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On a Construction of Modular *GMS*-algebras

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Abstract

In this paper we investigate the class of all modular *GMS*-algebras which contains the class of *MS*-algebras. We construct modular *GMS*-algebras from the variety \mathbf{K}_2 by means of K_2 -quadruples. We also characterize isomorphisms of these algebras by means of K_2 -quadruples.

Key words: *MS*-algebras, *GMS*-algebras, K_2 -algebras, Kleene algebras, isomorphisms.

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1 Introduction

T. S. Blyth and J. C. Varlet [2] have studied the variety of *MS*-algebras as a common abstraction of de Morgan algebras and Stone algebras. D. Ševčovič [12] investigated a larger variety of algebras containing *MS*-algebras, the so-called generalized *MS*-algebras (*GMS*-algebras). In such algebras the distributive identity need not be necessarily satisfied. In [4] T. S. Blyth and J. C. Varlet presented a construction of some *MS*-algebras from the subvariety \mathbf{K}_2 (the so-called K_2 -algebras) from Kleene algebras and distributive lattices. This was a construction by means of triples which were successfully used in construction of Stone algebras (see [6], [7]), distributive p -algebras (see [9]), modular p -algebras (see [10]), etc. T. S. Blyth and J. V. Varlet [5] improved their construction from [4] by means of quadruples and they showed that each member of \mathbf{K}_2 can be constructed in this way. In [8] M. Haviar presented a simple quadruple construction of K_2 -algebras which works with pairs of elements only. He also proved that there exists a one-to-one correspondence between locally bounded K_2 -algebras and decomposable K_2 -quadruples. Recently, A. Badawy, D. Guffová and M. Haviar [1] introduced the class of decomposable *MS*-algebras. They

presented a triple construction of decomposable *MS*-algebras. Moreover, they proved that there exists a one-to-one correspondence between the decomposable *MS*-algebras and the decomposable *MS*-triples.

The aim of this paper is to investigate a subvariety of *GMS*-algebras containing the variety of *MS*-algebras, the so-called modular *GMS*-algebras. We construct modular *GMS*-algebras from the variety \mathbf{K}_2 (K_2 -algebras) from Kleene algebras and modular lattices by means of K_2 -quadruples. Also we define an isomorphism between two K_2 -quadruples and we show that two K_2 -algebras are isomorphic if and only if their associated K_2 -quadruples are isomorphic.

2 Preliminaries

An *MS*-algebra is an algebra $(L; \vee, \wedge, \circ, 0, 1)$ of type $(2, 2, 1, 0, 0)$ where $(L; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the unary operation \circ satisfies

$$x \leq x^{\circ\circ}, \quad (x \wedge y)^{\circ} = x^{\circ} \vee y^{\circ}, \quad 1^{\circ} = 0.$$

The class \mathbf{MS} of all *MS*-algebras forms a variety. The members of the subvariety \mathbf{M} of \mathbf{MS} defined by the identity $x = x^{\circ\circ}$ are called de Morgan algebras and the members of the subvariety \mathbf{K} of \mathbf{M} defined by the identity $x \wedge x^{\circ} \leq y \vee y^{\circ}$ are called Kleene algebras. The subvariety \mathbf{K}_2 of \mathbf{MS} is defined by the additional two identities

$$x \wedge x^{\circ} = x^{\circ\circ} \wedge x^{\circ}, \quad x \wedge x^{\circ} \leq y \vee y^{\circ}.$$

The class \mathbf{S} of all Stone algebras is a subvariety of \mathbf{MS} and is characterized by the identity $x \wedge x^{\circ} = 0$. The subvariety \mathbf{B} of \mathbf{MS} characterized by the identity $x \vee x^{\circ} = 1$ is the class of Boolean algebras.

A generalized de Morgan algebra (or *GM*-algebra) is a universal algebra $(L; \vee, \wedge, \bar{}, 0, 1)$ where $(L; \vee, \wedge, 0, 1)$ is a bounded lattice and the unary operation of involution $\bar{}$ satisfies the identities

$$GM_1: x = x^{\bar{\bar{}}}, \quad GM_2: (x \wedge y)^{\bar{}} = x^{\bar{}} \vee y^{\bar{}}, \quad GM_3: 1^{\bar{}} = 0.$$

A modular *GM*-algebra L is a *GM*-algebra where $(L; \vee, \wedge, 0, 1)$ is a modular lattice. A modular generalized Kleene algebra (modular *GK*-algebra) L is a modular *GM*-algebra satisfying the identity $x \wedge x^{\circ} \leq x \vee y^{\circ}$.

A generalized *MS*-algebra (or *GMS*-algebra) is a universal algebra $(L; \vee, \wedge, \circ, 0, 1)$ where $(L; \vee, \wedge, 0, 1)$ is a bounded lattice and the unary operation \circ satisfies the identities

$$GMS_1: x \leq x^{\circ\circ}, \quad GMS_2: (x \wedge y)^{\circ} = x^{\circ} \vee y^{\circ}, \quad GMS_3: 1^{\circ} = 0.$$

The class of all *GM*-algebras is a subvariety of the variety of all *GMS*-algebras.

A modular *GMS*-algebra is a *GMS*-algebra $(L; \vee, \wedge, \circ, 0, 1)$ where $(L; \vee, \wedge, 0, 1)$ is a modular lattice.

The class of all modular GMS-algebras forms a variety. The class **MS** is a subvariety of the variety of all modular GMS-algebras. Then the varieties **B**, **M**, **S** and **K₂** are subvarieties of the variety of all modular GMS-algebras.

The class **S** of all modular S-algebras is a subvariety of the variety of all modular GMS-algebras and is characterized by the identity $x \wedge x^\circ = 0$. It is known that the class **S** is a subvariety of **S**.

The main immediate consequences of these axioms are summarized in the following result.

