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ON THE STABILITY OF THE ALE SPACE-TIME DISCONTINUOUS GALERKIN METHOD FOR NONLINEAR CONVECTION-DIFFUSION PROBLEMS IN TIME-DEPENDENT DOMAINS

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Abstract. The paper is concerned with the analysis of the space-time discontinuous Galerkin method (STDGM) applied to the numerical solution of the nonstationary nonlinear convection-diffusion initial-boundary value problem in a time-dependent domain formulated with the aid of the arbitrary Lagrangian-Eulerian (ALE) method. In the formulation of the numerical scheme we use the nonsymmetric, symmetric and incomplete versions of the space discretization of diffusion terms and interior and boundary penalty. The nonlinear convection terms are discretized with the aid of a numerical flux. The space discretization uses piecewise polynomial approximations of degree not greater than p with an integer $p \geqslant 1$. In the theoretical analysis, the piecewise linear time discretization is used. The main attention is paid to the investigation of unconditional stability of the method.

Keywords: nonstationary nonlinear convection-diffusion equations; time-dependent domain; ALE method; space-time discontinuous Galerkin method; unconditional stability

MSC 2010: 65M60, 65M99

1. Introduction

Most of works on the theory and numerical solution of nonstationary partial differential equations are considered and analyzed in space domains independent of time. However, problems described by partial differential equations in deformable domains Ω_t, which change their shape in dependence on time $t \in [0, T]$, play an important role in various fields of science and technology. Particularly, we can mention problems of fluid-structure interaction, when the boundary of the domain occupied by the moving

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fluid is deformed in dependence on time according to the deformation of an elastic body adjacent to the fluid. There are several techniques how to solve numerically initial-boundary value problems in time-dependent domains. We can mention, e.g., the immersed boundary method or fictitious domain method ([7], [33]). Another, rather popular technique is the arbitrary Lagrangian-Eulerian (ALE) method ([18]), which will be applied in this paper to the numerical solution of nonstationary nonlinear convection-diffusion problems in a time-dependent domain. In several works ([11], [12], [24], [25], [27], [30]) we used the ALE method with success for numerical solving compressible Navier-Stokes equations in the framework of fluid-structure interaction problems. The space discretization was carried out by the discontinuous Galerkin method (DGM). For the time discretization we used either the backward difference formula (BDF) method or the DGM in time. In the latter case, we get the space-time discontinuous Galerkin method (STDGM).

There is a number of works devoted to the theory and applications of the DGM. Let us mention, e.g., [2], [3], [5], [6], [9], [14], [15], [16], [17], [22], [23], [31], [32], [34]. The numerical simulation of strongly nonstationary transient problems requires the application of numerical schemes of high order of accuracy both in space and in time. It appears suitable to use the discontinuous Galerkin discretization with respect to space as well as time for the construction of numerical schemes with high accuracy in space and time for the solution of nonlinear nonstationary problems.

The discontinuous Galerkin time discretization was introduced and analyzed, e.g., in [19] for the solution of ordinary differential equations. In [1], [13], [20], [21], [35] and [36] the solution of parabolic problems is carried out with the aid of conforming finite elements in space combined with the DG time discretization. See also the monograph [37]. In [23], the STDGM was analyzed for a linear nonstationary convection-diffusion-reaction problem. The paper [26] is devoted to the theory of error estimates for the STDGM applied to a nonstationary convection-diffusion problem with a nonlinear convection and linear diffusion. In paper [10], the theory of the STDGM was developed for the case with nonlinear convection as well as diffusion. The paper [4] is a continuation of the works [26] and [10] devoted to proving unconditional stability of the STDGM. In all the above mentioned theoretical papers, the space domain is independent of time.

There are several papers devoted to the analysis of linear convection-diffusion problems in time-dependent domains, formulated with the aid of the ALE method. We can mention [28], [29] and [8]. The last paper is concerned with the stability analysis of the time DGM without space discretization.

The presented paper represents the generalization of results from [4] to the STDGM for the numerical solution of a nonstationary nonlinear convection-diffusion problem in a time-dependent domain, formulated with the aid of the ALE method.

In Section 2 we formulate the continuous problem. Section 3 is devoted to the description of the space semidiscretization. Section 4 is concerned with the complete space-time DG discretization. The main results are contained in Section 5, where the stability of the STDGM is proved.

