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Reticulation of a 0-distributive Lattice

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Abstract

A congruence relation θ on a 0-distributive lattice is defined such that the quotient lattice L/θ is a distributive lattice and the prime spectrum of L and of L/θ are homeomorphic. Also it is proved that the minimal prime spectrum (maximal spectrum) of L is homeomorphic with the minimal prime spectrum (maximal spectrum) of L/θ.

Key words: 0-distributive lattice, ideal, prime ideal, congruence relation, prime spectrum, minimal prime spectrum, maximal spectrum.

2010 Mathematics Subject Classification: 06D99

1 Introduction

For topological concepts which have now become commonplace the reader is referred to [7] and for lattice theoretic concepts the reader is referred to [6]. As a generalization of a distributive lattice and a pseudo-complemented lattice Varlet [11] has introduced the concept of a 0-distributive lattice. A 0-distributive lattice is a lattice L with 0 in which for all $a, b, c \in L$, $a \wedge b = 0 = a \wedge c$ implies $a \wedge (b \vee c) = 0$. A large part of the theory of filters in distributive lattices can be extended to 0-distributive lattices as pointed out in [12], [9] and [3]. A detailed study of the space of prime filters of a 0-distributive lattice is carried out in [3]. The set $\wp(L)$ of all prime filters of a bounded 0-distributive lattice L together with the hull kernel topology, i.e. the topology for which $\{V(x) \mid x \in L\}$ is a base, where $V(x) = \{P \in \wp(L) \mid x \notin P\}$, is called the prime spectrum of L.

The reticulation of an algebra was first defined by Simmons [10] for commutative rings and then for MV-algebras by Belluce [1]. Later it was extended to non-commutative rings [2], BL-algebras [8], residuated lattices [5] and Heyting algebras [4]. In each of the papers cited above except [5], although it is not explicitly defined this way, the reticulation of an algebra A is a pair $(L(A), \lambda)$

consisting of a bounded distributive lattice $L(A)$ and a surjection $\lambda\colon A \to L(A)$ such that the function given by the inverse image of λ induces a homeomorphism between the prime spectrum of $L(A)$ and that of A. This construction allows many properties to be transferred between $L(A)$ and A.

In this paper we intend to furnish the reticulation for a 0-distributive lattice, following the method of construction used for Heyting algebras by Dan [4].

2 Reticulation of a 0-distributive lattice

For any bounded lattice L, $\mathcal{F}(L)$ denotes the set of all filters of L. Obviously, $\langle \mathcal{F}(L), \wedge, \vee \rangle$ is a complete bounded lattice where $F \wedge J = F \cap J$ and $F \vee J = [F \cup J)$. Now we state lemmas which we need for the development of this paper.

Lemma 2.1 *Let L_1 and L_2 denote bounded lattices, $f\colon L_1 \to L_2$ be an onto $\{0,1\}$-homomorphism. Let F and J be any two filters of L_1. Then we have*

1. *$f(F)$ is a filter in L_2;*
2. *$F \subseteq J$ implies $f(F) \subseteq f(J)$;*
3. *$f(F \wedge J) = f(F) \wedge f(J)$;*
4. *$f(F \vee J) = f(F) \vee f(J)$;*
5. *$f([x)) = [f(x))$ for any $x \in L_1$.*

Lemma 2.2 *Let L_1 and L_2 denote bounded lattices, $f\colon L_1 \to L_2$ be an onto $\{0,1\}$-homomorphism. Let $\{F_\alpha \mid \alpha \in \Delta\}$ be a family of filters in L_1. Then we have $f\left(\bigvee_{\alpha\in\Delta} F_\alpha\right) = \bigvee_{\alpha\in\Delta} f(F_\alpha)$.*

By combining the results of Lemma 2.1 and Lemma 2.2 we have the following result

Remark 2.3 Let L_1 and L_2 denote bounded lattices. Let $f\colon L_1 \to L_2$ be an onto $\{0,1\}$-homomorphism. Let $\mathcal{F}(L_1)$ and $\mathcal{F}(L_2)$ denote the lattice of filters of L_1 and L_2 respectively. Then f induces a lattice homomorphism $\psi\colon \mathcal{F}(L_1) \to \mathcal{F}(L_2)$ defined by $\psi(F) = f(F)$ which preserves arbitrary joins and the property of being principal filter.