Lemma 2.1 *Let L be a GMS-algebra. Then we have*

- (1) $0^\circ = 1$,
- (2) $x \leq y$ implies $x^\circ \geq y^\circ$,
- (3) $x^\circ = x^{\circ\circ}$,
- (4) $(x \vee y)^\circ = x^\circ \wedge y^\circ$,
- (5) $(x \wedge y)^{\circ\circ} = x^{\circ\circ} \wedge y^{\circ\circ}$,
- (6) $(x \vee y)^{\circ\circ} = x^{\circ\circ} \vee y^{\circ\circ}$.

Consequently, if L is a modular GMS-algebra, then the set $L^{\circ\circ} = \{x \in L : x^{\circ\circ} = x\}$ is a modular GM-algebra and a subalgebra of L such that the mapping $x \mapsto x^{\circ\circ}$ is a homomorphism of L onto $L^{\circ\circ}$, and $D(L) = \{x \in L : x^\circ = 0\}$ is a filter of L , the elements of which are called dense.

For an arbitrary lattice L , the set $F(L)$ of all filters of L ordered under set inclusion is a lattice. It is known that $F(L)$ is a modular lattice if and only if L is modular. Let $a \in L$; $[a]$ denotes the filter of L generated by a .

For any modular GMS-algebra L , the relation Φ defined by

$$x \equiv y (\Phi) \quad \Leftrightarrow \quad x^{\circ\circ} = y^{\circ\circ}$$

is a congruence relation on L and $L/\Phi \cong L^{\circ\circ}$ holds. Each congruence class contains exactly one element of $L^{\circ\circ}$ which is the largest element in the congruence class, the largest element of $[x]\Phi$ is $x^{\circ\circ}$ which is denoted by $\max[x]\Phi$. Hence Φ partition L into $\{F_c : c \in L^{\circ\circ}\}$, where $F_c = \{x \in L : x^{\circ\circ} = c\}$. Obviously, $F_0 = \{0\}$ and $F_1 = \{x \in L : x^{\circ\circ} = 1\} = D(L)$.

Now we introduce certain modular GMS-algebras, which are called \underline{K}_2 -algebras.

Definition 2.2 A modular GMS-algebra L is called a \underline{K}_2 -algebra if $L^{\circ\circ}$ is a distributive lattice and L satisfies the identities $x \wedge x^\circ = x^{\circ\circ} \wedge x^\circ$ and $x \wedge x^\circ \leq y \vee y^\circ$.

The class \underline{K}_2 of all \underline{K}_2 -algebras contains the class **K₂**. Clearly, the classes **S**, **S**, **M**, **K** and **B** are subclasses of the class \underline{K}_2 .

Theorem 2.3 *Let $L \in \underline{K}_2$. Then*

- (1) $x = x^{\circ\circ} \wedge (x \vee x^\circ)$ for every $x \in L$,

- (2) $L^{\circ\circ} = \{x \in L : x = x^{\circ\circ}\}$ is a Kleene algebra,
(3) $L^\wedge = \{x \wedge x^\circ : x \in L\} = \{x \in L : x \leq x^\circ\}$ is an ideal of L ,
(4) $L^\vee = \{x \vee x^\circ : x \in L\} = \{x \in L : x \geq x^\circ\}$ is a filter of L ,
(5) $D(L) = \{x \in L : x^\circ = 0\}$ is a filter of L and $D(L) \subseteq L^\vee$.

Proof (1) Since $x \leq x^{\circ\circ}$, then by modularity of L we get

$$\begin{aligned} x^{\circ\circ} \wedge (x \vee x^\circ) &= (x^{\circ\circ} \wedge x^\circ) \vee x \\ &= (x \wedge x^\circ) \vee x \text{ by Definition 2.2} \\ &= x. \end{aligned}$$

(2) It is obvious.

(3) Clearly $0 \in L^\wedge$. Let $x, y \in L^\wedge$. Then $x \leq x^\circ$ and $y \leq y^\circ$. By Definition 2.2, we get $x = x \wedge x^\circ \leq y \vee y^\circ = y^\circ$. It follows that $x^\circ \geq y^{\circ\circ} \geq y$. Then $x^\circ \wedge y^\circ \geq x, y$ implies $x^\circ \wedge y^\circ \geq x \vee y$. Now

$$(x \vee y) \wedge (x \vee y)^\circ = (x \vee y) \wedge (x^\circ \wedge y^\circ) = x \vee y.$$

Consequently $x \vee y \leq (x \vee y)^\circ$ and $x \vee y \in L^\wedge$. Let $x \in L^\wedge$ be such that $z \leq x$ for some $z \in L$. Then $z \leq x \leq x^\circ \leq z^\circ$. Hence $z \in L^\wedge$. Then L^\wedge is an ideal of L .

(4) By duality of (3).

(5) It is obvious. □

Corollary 2.4 *Let L be a modular GMS-algebra. Then for all $x \in L$ the following conditions are equivalent:*

- (1) $x = x^{\circ\circ} \wedge (x \vee x^\circ)$,
(2) $x \wedge x^\circ = x^{\circ\circ} \wedge x^\circ$.

Now we reformulate the definition of polarization given in [Definition 1(iii), 11] as follows.

Definition 2.5 Let K be a Kleene algebra and D be a modular lattice with 1. A mapping $\varphi: K \rightarrow F(D)$ is called a polarization if φ is a (0,1)-homomorphism such that $a\varphi = D$ for every $a \in K^\vee$ and $a\varphi$ is a principal filter of D for every $a \in K^\wedge$.

