a monic polynomial of degree n, let $\varphi \in M_n(R)$. If $h(\varphi) = 0$ and there exists a factorization $h = h_0h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1 \cup \mathbb{J}_{-1}$, then φ is very $J^{\#}$-clean.*

Proof. By hypothesis, there exists a factorization $h = h_0h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1 \cup \mathbb{J}_{-1}$. If $h_1 \in \mathbb{J}_1$, then φ is strongly $J^{\#}$-clean by Lemma 6, and so φ is very $J^{\#}$-clean. Hence we assume that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_{-1}$. Then

$$h_0 \equiv x^{\deg(h_0)} \pmod{J} \quad \text{and} \quad h_1 \equiv (x-(-1))^{\deg(h_1)} \pmod{J}.$$

Set $t := -x$ and $g(t) := (-1)^{\deg(h)}h(-t)$. Then $g(t)$ factors as $g = g_0g_1$, where

$$g_0(t) = (-1)^{\deg(h_0)}h_0(-t) \quad \text{and} \quad g_1(t) = (-1)^{\deg(h_1)}h_1(-t).$$

Note that $\deg(g_0) = \deg(h_0)$ and $\deg(g_1) = \deg(h_1)$. Since $h_0 \equiv x^{\deg(h_0)} \pmod{J}$, we see that

$$g_0(t) = (-1)^{\deg(h_0)}h_0(-t) \equiv (-1)^{\deg(h_0)}x^{\deg(h_0)} \equiv t^{\deg(g_0)} \pmod{J},$$

and so $g_0 \in \mathbb{J}_0$. Further, as $h_1 \equiv (x-(-1))^{\deg(h_1)} \pmod{J}$, we have

$$\begin{aligned} g_1(t) &= (-1)^{\deg(h_1)}h_1(-t) \equiv (-1)^{\deg(h_1)}(-t-(-1))^{\deg(h_1)} \\ &\equiv (t-1)^{\deg(g_1)} \pmod{J}, \end{aligned}$$

and so $g_1 \in \mathbb{J}_1$. We observe that $g(-\varphi) = 0$ because $h(\varphi) = 0$. In view of Lemma 6, $-\varphi \in M_n(R)$ is strongly $J^{\#}$-clean. That is, φ is very $J^{\#}$-clean. The proof is completed. □

Theorem 7. *Let R be a commutative local ring and $h \in R[x]$ a monic polynomial of degree n. Then the following are equivalent.*

(1) Every $\varphi \in M_n(R)$ with $\chi(\varphi) = h$ is very $J^{\#}$-clean.

(2) There exists a factorization $h = h_0h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1 \cup \mathbb{J}_{-1}$.

Proof. (1) $\Rightarrow$ (2) Since φ is very $J^{\#}$-clean, φ or $-\varphi$ is strongly $J^{\#}$-clean. If φ is strongly $J^{\#}$-clean, then there exists a factorization $h = h_0h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$ by Lemma 7. Suppose $-\varphi$ is strongly $J^{\#}$-clean. It follows by Lemma 7 that $g(t) := \chi(-\varphi)$ factors as $g = g_0g_1$ where $g_0 \in \mathbb{J}_0$ and $g_1 \in \mathbb{J}_1$. This implies

$$h(x) = \chi(\varphi) = (-1)^{\deg(h)}g(-x) = (-1)^{\deg(h)}g_0(-x)g_1(-x)\,.$$

Set $h_0(x) = (-1)^{\deg(g_0)}g_0(-x)$ and $h_1(x) = (-1)^{\deg(g_1)}g_1(-x)$. Then $h = h_0h_1$. Since $g_0(t) \equiv t^{\deg(g_0)} \pmod J$, we get

$$h_0(x) = (-1)^{\deg(g_0)}g_0(-x) \equiv x^{\deg(g_0)} \pmod J\,,$$

hence $h_0 \in \mathbb{J}_0$. In addition, as $g_1(t) \equiv (t-1)^{\deg(g_1)} \pmod J$, we see that

$$h_1(x) = (-1)^{\deg(g_1)}g_1(-x) \equiv (x+1)^{\deg(g_1)} \pmod J\,.$$

This gives $h_1 \in \mathbb{J}_{-1}$. That is, $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1 \cup \mathbb{J}_{-1}$, as asserted.

(2) $\Rightarrow$ (1) For any $\varphi \in M_2(R)$ with $\chi(\varphi) = h$, we have $h(\varphi) = 0$ by the Cayley-Hamilton Theorem. In light of Lemma 8, φ is very $J^{\#}$-clean. □

Corollary 2. *[7, Corollary 2.8] Let R be a commutative local ring and $\varphi \in M_2(R)$. Then φ is strongly $J^{\#}$-clean if and only if*

(1) $\chi(\varphi) \equiv x^2 \pmod J$*; or*

(2) $\chi(\varphi) \equiv (x-1)^2 \pmod J$*; or*

(3) $\chi(\varphi)$ *has a root in* J *and a root in* $1 + J$*.*

In analogy with Corollary 2, we have the following result.