For a $\{0,1\}$-lattice homomorphism $f\colon L_1 \to L_2$, for $S \subseteq L_2$ we define $f^{-1}(S) = \{x \in L_1 \mid f(x) \in S\}$. Then we have

Lemma 2.4 *Let L_1 and L_2 denote bounded lattices, $f\colon L_1 \to L_2$ be an onto $\{0,1\}$-homomorphism. For any filters S and T of L_2, the following statements hold.*

1. *$f^{-1}(S)$ is a filter in L_1;*
2. *If S is a prime filter of L_2, then $f^{-1}(S)$ is a prime filter of L_1;*
3. *$S \subseteq T$ implies $f^{-1}(S) \subseteq f^{-1}(T)$.*

Note that under an onto $\{0,1\}$-homomorphism $f\colon L_1 \to L_2$, the image of a prime filter of L_1 need not be a prime filter of L_2. For this consider the following example. Let $L_1 = \{0, a, b, 1\}$ and $L_2 = \{0', 1'\}$ be two lattices whose Hasse diagrams are as shown in the Figure 1. Define the mapping $f\colon L_1 \to L_2$ by $f(b) = f(0) = 0'$ and $f(a) = f(1) = 1'$.

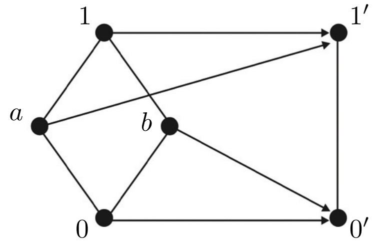

Fig. 1

Here, $P = \{1, b\}$ is a prime filter of L_1, but $f(P) = L_2$ is not prime.

Now onwards L will denote a bounded 0-distributive lattice unless otherwise stated. $\wp(L)$ denotes the set of all prime filters of L. For each $x \in L$ define $V(x) = \{P \in \wp(L) \mid x \notin P\}$. Define a relation θ on L by

$$x \equiv y(\theta) \quad \text{if and only if} \quad V(x) = V(y),$$

i.e., $x \equiv y(\theta)$ iff $x \notin P \Leftrightarrow y \notin P$ for $P \in \wp(L)$, or $x \equiv y(\theta)$ iff $x \in P \Leftrightarrow y \in P$ for $P \in \wp(L)$. It is easy to verify that this relation θ is a congruence relation (see [6]). Further we have

Theorem 2.5 *The quotient lattice L/θ is a distributive lattice.*

Proof For any $a, b, c \in L$, to prove that

$$[a]^\theta \wedge \left([b]^\theta \vee [c]^\theta\right) = \left([a]^\theta \wedge [b]^\theta\right) \vee \left([a]^\theta \wedge [c]^\theta\right),$$

let $x \in [a \wedge (b \vee c)]^\theta$. By definition of θ,

$$\begin{aligned}
&x \in P \Leftrightarrow a \wedge (b \vee c) \in P && \text{for } P \in \wp(L)\\
&x \in P \Leftrightarrow a \in P \text{ and } b \vee c \in P && \text{for } P \in \wp(L)\\
&x \in P \Leftrightarrow a \wedge b \in P \text{ or } a \wedge c \in P && \text{for } P \in \wp(L)\\
&x \in P \Leftrightarrow (a \wedge b) \vee (a \wedge c) \in P && \text{for } P \in \wp(L)
\end{aligned}$$

This shows that $x \in [(a \wedge b) \vee (a \wedge c)]^\theta$. Hence

$$[a \wedge (b \vee c)]^\theta \subseteq [(a \wedge b) \vee (a \wedge c)]^\theta .$$