3 The triple associated with a \underline{K}_2 -algebra

Let $L \in \underline{\mathbf{K}}_2$. L^\vee is a filter of L , and L^\wedge is a modular lattice with the largest element 1. So $F(L^\vee)$ is also a modular lattice. Consider the map $\varphi(L): L^{\circ\circ} \rightarrow F(L^\vee)$ defined by the following way

$$a\varphi(L) = \{x \in L^\vee : x \geq a^\circ\} = [a^\circ] \cap L^\vee, \quad a \in L^{\circ\circ}.$$

Lemma 3.1 *Let $L \in \underline{\mathbf{K}}_2$. Then $\varphi(L)$ is a polarization of $L^{\circ\circ}$ into $F(L^\vee)$.*

Proof It is easy to check that $0\varphi(L) = [1]$, $1\varphi(L) = L^\vee$ and $(a \wedge b)\varphi(L) = a\varphi(L) \cap b\varphi(L)$. Now we show that $(a \vee b)\varphi(L) = a\varphi(L) \vee b\varphi(L)$. Since $a, b \leq a \vee b$, then $a\varphi(L) \vee b\varphi(L) \subseteq (a \vee b)\varphi(L)$. For the converse, let $t \in (a \vee b)\varphi(L) = [a^\circ \wedge b^\circ] \cap L^\vee$. Put $x = a \vee (a^\circ \wedge t)$. Then $x^\circ = a^\circ \wedge (a \vee t^\circ) = (a^\circ \wedge a) \vee (a^\circ \wedge t^\circ) \leq a \vee (a^\circ \wedge t) = x$ since $L^{\circ\circ}$ is distributive and $t^\circ \leq t$. Thus $x \in L^\vee$. Moreover,

$$a^\circ \wedge (b^\circ \vee x) = a^\circ \wedge (b^\circ \vee (a \vee (a^\circ \wedge t))) = (a^\circ \wedge (a \vee b^\circ)) \vee (a^\circ \wedge t) \leq t,$$

since $a^\circ \wedge (a \vee b^\circ) = (a^\circ \wedge a) \vee (a^\circ \wedge b^\circ) \leq t$. Now, $t \in [a^\circ] \vee [b^\circ \vee x] \subseteq [a^\circ] \vee ([b^\circ] \cap L^\vee)$. But $t \in L^\vee$ and $F(L)$ is a modular lattice, hence

$$t \in ([a^\circ] \vee ([b^\circ] \cap L^\vee)) \cap L^\vee = ([a^\circ] \cap L^\vee) \vee ([b^\circ] \cap L^\vee) = a\varphi(L) \vee b\varphi(L).$$

Thus $\varphi(L)$ is (0,1)-lattice homomorphism. If $a \in L^{\circ\circ}$, then $(a \vee a^\circ)\varphi(L) = [a^\circ \wedge a] \cap L^\vee = L^\vee$ and $(a \wedge a^\circ)\varphi(L) = [a^\circ \vee a]$. Then φ is a polarization. \square

Definition 3.2 A triple (K, D, φ) is said to be a \underline{K}_2 -triple if

- (1) $(K; \vee, \wedge, 0, 1)$ is a Kleene algebra,
- (2) D is a modular lattice with 1,
- (3) $\varphi: K \rightarrow F(D)$ is a polarization.

Let L be a \underline{K}_2 -algebra. Then $(L^{\circ\circ}, L^\vee, \varphi(L))$ is the triple associated with L and this triple is a \underline{K}_2 -triple.

Lemma 3.3 Let (K, D, φ) be a \underline{K}_2 -triple. Then we have

$$a\varphi \cap (b\varphi \vee c\varphi) = (a\varphi \cap b\varphi) \vee (a\varphi \cap c\varphi) \text{ for every } a, b, c \in K.$$

Lemma 3.4 Let (K, D, φ) be a \underline{K}_2 -triple. Then we have

(i) for every $a \in K$ and for every $y \in D$ there exists an element $t \in D$ such that

$$a\varphi \cap [y] = [t],$$

(ii) for every $a \in K$ and for every $y \in D$ there exists an element $t \in a^\circ\varphi$ such that

$$a\varphi \vee [y] = a\varphi \vee [t],$$

(iii) for every $a, b \in K$ and for every $y \in D$ there exists an element $t \in D$ such that

$$((a\varphi \cap b^\circ\varphi) \vee [y]) \cap (a^\circ\varphi \vee b\varphi \vee [y]) = [t].$$

Proof For any $a \in K$, there is $d_a \in D$ such that $(a \wedge a^\circ)\varphi = a\varphi \cap a^\circ\varphi = [d_a]$ as $a \wedge a^\circ \in K^\wedge$ and φ is a polarization. Recall that $F(D)$ is a modular lattice.

(i). For all $a \in K, a \wedge a^\circ \in K^\wedge, a \vee a^\circ \in K^\vee$. Then there exists $d_a \in D$ such that $a\varphi \cap a^\circ\varphi = [d_a]$ and $a\varphi \vee a^\circ\varphi = (a \vee a^\circ)\varphi = D$. Therefore, there exist elements $x_1 \in a\varphi$ and $z_1 \in a^\circ\varphi$ such that $x_1, z_1 \leq d_a$ and $x_1 \wedge z_1 \leq y$.

We notice that $x_1 \vee z_1 \in a\varphi \cap a^\circ\varphi$. Hence $x_1 \vee z_1 = d_a$. We claim $t = x_1 \vee y$. Clearly $t \in a\varphi \cap [y]$. Conversely, let $v \in a\varphi \cap [y]$. Then

$$\begin{aligned} v &\geq (v \wedge x_1) \vee y \\ &= ((v \wedge x_1) \vee (x_1 \wedge z_1)) \vee y \\ &= (((v \wedge x_1) \vee z_1) \wedge x_1) \vee y \text{ by modularity of } D \\ &= (d_a \wedge x_1) \vee y \\ &= x_1 \vee y \text{ as } (v \wedge x_1) \vee z_1 = d_a \geq x_1. \end{aligned}$$

Hence $v \geq x_1 \vee y = t$, and therefore $a\varphi \cap [y] = [t]$.