Corollary 3. *Let R be a commutative local ring and $\varphi \in M_2(R)$. Then $-\varphi$ is strongly $J^{\#}$-clean if and only if*

(1) $\chi(\varphi) \equiv x^2 \pmod J$*; or*

(2) $\chi(\varphi) \equiv (x+1)^2 \pmod J$*; or*

(3) $\chi(\varphi)$ *has a root in* J *and a root in* $-1 + J$*.*

Proof. Suppose that $-\varphi$ is strongly $J^{\#}$-clean. As in the proof of Theorem 7, there exists a factorization $\chi(\varphi) = h_0h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_{-1}$. Consider the following cases:

Case I. $\deg(h_0) = 2$ and $\deg(h_1) = 0$. Then $h_0 = \chi(\varphi) = x^2 - \mathrm{tr}(\varphi)x + \det(\varphi)$ and $h_1 = 1$. As $h_0 \in \mathbb{J}_0$, it follows from Lemma 5 that $\varphi \in J^{\#}\big(M_2(R)\big)$ or equivalently, $\chi(\varphi) \equiv x^2 \pmod J$.

Case II. $\deg(h_0) = 0$ and $\deg(h_1) = 2$. Then $h_1(x) = \chi(\varphi) \equiv (x+1)^2 \pmod J$ because $h_1 \in \mathbb{J}_{-1}$.

Case III. $\deg(h_0) = 1$ and $\deg(h_1) = 1$. Then $h_0 = x - \alpha$ and $h_1 = x - \beta$. Since $h_0 \in \mathbb{J}_0$, we see that $h_0 \equiv x \pmod J$, and then $\alpha \in J$. As $h_1 \in \mathbb{J}_{-1}$, we have

$h_1 \equiv x+1 \pmod{J}$, and so $\beta \in -1+J$. Therefore $\chi(\varphi)$ has a root in J and a root in $-1+J$.

For the reverse implication, if (1) or (2) is valid, then $-\varphi \in J^{\#}\big(M_2(R)\big)$ or $I_2+\varphi \in J^{\#}\big(M_2(R)\big)$. This implies that $-\varphi$ is strongly $J^{\#}$-clean. Suppose that $\chi(\varphi)$ has a root in J and a root in $-1+J$ and $-\varphi$, $I_2+\varphi \notin J\big(M_2(R)\big)$. By Remark 3, we know that $\chi(\varphi)(-x)=\chi(-\varphi)(x)$. In this case, $\chi(\varphi)$ has a root in J and a root in $1+J$. According to [6, Theorem 16.4.31], φ is strongly J-clean, and therefore it is strongly $J^{\#}$-clean. $\square$

For instance, choose $\varphi = \begin{pmatrix} 0 & 7 \\ 8 & 1 \end{pmatrix} \in M_2(\mathbb{Z}_9)$. Note that $J(\mathbb{Z}_9)=3\mathbb{Z}_9$. Then $\chi(\varphi)=x^2+x+7=(x+1)^2+6x+6$. Hence $\chi(\varphi)\equiv (x+1)^2 \pmod{J(\mathbb{Z}_9)}$, and so $\varphi \in M_2(\mathbb{Z}_9)$ is very $J^{\#}$-clean by Corollary 3.

In the next, we investigate very $J^{\#}$-clean matrices over power series rings.

Theorem 8. *Let R be a commutative local ring. Then the following are equivalent.*

(1) $A(x) \in M_2\big(R[[x]]\big)$ is very $J^{\#}$-clean.

(2) $A(0) \in M_2(R)$ is very $J^{\#}$-clean.

Proof. (1) $\Rightarrow$ (2) Since $A(x)$ is very $J^{\#}$-clean in $M_2\big(R[[x]]\big)$, there exist an

$$E(x)=E^2(x) \in M_2\big(R[[x]]\big) \quad \text{and} \quad V(x) \in J^{\#}\big(M_2(R[[x]])\big)$$

such that $E(x)V(x)=V(x)E(x)$, and

$$A(x)=E(x)+V(x) \quad \text{or} \quad A(x)=-E(x)+V(x)\,.$$

This implies that $E(0)V(0)=V(0)E(0)$ and

$$A(0)=E(0)+V(0) \quad \text{or} \quad A(0)=-E(0)+V(0)\,,$$

where $E(0)=E^2(0) \in M_2(R)$ and $V(0) \in J^{\#}\big(M_2(R)\big)$. Hence $A(0)$ is very $J^{\#}$-clean in $M_2(R)$.

(2) $\Rightarrow$ (1) Since $R[[x]]/J\big(R[[x]]\big) \cong R/J$ and R is local, $R[[x]]$ is local. Assume that $-A(0)$ is strongly $J^{\#}$-clean. Then

- $-A(0) \in J^{\#}\big(M_2(R)\big)$;
- or $I_2+A(0) \in J^{\#}\big(M_2(R)\big)$;
- or the characteristic polynomial $\chi\big(A(0)\big)=y^2-\mu y+\lambda$ has a root $\alpha \in -1+J$ and a root $\beta \in J$.