Always, $[(a \wedge b) \vee (a \wedge c)]^\theta \subseteq [a \wedge (b \vee c)]^\theta$. Hence

$$[a \wedge (b \vee c)]^\theta = [(a \wedge b) \vee (a \wedge c)]^\theta .$$

Therefore L/θ is a distributive lattice. □

Let $\lambda\colon L \to L/\theta$ be the canonical mapping defined by $\lambda(x) = [x]^\theta$. Then λ is an onto $\{0,1\}$-homomorphism. For any filter F of L, $\lambda(F) = F^*$ is a filter of L/θ (see Lemma 2.1(1)). Further

$$F^* = \{[x]^\theta \in L/\theta \mid [x]^\theta = [y]^\theta \text{ for some } y \in F\}.$$

Similarly for any filter K of L/θ, $\lambda^{-1}(K) = K_*$ is a filter of L (see Lemma 2.4(1)) and $K_* = \bigcup_{[x]^\theta \in K} [x]^\theta$.

If L is a bounded 0-distributive lattice, then we have

Theorem 2.6 *The mapping $\psi\colon \mathcal{F}(L) \to \mathcal{F}(L/\theta)$ induced by λ and defined by $\psi(F) = F^* = \lambda(F)$ is an onto lattice homomorphism which preserves*

1. *arbitrary joins,*
2. *being a principal filter,*
3. *being a prime filter.*

Proof The proofs of the statements (1) and (2) follow from the Remark 2.3. We now prove the statement (3) only. Let P be a prime filter on L. To prove that $\lambda(P) = P^*$ is a prime filter of L/θ, let $[x]^\theta \vee [y]^\theta \in P^*$ for some $x, y \in L$. Therefore $[x \vee y]^\theta = [t]^\theta$ for some $t \in P$. Then $x \vee y \equiv t(\theta)$ for some $t \in P$. By the definition of θ, $x \vee y \in P$. As P is a prime filter, $x \in P$ or $y \in P$. But then $[x]^\theta \in P^*$ or $[y]^\theta \in P^*$. This shows that $P^* = \lambda(P)$ is a prime filter. □

Remark 2.7 For any filter K of L/θ, $(K_*)^* = K$, but for a filter F of L, $(F^*)_* \neq F$ in general.

For this consider the following example. Let $L = \{0, a, b, c, d, e, 1\}$ be the lattice whose Hasse diagram is as given in Figure 2.

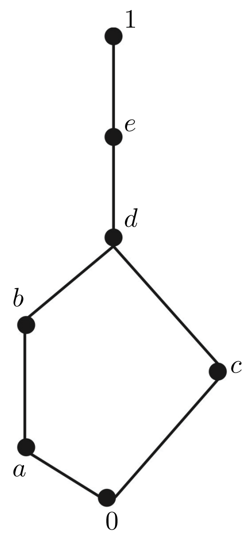

Fig. 2

Then

$$\mathcal{F}(L) = \{\{1\}, \{1, e\}, \{1, e, d\}, \{1, e, d, b\}, \{1, e, d, b, a\}, \{1, e, d, c\}, L\}$$

and

$$\wp(L) = \{\{1\}, \{1, e\}, \{1, e, d, b, a\}, \{1, e, d, c\}\}.$$

Define θ on L by

$$x \equiv y(\theta) \Leftrightarrow V(x) = V(y).$$

Since

$$\begin{aligned} V(1) &= \emptyset, \\ V(e) &= \{\{1\}\}, \\ V(d) &= \{\{1\}, \{1, e\}\}, \\ V(b) &= \{\{1\}, \{1, e\}, \{1, e, d, c\}\}, \\ V(a) &= \{\{1\}, \{1, e\}, \{1, e, d, c\}\}, \\ V(c) &= \{\{1\}, \{1, e\}, \{1, e, a, b, d\}\}, \\ V(0) &= \wp(L), \end{aligned}$$

then $1 \equiv 1(\theta)$, $e \equiv e(\theta)$, $c \equiv c(\theta)$, $d \equiv d(\theta)$, $b \equiv a(\theta)$ and $0 \equiv 0(\theta)$. Consider the filter $F = \{1, e, b, d\}$. Then $F^* = \left\{[1]^\theta, [e]^\theta, [b]^\theta, [d]^\theta\right\}$ and $(F^*)_* = \{1, e, b, a, d\}$. This shows that $(F^*)_* \neq F$.