(ii). It is enough to show that $a^\circ\varphi \cap (a\varphi \vee [y]) = [t]$, for some $t \in D$ since then $t \in a^\circ\varphi$ and $[t] \vee a\varphi = (a^\circ\varphi \cap (a\varphi \vee [y])) \vee a\varphi = (a\varphi \vee [y]) \cap (a^\circ\varphi \vee a\varphi) = a\varphi \vee [y]$, from modularity of $F(D)$. Let $x_1 \in a\varphi$, $z_1 \in a^\circ\varphi$, $x_1 \wedge z_1 \leq y$ and $x_1, z_1 \leq d_a$. We claim that $t = z_1 \vee (x_1 \wedge y)$. Evidently, $t \in a^\circ\varphi \cap (a\varphi \vee [y])$. Conversely, let $v \in a^\circ\varphi \cap (a\varphi \vee [y])$. Then $v \geq v \wedge z_1 \in a^\circ\varphi$ and there is $x \in a\varphi$ with $v \geq x \wedge y \geq (x \wedge x_1) \wedge y$. Denote $z_0 = v \wedge z_1$ and $x_0 = x \wedge x_1$. Hence

$$v \geq (x_0 \wedge y) \vee z_0 \geq (x_0 \wedge x_1 \wedge z_1) \vee z_0 = (x_0 \wedge z_1) \vee z_0 = (x_0 \vee z_0) \wedge z_1 = z_1,$$

because $x_0 \vee z_0 = d_a \geq z_1$. This implies

$$\begin{aligned} v &\geq (x_0 \wedge y) \vee z_1 \\ &= (x_0 \wedge y) \vee (x_1 \wedge z_1) \vee z_1 \\ &= ((x_0 \vee (x_1 \wedge z_1)) \wedge y) \vee z_1 \\ &= ((x_0 \vee z_1) \wedge x_1 \wedge y) \vee z_1 \\ &= (x_1 \wedge y) \vee z_1 \text{ as } x_0 \vee z_1 = d_a \geq x_1 \wedge y \\ &= t. \end{aligned}$$

So, $v \geq t$ and $a^\circ\varphi \cap (a\varphi \vee [y]) = [t]$.

(iii). From (ii) there exists $y_1 \in a\varphi$ such that $[y_1] \vee a^\circ\varphi = [y] \vee a^\circ\varphi$. Using Lemma 3.3 and modularity of $F(D)$, we get

$$\begin{aligned} &((a\varphi \cap b^\circ\varphi) \vee [y]) \cap (a^\circ\varphi \vee b\varphi \vee [y]) \\ &= ((a\varphi \cap b^\circ\varphi) \cap (a^\circ\varphi \vee b\varphi \vee [y])) \vee [y] \\ &= ((a\varphi \cap b^\circ\varphi) \cap (a^\circ\varphi \vee b\varphi \vee [y_1])) \vee [y] \\ &= (b^\circ\varphi \cap (a\varphi \cap (a^\circ\varphi \vee b\varphi \vee [y_1]))) \vee [y] \\ &= (b^\circ\varphi \cap ((a\varphi \cap (a^\circ\varphi \vee b\varphi)) \vee [y_1])) \vee [y] \\ &= (b^\circ\varphi \cap ([d_a] \vee (a\varphi \cap b\varphi) \vee [y_1])) \vee [y] \\ &= (b^\circ\varphi \cap (a\varphi \cap (b\varphi \vee [y_1 \wedge d_a]))) \vee [y] \\ &= (a\varphi \cap [t_1]) \vee [y] \\ &= [t_2] \vee [y] \\ &= [t_2 \wedge y]. \end{aligned}$$

where $t_1, t_2 \in D$ are such elements that $b^\circ\varphi \cap (b\varphi \vee [y_1 \wedge d_a]) = [t_1]$ (see the proof of (ii)), $a\varphi \cap [t_1] = [t_2]$ from (i). Thus $t = t_2 \wedge y$. \square

Theorem 3.5 *Let (K, D, φ) be a \underline{K}_2 -triple. Then for any $a, b \in K$ and $x, y \in D$ there exists an element $t \in D$ such that*

$$(a^\circ \varphi \vee [x]) \cap (b^\circ \varphi \vee [y]) = (a \vee b)^\circ \varphi \vee [t].$$

Proof Let $a, b \in K$ and $x, y \in D$. It is enough to show that there is $t \in D$ such that

$$(a^\circ \varphi \vee [x]) \cap (b^\circ \varphi \vee [y]) \cap (a \wedge b) \varphi = [t]$$

because then

$$\begin{aligned} [t] \vee (a \vee b)^\circ \varphi &= ((a^\circ \varphi \vee [x]) \cap (b^\circ \varphi \vee [y]) \cap (a \wedge b) \varphi) \vee (a \vee b)^\circ \varphi \\ &= (a^\circ \varphi \vee [x]) \cap (b^\circ \varphi \vee [y]) \cap ((a \wedge b) \varphi \vee (a \vee b)^\circ \varphi) \\ &= (a^\circ \varphi \vee [x]) \cap (b^\circ \varphi \vee [y]) \end{aligned}$$

by modularity of $F(D)$ and since $(a \vee b) \varphi \vee (a \vee b)^\circ \varphi = D$. In accordance with Lemma 3.4, we can suppose $x \in a\varphi$ and $y \in b\varphi$. Then by Lemma 3.3 and by modularity of $F(D)$,

$$\begin{aligned} &(a^\circ \varphi \vee [x]) \cap (b^\circ \varphi \vee [y]) \cap (a \vee b) \varphi \\ &= ((a^\circ \varphi \vee [x]) \cap (a\varphi \vee b\varphi)) \cap ((b^\circ \varphi \vee [y]) \cap (a\varphi \vee b\varphi)) \\ &= ((a^\circ \varphi \cap (a\varphi \vee b\varphi)) \vee [x]) \cap ((b^\circ \varphi \cap (a\varphi \vee b\varphi)) \vee [y]) \\ &= ((a^\circ \varphi \cap a\varphi) \vee (a^\circ \varphi \cap b\varphi) \vee [x]) \cap ((b^\circ \varphi \cap a\varphi) \vee (b^\circ \varphi \cap b\varphi) \vee [y]) \\ &= ([d_a \wedge x] \vee (a^\circ \varphi \cap b\varphi)) \cap ([d_b \wedge y] \vee (b^\circ \varphi \cap a\varphi)) \end{aligned}$$