If $-A(0) \in J^{\#}\big(M_2(R)\big)$, then

$$-A(x) \in J^{\#}\big(M_2(R[[x]])\big)\,.$$

If $I_2 + A(0) \in J^{\#}\big(M_2(R)\big)$, then

$$I_2 + A(x) \in J^{\#}\big(M_2(R[[x]])\big)\,.$$

Otherwise, we write $y = \sum\limits_{i=0}^{\infty} b_i x^i$ and

$$\chi(-A(x)) = y^2 - \mu(x)y + \lambda(x)\,.$$

Then $y^2 = \sum\limits_{i=0}^{\infty} c_i x^i$ where $c_i = \sum\limits_{k=0}^{i} b_k b_{i-k}$. Let

$$\mu(x) = \sum_{i=0}^{\infty} \mu_i x^i, \quad \lambda(x) = \sum_{i=0}^{\infty} \lambda_i x^i \in R[[x]]$$

where $\mu_0 = \mu$ and $\lambda_0 = \lambda$. Then

$$y^2 - \mu(x)y + \lambda(x) = 0$$

holds in $R[[x]]$ if the following equations are satisfied:

$$\begin{aligned} b_0^2 - b_0\mu_0 + \lambda_0 &= 0;\\ (b_0b_1 + b_1b_0) - (b_0\mu_1 + b_1\mu_0) + \lambda_1 &= 0;\\ (b_0b_2 + b_1^2 + b_2b_0) - (b_0\mu_2 + b_1\mu_1 + b_2\mu_0) + \lambda_2 &= 0;\\ &\vdots \end{aligned}$$

Obviously, $\mu_0 = \alpha + \beta \in U$ and $\alpha - \beta \in U$. Let $b_0 = \alpha$. Since R is commutative and $2b_0 - \mu_0 = 2\alpha - \mu = \alpha - \beta$, there exists some $b_1 \in R$ such that

$$b_1(2b_0 - \mu_0) = b_0\mu_1 - \lambda_1.$$

Further, there exists some $b_2 \in R$ such that

$$b_2(2b_0 - \mu_0) = b_0\mu_2 + b_1\mu_1 - b_1^2 - \lambda_2.$$

By iteration of this process, we get $b_3, b_4, \dots$. Then $y^2 - \mu(x)y + \lambda(x) = 0$ has a root $y_0(x) \in -1 + J\big(R[[x]]\big)$. If $b_0 = \beta \in J$, analogously, we can show that $y^2 - \mu(x)y + \lambda(x) = 0$ has a root $y_1(x) \in J\big(R[[x]]\big)$. In light of Corollary 3, $-A(x)$ is strongly $J^{\#}$-clean. Similarly, we can prove that if $A(0)$ is strongly $J^{\#}$-clean, then $A(x)$ is strongly $J^{\#}$-clean by Corollary 2. Therefore $A(x)$ is very $J^{\#}$-clean in $M_2\big(R[[x]]\big)$. □

Example 7. Let $R = \mathbb{Z}_9[[x]]$ and

$$A(x) = \begin{pmatrix} \bar{0} & \bar{2} - \sum\limits_{n=1}^{\infty} (\bar{1} + \bar{5}^n)x^n \\ \bar{1} & \bar{1} - \sum\limits_{n=1}^{\infty} (\bar{1} + \bar{7}^n)x^n \end{pmatrix} \in M_2(R).$$

Then
$$A(0) = \begin{pmatrix} \bar{0} & \bar{2} \\ \bar{1} & \bar{1} \end{pmatrix} = -\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix} + \begin{pmatrix} \bar{1} & \bar{2} \\ \bar{1} & \bar{2} \end{pmatrix},$$
where $\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix}$ is an idempotent and
$$\begin{pmatrix} \bar{1} & \bar{2} \\ \bar{1} & \bar{2} \end{pmatrix} \in J^{\#}\big(M_2(\mathbb{Z}_9)\big)$$
because
$$\begin{pmatrix} \bar{1} & \bar{2} \\ \bar{1} & \bar{2} \end{pmatrix}^2 = \begin{pmatrix} \bar{3} & \bar{6} \\ \bar{3} & \bar{6} \end{pmatrix} \in J\big(M_2(\mathbb{Z}_9)\big).$$
Thus $A(x)$ is very $J^{\#}$-clean by Theorem 8. Note that $A(0)$ is not strongly J-clean.

Corollary 4. *Let R be a commutative local ring and $A(x) \in M_2\big(R[[x]]/(x^m)\big)$ $(m \geq 1)$. Then the following are equivalent.*

(1) $A(x) \in M_2\big(R[[x]]/(x^m)\big)$ is very $J^{\#}$-clean.

(2) $A(0) \in M_2(R)$ is very $J^{\#}$-clean.

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1) Let $\psi\colon R[[x]] \to R[[x]]/(x^m)$ denote the natural homomorphism. Then ψ induces the surjective ring homomorphism
$$\psi^*\colon M_2\big(R[[x]]\big) \to M_2\big(R[[x]]/(x^m)\big).$$
Then there exists $B(x) \in M_2\big(R[[x]]\big)$ such that $\psi^*\big(B(x)\big) = A(x)$. Then Theorem 8 completes the proof. ☐

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