Though $(F^*)_* \neq F$ in general, under the condition of primeness on F we have

Theorem 2.8 *For any prime filter P of L, $(P^*)_* = P$.*

Proof Let P be a prime filter of L. We have: $x \in (P^*)_* \Rightarrow \lambda(x) \in P^* \Rightarrow [x]^\theta \in P^* \Rightarrow [x]^\theta = [y]^\theta$ for some $y \in P$, i.e. $x \equiv y(\theta)$ and $y \in P$. By the definition of θ, $x \equiv y(\theta)$ iff $(x \in P \Leftrightarrow y \in P$ for any $P \in \wp(L))$. Hence $x \in P$. This shows that $(P^*)_* \subseteq P$. Conversely, assume that $x \in P$. Then $\lambda(x) = [x]^\theta \in P^*$. As $x \in \lambda^{-1}([x]^\theta)$ we get $x \in (P^*)_*$. This shows that $P \subseteq (P^*)_*$. Hence $P = (P^*)_*$. □

From Theorem 2.8 we get

Corollary 2.9 *The following statements hold.*

1. *If Q is a minimal prime filter of L, then Q^* is a minimal prime filter of L/θ.*
2. *If T is a minimal prime filter of L/θ, then T_* is a minimal prime filter of L.*
3. *If M is a maximal filter of L, then M^* is a maximal filter of L/θ.*
4. *Let K be a maximal filter of L/θ. Then K_* is a maximal filter of L.*

Proof (1) Let Q be a minimal prime filter of L. Then Q^* is a prime filter of L/θ (by Theorem 2.6). Let T be a prime filter in L/θ contained in Q^*. Thus $T \subseteq Q^* \Rightarrow T_* \subseteq (Q^*)_*$ (by Lemma 2.4(3)) $\Rightarrow T_* \subseteq Q$ (by Theorem 2.8). Now, T_* is a prime filter of L (by Lemma 2.4(2)) and Q is a minimal prime filter of L, whence $T_* = Q$, and so $T = Q^*$. This shows that Q^* is a minimal prime filter of L/θ.

(2) Let T be a minimal prime filter of L/θ. Then T_* is a prime filter of L (by Lemma 2.4(2)). Let P be a prime filter of L contained in T_*. Then $P \subseteq T_* \Rightarrow P^* \subseteq (T_*)^*$ (by Lemma 2.1(2)) $\Rightarrow P^* \subseteq T$ (by Remark 2.7). As T is a minimal prime filter of L/θ and P^* is a prime filter of L/θ (by Theorem 2.6(3)) we get $P^* = T$. Hence $P = T_*$ (by Theorem 2.8). This proves that T_* is a minimal prime filter of L.

(3) Let M be a maximal filter of L. First we prove that M is prime. If M is not a prime filter, then there exist $a, b \in L$ such that $a \vee b \in M$ with $a \notin M$ and $b \notin M$. As M is maximal, there exist $c, d \in M$ such that $a \wedge c = 0$ and $b \wedge d = 0$. But then $(a \vee b) \wedge (c \wedge d) = 0$ (L being a 0-distributive lattice) will imply $0 \in M$; which is absurd. Hence M is a prime filter.

By Lemma 2.1(1), M^* is a filter of L/θ. Let there exist a proper filter, say T, in L/θ containing M^*. But then $M = (M^*)_* \subseteq T_*$ (by Lemma 2.4(3) and Theorem 2.8). As T_* is a filter of L (see Lemma 2.4(1)) and M is a maximal filter of L, we get $M = T_*$. Hence $M^* = (T_*)^* = T$ (by Remark 2.7). This shows that M^* is a maximal filter of L/θ.