where d_a, d_b are as in the proof of Lemma 3.4. Denote $x_0 = x \wedge d_a, y_0 = y \wedge d_b$ and $x_0 \wedge y_0 = z$. We first show that

$$((a\varphi \cap b^\circ \varphi) \vee [z]) \cap ((a^\circ \varphi \cap b\varphi) \vee [z]) = [p],$$

for some $p \in D$. Since $a^\circ \varphi \vee b\varphi \supseteq a^\circ \varphi \cap b\varphi$, we can write

$$\begin{aligned} &((a\varphi \cap b^\circ \varphi) \vee [z]) \cap ((a^\circ \varphi \cap b\varphi) \vee [z]) \\ &= ((a\varphi \cap b^\circ \varphi) \vee [z]) \cap (a^\circ \varphi \vee b\varphi \vee [z]) \cap ((a^\circ \varphi \cap b\varphi) \vee [z]) \\ &= [q] \cap ((a^\circ \varphi \cap b\varphi) \vee [z]) \end{aligned}$$

where $[q] = ((a\varphi \cap b^\circ \varphi) \vee [z]) \cap (a^\circ \varphi \vee b\varphi \vee [z])$, by Lemma 3.4 (iii). Evidently $[q] \supseteq [z]$. Hence by modularity we get

$$\begin{aligned} &[q] \cap ((a^\circ \varphi \cap b\varphi) \vee [z]) \\ &= ([q] \cap a^\circ \varphi \cap b\varphi) \vee [z] \\ &= ([q] \cap (a^\circ \wedge b) \varphi) \vee [z] \\ &= [t_1] \vee [z] \text{ where } [q] \cap (a^\circ \wedge b) \varphi = [t_1] \text{ by Lemma 4.3(i)} \\ &= [t_1 \wedge z] \\ &= [p] \text{ where } p = t_1 \wedge z. \end{aligned}$$

Since $[p] \supseteq [z] \supseteq [x_0], [y_0]$ and $F(D)$ is modular, we have

$$\begin{aligned}
& ([x_0] \vee (a^\circ \varphi \cap b\varphi)) \cap ([y_0] \vee (b^\circ \varphi \cap a\varphi)) \\
&= ([p] \cap ([x_0] \vee (a^\circ \varphi \cap b\varphi))) \cap ([p] \cap ([y_0] \vee (b^\circ \varphi \cap a\varphi))) \\
&= (([p] \cap (a^\circ \varphi \cap b\varphi)) \vee [x_0]) \cap (([p] \cap (b^\circ \varphi \cap a\varphi)) \vee [y_0]) \\
&= ([v] \vee [x_0]) \cap ([w] \vee [y_0]) \text{ for some } v, w \in D \\
&= [(u \wedge x_0) \vee (w \wedge y_0)] \\
&= [t],
\end{aligned}$$

where $[v] = [p] \cap a^\circ \varphi \cap b\varphi$, $[w] = [p] \cap b^\circ \varphi \cap a\varphi$ and $t = (u \wedge x_0) \vee (w \wedge y_0) \in D$. \square

4 \underline{K}_2 -construction

In this section we generalize the construction of [3, 4] from the so-called K_2 -algebras to \underline{K}_2 -algebras. Also we prove that there exists a one-to-one correspondence between \underline{K}_2 -algebras and \underline{K}_2 -quadruples.

Definition 4.1 A \underline{K}_2 -quadruple is (K, D, φ, γ) where

- (i) (K, D, φ) is a \underline{K}_2 -triple, and
- (ii) γ is a monomial congruence on D , that is every γ class $[y]\gamma$ has a largest element $(\max[y]\gamma)$.

Let $L \in \underline{\mathbf{K}}_2$. Then $(L^\circ, L^\vee, \varphi(L))$ is a K_2 -triple. Let $\gamma(L)$ be the restriction of the congruence Φ on L^\vee . Since $\max[x]\gamma = x^\circ$, for every $x \in L^\vee$. Then $\gamma(L)$ is a monomial congruence on L^\vee . We say that $(L^\circ, L^\vee, \varphi(L), \gamma(L))$ is the quadruple associated with L and this quadruple is a \underline{K}_2 -quadruple.

Theorem 4.2 Let (K, D, φ, γ) be a \underline{K}_2 -quadruple. Then

$$L = \{(a, a^\circ \varphi \vee [x]) : a \in K, x \in D, \max[x]\gamma \in a^\circ \varphi\}$$

is a \underline{K}_2 -algebra if we define

$$\begin{aligned}
(a, a^\circ \varphi \vee [x]) \wedge (b, b^\circ \varphi \vee [y]) &= (a \wedge b, (a^\circ \varphi \vee [x]) \vee (b^\circ \varphi \vee [y])), \\
(a, a^\circ \varphi \vee [x]) \vee (b, b^\circ \varphi \vee [y]) &= (a \vee b, (a^\circ \varphi \vee [x]) \cap (b^\circ \varphi \vee [y])), \\
(a, a^\circ \varphi \vee [x])^\circ &= (a^\circ, a\varphi), \\
1_L &= (1, [1]), \\
0_L &= (0, D).
\end{aligned}$$

Moreover, $L^\circ \cong K$.

Proof Let $F_d(D)$ denote the dual lattice to the modular lattice $F(D)$ of all filters of D . Evidently, L is a subset of the direct product $K \times F_d(D)$. We show

first that L is a sublattice of $K \times F_d(D)$. Let $(a, a^\circ\varphi \vee [x]), (b, b^\circ\varphi \vee [y]) \in L$. Then

$$(a, a^\circ\varphi \vee [x]) \wedge (b, b^\circ\varphi \vee [y]) = (a \wedge b, (a \wedge b)^\circ\varphi \vee [x \wedge y]) \in L,$$

because of φ is a lattice homomorphism and

$$\max[x \wedge y]\gamma = \max[x]\gamma \wedge \max[y]\gamma \in a^\circ\varphi \vee b^\circ\varphi = (a \wedge b)^\circ\varphi.$$

Moreover,

$$\begin{aligned} & (a, a^\circ\varphi \vee [x]) \vee (b, b^\circ\varphi \vee [y]) \\ &= (a \vee b, (a^\circ\varphi \vee [x]) \cap (b^\circ\varphi \vee [y])) \\ &= (a \vee b, (a \vee b)^\circ\varphi \vee [t]) \text{ for some } t \in D, \text{ by Theorem 3.5.} \end{aligned}$$

Now we prove that $\max[x]\gamma \in a^\circ\varphi$ and $\max[y]\gamma \in b^\circ\varphi$ implies $\max[t]\gamma \in (a \vee b)^\circ\varphi$. From the proof of Theorem 3.5, $t = (v \wedge x_0) \vee (w \wedge y_0)$ where $v \in a^\circ\varphi$, $w \in b^\circ\varphi$, $x_0 = x \wedge d_a$ and $y_0 = y \wedge d_a$. Then

$$t = (v \wedge x \wedge d_a) \vee (w \wedge y \wedge d_b) = (x \wedge v_0) \vee (y \wedge w_0)$$

where $v_0 = v \wedge d_a \in a^\circ\varphi$ and $w_0 = w \wedge d_b \in b^\circ\varphi$. Then

$$\max[t]\gamma \geq (\max[x]\gamma \wedge \max[v_0]\gamma) \vee (\max[y]\gamma \wedge \max[w_0]\gamma) \in a^\circ\varphi \cap b^\circ\varphi = (a \vee b)^\circ\varphi,$$

because of $\max[v_0]\gamma \geq v_0 \in a^\circ\varphi$ and $\max[w_0]\gamma \geq w_0 \in b^\circ\varphi$ implies $\max[v_0]\gamma \in a^\circ\varphi$ and $\max[w_0]\gamma \in b^\circ\varphi$, respectively. Then $(a \vee b, (a \vee b)^\circ\varphi \vee [t]) \in L$. Therefore L is a sublattice of $K \times F_d(D)$. Hence L is a modular lattice. The order of L is given by

$$(a, a^\circ\varphi \vee [x]) \leq (b, b^\circ\varphi \vee [y]) \text{ iff } a \leq b \text{ and } a^\circ\varphi \vee [x] \supseteq b^\circ\varphi \vee [y].$$

L is bounded and

$$(0, D) \leq (a, a^\circ\varphi \vee [x]) \leq (1, [1]).$$

In addition,

$$\begin{aligned} & (a, a^\circ\varphi \vee [x]) \leq (a, a^\circ\varphi) = (a, a^\circ\varphi \vee [x])^\circ, \\ & ((a, a^\circ\varphi \vee [x]) \wedge (b, b^\circ\varphi \vee [y]))^\circ = (a, a^\circ\varphi \vee [x])^\circ \vee (b, b^\circ\varphi \vee [y])^\circ, \\ & (1, [1])^\circ = (0, D). \end{aligned}$$

Then L is a modular GMS-algebra. Also we get

$$\begin{aligned} & (a, a^\circ\varphi \vee [x]) \wedge (a, a^\circ\varphi \vee [x])^\circ \\ &= (a \wedge a^\circ, a^\circ\varphi \vee [x] \vee a\varphi) \\ &= (a \wedge a^\circ, a^\circ\varphi \vee a\varphi) \text{ as } [x] \subseteq a\varphi \vee a^\circ\varphi = D \\ &= (a, a^\circ\varphi) \wedge (a^\circ, a\varphi) \\ &= (a, a^\circ\varphi \vee [x])^\circ \wedge (a, a^\circ\varphi \vee [x])^\circ, \end{aligned}$$

and

$$(a, a^\circ\varphi \vee [x]) \wedge (a, a^\circ\varphi \vee [x])^\circ \leq (b, b^\circ\varphi \vee [y]) \vee (b, b^\circ\varphi \vee [y])^\circ.$$

Hence $L \in \underline{\mathbf{K}}_2$. Now,

$$L^{\circ\circ} = \{(a, a^\circ\varphi \vee [x])^{\circ\circ} : (a, a^\circ\varphi \vee [x]) \in L\} = \{(a, a^\circ\varphi) : a \in K\} \cong K$$

under the isomorphism $(a, a^\circ\varphi) \mapsto a$. Then $L^{\circ\circ}$ is a Kleene algebra. Therefore L is a $\underline{\mathbf{K}}_2$ -algebra. \square

Corollary 4.3 *From Theorem 4.2, we have*

- (1) $L^\vee = \{(a, a^\circ\varphi \vee [x]) \in L : a \in K^\vee, x \in D\}$,
- (2) $D(L) = \{(1, [x]) : x \in [1]\gamma, x \in D\}$.

Corollary 4.4 *Let (K, D, φ, γ) be a $\underline{\mathbf{K}}_2$ -quadruple. Then*

- (1) *If D is a distributive lattice, then L described by Theorem 4.2 is a \mathbf{K}_2 -algebra;*
- (2) *If K is a Boolean algebra and $\gamma = \iota$, then L described by Theorem 4.2 is a modular S -algebra;*
- (3) *If K is a Boolean algebra, D is a distributive lattice and $\gamma = \iota$, then L described by Theorem 4.2 is a Stone algebra.*

We say that $L \in \underline{\mathbf{K}}_2$ from Theorem 4.2 is associated with the $\underline{\mathbf{K}}_2$ -quadruple (K, D, φ, γ) and the construction of L described in Theorem 4.2 will be called a $\underline{\mathbf{K}}_2$ -construction.