(4) Let K be a maximal filter of L/θ. As L/θ is a distributive lattice (see Theorem 2.5), K is a prime filter of L/θ. Hence K_* is a proper filter of L. As $0 \in L$, K_* is contained in some maximal filter of L, say M. Then L being a 0-distributive lattice, M is a prime filter of L (see the proof of (3)) and hence $(M^*)_* = M$ (by Theorem 2.8). Further, $K_* \subseteq M \Rightarrow (K_*)^* \subseteq M^*$ (see Lemma 2.1(2)) $\Rightarrow K \subseteq M^*$ (see Remark 2.7). As M^* is a filter of L/θ (Lemma 2.1(1)) by maximality of K, we get $K = M^*$. Hence $K_* = (M^*)_* = M$, which shows that K_* is a maximal filter of L. □

Remark 2.10 Note that Theorem 2.8 establishes a bijection between $\wp(L)$ and $\wp(L/\theta)$ which preserves the property of being a minimal prime filter and the property of being a maximal filter.

Let $\wp(L)$ and $\wp(L/\theta)$ denote the set of all prime filters of L and of L/θ, respectively. Equip $\wp(L)$ and $\wp(L/\theta)$ with the Stone topologies τ and τ', respectively. The base for the open sets for τ is the family $\mathfrak{B} = \{V(x) \mid x \in L\}$ where $V(x) = \{P \in \wp(L) \mid x \notin P\}$. The base for the open sets for τ' is the family $\mathfrak{B}' = \{X([x]^\theta) \mid x \in L\}$ where $X([x]^\theta) = \{P \in \wp(L/\theta) \mid [x]^\theta \in P\}$. For any family $\mathcal{K} \subseteq \wp(L)$ we write $\mathcal{K}^* = \{P^* \mid P \in \mathcal{K}\}$ with these notations we have

Theorem 2.11 *The following statements are true for any $x, y, x_\alpha \in L$.*

1. $V(x)^* = X([x]^\theta)$.
2. $(V(x) \cap V(y))^* = V(x)^* \cap V(y)^*$.
3. $\left(\bigcup_{\alpha \in \Delta} V(x_\alpha)\right)^* = \bigcup_{\alpha \in \Delta} V(x_\alpha)^* = \bigcup_{\alpha \in \Delta} X([x_\alpha]^\theta)$.

Proof (1) We have for any $x \in L$, $V(x)^* = \{P^* \mid P \in V(x)\} = \{P^* \mid x \notin P\}$. Let $P \in \wp(L)$ such that $P^* \in V(x)^*$. Then $x \notin P$. If $[x]^\theta \in P^*$, then there exists $y \in P$ such that $[x]^\theta = [y]^\theta$ (since $P^* = \lambda(P)$). Hence $x \equiv y(\theta)$. As $y \in P$ and $P \in \wp(L)$ we must get $x \in P$; a contradiction. Hence $[x]^\theta \notin P^*$, i.e. $P^* \in X([x]^\theta)$ (P is a prime filter in $L \Rightarrow P^*$ is a prime filter in L/θ by Theorem 2.6(3)). Hence $V(x)^* \subseteq X([x]^\theta)$.

Conversely, let $T \in X([x]^\theta)$, i.e. $T \in \wp(L/\theta)$ such that $[x]^\theta \notin T$. But then $x \notin T_*$ $(= \lambda^{-1}(T))$. Furthermore, $T \in \wp(L/\theta) \Rightarrow T_* \in \wp(L)$ (see Lemma 2.4(2)). As $T_* \in \wp(L)$ and $x \notin T_*$, we get $T_* \in V(x)$. Hence $(T_*)^* \in V(x)^*$. Therefore $T \in V(x)^*$ (see Remark 2.7). This shows that $X([x]^\theta) \subseteq V(x)^*$. Combining both inclusions, we get $X([x]^\theta) = V(x)^*$.

(2) For any $x, y \in L$ we have

$$\begin{aligned}(V(x) \cap V(y))^* &= V(x \wedge y)^* = X([x \wedge y]^\theta) = X([x]^\theta \wedge [y]^\theta) \\ &= X([x]^\theta) \cap X([y]^\theta) = V(x)^* \cap V(y)^*.\end{aligned}$$