Theorem 4.5 *Let $L \in \underline{\mathbf{K}}_2$. Let $(L^{\circ\circ}, L^\vee, \varphi(L), \gamma(L))$ be the $\underline{\mathbf{K}}_2$ -quadruple associated with L . Then L_1 associated with $(L^{\circ\circ}, L^\vee, \varphi(L), \gamma(L))$ is isomorphic to L .*

Proof For every $x \in L$, $x = x^{\circ\circ} \wedge (x \vee x^\circ)$ and by modularity of $F(L)$, we observe

$$x^\circ\varphi(L) \vee [x \vee x^\circ] = ([x^{\circ\circ}] \cap L^\vee) \vee [x \vee x^\circ] = L^\vee \cap ([x^{\circ\circ}] \vee [x \vee x^\circ]) = L^\vee \cap [x].$$

We shall prove that the mapping $f: L \rightarrow L_1$ defined by

$$xf = (x^{\circ\circ}, x^\circ\varphi(L) \vee [x \vee x^\circ]) = (x^{\circ\circ}, L^\vee \cap [x])$$

is the described isomorphism. Obviously $xf \in L_1$, since $\max[x \vee x^\circ]\gamma(L) = (x \vee x^\circ)^{\circ\circ} = x^{\circ\circ} \vee x^\circ \in [x^{\circ\circ}] \cap L^\vee = x^\circ\varphi(L)$. For every $x, y \in L$,

$$\begin{aligned} (x \wedge y)f &= ((x \wedge y)^{\circ\circ}, (x \wedge y)^\circ\varphi(L) \vee [(x \wedge y) \vee (x \wedge y)^\circ]) \\ &= ((x \wedge y)^{\circ\circ}, [x \wedge y] \cap L^\vee), \\ xf \wedge yf &= (x^{\circ\circ}, x^\circ\varphi(L) \vee [x \vee x^\circ]) \wedge (y^{\circ\circ}, y^\circ\varphi(L) \vee [y \vee y^\circ]) \\ &= (x^{\circ\circ} \wedge y^{\circ\circ}, x^\circ\varphi(L) \vee [x \vee x^\circ] \vee y^\circ\varphi(L) \vee [y \vee y^\circ]). \end{aligned}$$

Since $x = x^{\circ\circ} \wedge (x \vee x^\circ)$, $y = y^{\circ\circ} \wedge (y \vee y^\circ)$ and $\varphi(L)$ is a polarity (see Lemma 3.1), then by modularity of $F(L)$, we have

$$\begin{aligned}
x^\circ \varphi(L) \vee [x \vee x^\circ] \vee y^\circ \varphi(L) \vee [y \vee y^\circ] \\
&= (x \wedge y)^\circ \varphi(L) \vee [(x \vee x^\circ) \wedge (y \vee y^\circ)] \\
&= ((x \wedge y)^{\circ\circ} \cap L^\vee) \vee [(x \vee x^\circ) \wedge (y \vee y^\circ)] \\
&= L^\vee \cap ((x \wedge y)^{\circ\circ} \vee [(x \vee x^\circ) \wedge (y \vee y^\circ)]) \\
&= L^\vee \cap [x^{\circ\circ} \wedge y^{\circ\circ} \wedge (x \vee x^\circ) \wedge (y \vee y^\circ)] \\
&= L^\vee \cap [x \wedge y].
\end{aligned}$$

Then $(x \wedge y)f = xf \wedge yf$. Also,

$$\begin{aligned}
(x \vee y)f &= ((x \vee y)^{\circ\circ}, [x \vee y] \cap L^\vee) \\
&= (x^{\circ\circ} \vee y^{\circ\circ}, [x] \cap [y] \cap L^\vee) \\
&= (x^{\circ\circ} \vee y^{\circ\circ}, ([x] \cap L^\vee) \cap ([y] \cap L^\vee)) \\
&= (x^{\circ\circ}, [x] \cap L^\vee) \vee (y^{\circ\circ}, [y] \cap L^\vee) \\
&= xf \vee yf
\end{aligned}$$

and $0f = (0, L^\vee)$, $1f = (1, [1])$. Then f is a (0,1)-lattice homomorphism. Now,

$$\begin{aligned}
(xf)^\circ &= (x^{\circ\circ}, x^\circ \varphi(L) \vee [x \vee x^\circ])^\circ \\
&= (x^\circ, x^{\circ\circ} \varphi(L)) \\
&= (x^\circ, [x^\circ] \cap L^\vee) \\
&= x^\circ f,
\end{aligned}$$

hence f is a homomorphism of \underline{K}_2 -algebras.

Now assume $x_1 f = x_2 f$. Then $(x_1^{\circ\circ}, [x_1] \cap L^\vee) = (x_2^{\circ\circ}, [x_2] \cap L^\vee)$. It follows that $x_1^{\circ\circ} = x_2^{\circ\circ}$ and $[x_1] \cap L^\vee = [x_2] \cap L^\vee$. Now

$$\begin{aligned}
[x_1] &= [x_1^{\circ\circ} \wedge (x_1 \vee x_1^\circ)] \\
&= [x_1^{\circ\circ}] \vee [x_1 \vee x_1^\circ] \\
&= [x_1^{\circ\circ}] \vee (L^\vee \cap [x_1 \vee x_1^\circ]) \text{ as } x_1 \vee x_1^\circ \in L^\vee \\
&= [x_1^{\circ\circ}] \vee (L^\vee \cap [x_1] \cap [x_1^\circ]) \\
&= [x_2^{\circ\circ}] \vee (L^\vee \cap [x_2] \cap [x_2^\circ]) \\
&= [x_2^{\circ\circ}] \vee (L^\vee \cap [x_2 \vee x_2^\circ]) \\
&= [x_2^{\circ\circ}] \vee [x_2 \vee x_2^\circ] \text{ as } x_2 \vee x_2^\circ \in L^\vee \\
&= [x_2^{\circ\circ} \wedge (x_2 \vee x_2^\circ)] \\
&= [x_2].
\end{aligned}$$

Consequently, $x_1 = x_2$ and f is injective. It remains to prove that f is surjective. Let $(x^{\circ\circ}, x^\circ \varphi(L) \vee [z]) \in L_1$, that is $z^{\circ\circ} = \max[z] \gamma(L) \in x^\circ \varphi(L) = [x^{\circ\circ}] \cap L^\vee$. Then by modularity of $F(L)$ we get

$$(x^{\circ\circ}, x^\circ \varphi(L) \vee [z]) = (x^{\circ\circ}, ([x^{\circ\circ}] \cap L^\vee) \vee [z]) = (x^{\circ\circ}, L^\vee \cap [x^{\circ\circ} \wedge z]).$$

Set $h = x^{\circ\circ} \wedge z$. Then $h^{\circ\circ} = x^{\circ\circ} \wedge z^{\circ\circ} = x^{\circ\circ}$ and consequently

$$(x^{\circ\circ}, x^{\circ} \varphi(L) \vee [z]) = (h^{\circ\circ}, [h] \cap L^{\vee}) = (h^{\circ\circ}, h^{\circ} \varphi(L) \vee [h \vee h^{\circ}]) = hf.$$

Thus f is an isomorphism. \square

5 Isomorphisms

In this section we define an isomorphism between two \underline{K}_2 -quadruples and we show that two \underline{K}_2 -algebras are isomorphic if and only if their associated \underline{K}_2 -quadruples are isomorphic.

Definition 5.1 An isomorphism of the \underline{K}_2 -quadruples (K, D, φ, γ) and $(K_1, D_1, \varphi_1, \gamma_1)$ is a pair (f, g) , where f is an isomorphism of K and K_1 , g is an isomorphism of D and D_1 such that $x \equiv y(\gamma)$ iff $xg \equiv yg(\gamma_1)$ for all $x, y \in D$ and the diagram

$$\begin{array}{ccc} K & \xrightarrow{\varphi} & F(D) \\ f \downarrow & & \downarrow F(g) \\ K_1 & \xrightarrow{\varphi_1} & F(D_1) \end{array}$$

commutes ($F(g)$ stands for the isomorphism of $F(D)$ and $F(D_1)$ induced by g).

Theorem 5.2 Let $L, M \in \underline{K}_2$. Then $L \cong M$ if and only if

$$(L^{\circ\circ}, L^{\vee}, \varphi(L), \gamma(L)) \cong (M^{\circ\circ}, M^{\vee}, \varphi(M), \gamma(M)).$$

Proof Let $\theta: L \rightarrow M$ be an isomorphism. We have two isomorphisms, $f: L^{\circ\circ} \rightarrow M^{\circ\circ}$ defined by $xf = x\theta$ and $g: L^{\vee} \rightarrow M^{\vee}$ defined by $xg = x\theta$. Now define $F(g): F(L^{\vee}) \rightarrow F(M^{\vee})$ by $AF(g) = \{a\theta: a \in A\}$.

For every $a \in L^{\circ\circ}$, we have

$$\begin{aligned} (af)\varphi(M) &= (a\theta)\varphi(M) = [(a\theta)^{\circ}] \cap M^{\vee}, \\ a\varphi(L)F(g) &= ([a^{\circ}] \cap L^{\vee})F(g) = \{y\theta: y \in [a^{\circ}] \cap L^{\vee}\} = [(a\theta)^{\circ}] \cap M^{\vee}. \end{aligned}$$

For $x, y \in L^{\vee}$, $x \equiv y(\gamma(L))$ iff $x^{\circ\circ} = y^{\circ\circ}$ iff $x^{\circ\circ}\theta = y^{\circ\circ}\theta$ iff $(xg)^{\circ\circ} = (y\theta)^{\circ\circ} = x^{\circ\circ}\theta = y^{\circ\circ}\theta = (y\theta)^{\circ\circ} = (yg)^{\circ\circ}$. Hence $xg \equiv yg(\gamma(M))$. Then (f, g) is a \underline{K}_2 -quadruple isomorphism. Conversely, we have to show that the isomorphism (f, g) of \underline{K}_2 -quadruples $(L^{\circ\circ}, L^{\vee}, \varphi(L), \gamma(L))$ and $(M^{\circ\circ}, M^{\vee}, \varphi(M), \gamma(M))$ implies the existence of an isomorphism $h: L \rightarrow M$, between \underline{K}_2 -algebras L, M constructed by \underline{K}_2 -construction. We claim that

$$(a, a^{\circ} \varphi(L) \vee [x])h = (af, (af)^{\circ} \varphi(M) \vee [xg])$$

is the desired isomorphism. Firstly we note that

$$(\max[x]\gamma(L))g = \max[xg]\gamma(M) \text{ for all } x \in L^{\vee}$$

Then

$$\max[xg]\gamma(M) = (\max[x]\gamma(L))g \in (a^\circ\varphi(L))F(g) = (af)^\circ\varphi(M)$$

as $\max[x]\gamma(L) \in a^\circ\varphi(L)$. Hence h is well defined.

Since f and $F(g)$ are isomorphisms, then we get

$$\begin{aligned} (a, a^\circ\varphi(L) \vee [x]) &\leq (b, b^\circ\varphi(L) \vee [y]) \\ \Leftrightarrow a &\leq b, a^\circ\varphi(L) \vee [x] \supseteq b^\circ\varphi(L) \vee [y] \\ \Leftrightarrow af &\leq bf, (a^\circ\varphi(L) \vee [x])F(g) \supseteq (b^\circ\varphi(L) \vee [y])F(g) \\ \Leftrightarrow af &\leq bf, (a^\circ\varphi(L))F(g) \vee [x]F(g) \supseteq (b^\circ\varphi(L))F(g) \vee [y]F(g) \\ \Leftrightarrow af &\leq bf, (af)^\circ\varphi(M) \vee [xg] \supseteq (bf)^\circ\varphi(M) \vee [yg] \\ \Leftrightarrow (af, &(af)^\circ\varphi(M) \vee [xg]) \leq (bf, (bf)^\circ\varphi(M) \vee [yg]) \\ \Leftrightarrow (a, &a^\circ\varphi(L) \vee [x])h \leq (b, b^\circ\varphi(L) \vee [y])h. \end{aligned}$$

Thus, since h is a bijection, h is an isomorphism. \square

In a subsequent paper, we shall consider homomorphisms, subalgebras and congruence pairs of \underline{K}_2 -algebras.

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