(3) Let $P^* \in \left(\bigcup_{\alpha \in \Delta} V(x_\alpha)\right)^*$. This implies $P \in \bigcup_{\alpha \in \Delta} V(x_\alpha)$. Hence $P \in V(x_{\alpha_0})$ for some $\alpha_0 \in \Delta$. But then $P^* \in V(x_{\alpha_0})^*$ implies $P^* \in \bigcup_{\alpha \in \Delta} V(x_\alpha)^*$. Thus $\left(\bigcup_{\alpha \in \Delta} V(x_\alpha)\right)^* \subseteq \bigcup_{\alpha \in \Delta} V(x_\alpha)^*$. Conversely, let $P^* \in \bigcup_{\alpha \in \Delta} V(x_\alpha)^*$. Then $P^* \in V(x_{\alpha_0})^*$ for some $\alpha_0 \in \Delta$. But then $P \in V(x_{\alpha_0})$ implies $P \in \bigcup_{\alpha \in \Delta} V(x_\alpha)$, and consequently, $P^* \in \left(\bigcup_{\alpha \in \Delta} V(x_\alpha)\right)^*$. Thus we get $\bigcup_{\alpha \in \Delta} V(x_\alpha)^* \subseteq \left(\bigcup_{\alpha \in \Delta} V(x_\alpha)\right)^*$. Combining both inclusions, we get

$$\bigcup_{\alpha \in \Delta} V(x_\alpha)^* = \left(\bigcup_{\alpha \in \Delta} V(x_\alpha)\right)^*.$$

□

Now we prove that the prime spectrum $\wp(L)$ of L and the prime spectrum $\wp(L/\theta)$ of L/θ are homeomorphic.

Theorem 2.12 *Define the mapping* $g\colon \wp(L/\theta) \to \wp(L)$ *by* $g(P) = P_* = \lambda^{-1}(P)$ *for every* $P \in \wp(L/\theta)$. *Then* g *is a homeomorphism.*

Proof Obviously, g is a well defined map (see Lemma 2.4(2)). Let $P \in \wp(L)$. Then $P^* \in \wp(L/\theta)$ (see Theorem 2.6(3)) and $g(P^*) = \lambda^{-1}(P^*) = (P^*)_* = P$ (see Theorem 2.8). This shows that g is onto. If $g(P_1) = g(P_2)$ for $P_1, P_2 \in \wp(L/\theta)$ then $P_1 = P_2$ (by Remark 2.7). Therefore g is one-one.

We know that $V(x)$ is a basic open set in the space $\wp(L)$ for any $x \in L$. Further, by Theorem 2.11(1) we get $g^{-1}(V(x)) = X([x]^\theta)$, which is basic open set in the space $\wp(L/\theta)$, and hence g is continuous.

We know $X([x]^\theta)$ is basic open set in the space $\wp(L/\theta)$ for any $x \in L$ and

$$\begin{aligned} g\left(X([x]^\theta)\right) &= \left\{g(P) \in \wp(L) \mid P \in \wp(L/\theta) \text{ and } [x]^\theta \notin P\right\} \\ &= \{P_* \in \wp(L) \mid \lambda(x) \notin P, P \in \wp(L/\theta)\} \\ &= \{P_* \in \wp(L) \mid x \notin \lambda^{-1}(P) = P_*\} \\ &= \{Q \in \wp(L) \mid x \notin Q\} = V(x),\end{aligned}$$

which is a basic open set in the space $\wp(L)$ for any $x \in L$. Therefore g is an open map. As g is a bijective, continuous and open map, g is a homeomorphism. □

Combining all the above results we have

Theorem 2.13 *The pair $(L/\theta, \lambda)$ forms a reticulation for a bounded 0-distributive lattice L.*

By Corollary 2.9, we get

Corollary 2.14 *The two spaces $\mathfrak{M}(L)$ and $\mathfrak{M}(L/\theta)$ (with restricted Stone topologies) are homeomorphic, where $\mathfrak{M}(L)$ and $\mathfrak{M}(L/\theta)$ denote the spaces of minimal prime filters of L and L/θ, respectively.*

Corollary 2.15 *The two spaces $\mathfrak{A}(L)$ and $\mathfrak{A}(L/\theta)$ (with restricted Stone topologies) are homeomorphic, where $\mathfrak{A}(L)$ and $\mathfrak{A}(L/\theta)$ denote the spaces of maximal filters of L and L/θ, respectively.*

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