2. Formulation of the continuous problem

In what follows, we shall use the standard notation $L^2(\omega)$ for the Lebesgue space, $H^k(\omega), W^{k,p}(\omega)$ for the Hilbert and Sobolev spaces over a bounded domain $\omega \subset \mathbb{R}^d$, and $C^1([0,T];\ W^{1,\infty}(\Omega_t))$ for the Bochner space of continuously differentiable functions in $[0,T]$ with values in $W^{1,\infty}(\Omega_t)$. We shall be concerned with an initial-boundary value nonlinear convection-diffusion problem in a time-dependent bounded polyhedral domain $\Omega_t \subset \mathbb{R}^2$, where $t \in [0,T]$, $T > 0$: Find a function $u = u(x,t)$ with $x \in \Omega_t$, $t \in (0,T)$ such that

$$\frac{\partial u}{\partial t} + \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} - \operatorname{div}(\beta(u)\nabla u) = g \quad \text{in } \Omega_t,\ t \in (0,T), \tag{2.1}$$

$$u = u_D \quad \text{on } \partial\Omega_t,\ t \in (0,T), \tag{2.2}$$

$$u(x,0) = u^0(x), \quad x \in \Omega_0. \tag{2.3}$$

We assume that $f_s \in C^1(\mathbb{R})$, $f_s(0) = 0$,

$$|f_s'| \leqslant L_f, \quad s = 1,\ldots,d, \tag{2.4}$$

and the function β is bounded and Lipschitz-continuous:

$$\beta\colon\ \mathbb{R} \to [\beta_0, \beta_1], \quad 0 < \beta_0 < \beta_1 < \infty, \tag{2.5}$$

$$|\beta(u_1) - \beta(u_2)| \leqslant L_\beta |u_1 - u_2| \quad \forall\, u_1, u_2 \in \mathbb{R}. \tag{2.6}$$

Problem (2.1)–(2.3) will be reformulated with the aid of the arbitrary Lagrangian-Eulerian (ALE) method. It is based on a regular one-to-one ALE mapping of the reference configuration Ω_0 onto the current configuration Ω_t:

$$\mathcal{A}_t\colon\ \overline{\Omega}_0 \to \overline{\Omega}_t, \quad X \in \overline{\Omega}_0 \to x = x(X,t) = \mathcal{A}_t(X) \in \overline{\Omega}_t, \quad t \in [0,T].$$

We assume that $\mathcal{A} \in C^1([0,T]; W^{1,\infty}(\Omega_t))$. We define the ALE velocity by

$$\tilde{\boldsymbol{z}}(X,t) = \frac{\partial}{\partial t}\mathcal{A}_t(X), \quad \boldsymbol{z}(x,t) = \tilde{\boldsymbol{z}}(\mathcal{A}_t^{-1}(x),t), \quad t \in [0,T],\ X \in \Omega_0,\ x \in \Omega_t.$$

Let

$$|\boldsymbol{z}(x,t)|,\ |\operatorname{div}\boldsymbol{z}(x,t)| \leqslant c_z \quad \text{for } x\in\Omega_t,\ t\in(0,T). \tag{2.7}$$

Further, we define the ALE derivative $D_t f = Df/Dt$ of a function $f = f(x,t)$ for $x\in\Omega_t$ and $t\in[0,T]$ as

$$D_t f(x,t) = \frac{D}{Dt} f(x,t) = \frac{\partial \tilde f}{\partial t}(X,t),$$

where $\tilde f(X,t) = f(\mathcal{A}_t(X),t),\ X\in\Omega_0$, and $x = \mathcal{A}_t(X)\in\Omega_t$. The use of the chain rule yields the relation

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \boldsymbol{z}\cdot\nabla f, \tag{2.8}$$

which allows us to reformulate problem (2.1)–(2.3) in the ALE form: Find $u = u(x,t)$ with $x\in\Omega_t,\ t\in(0,T)$, such that

$$D_t u + \sum_{s=1}^{d}\frac{\partial f_s(u)}{\partial x_s} - \boldsymbol{z}\cdot\nabla u - \operatorname{div}(\beta(u)\nabla u) = g \quad \text{in } \Omega_t,\ t\in(0,T), \tag{2.9}$$

$$u = u_D \quad \text{on } \partial\Omega_t, \tag{2.10}$$

$$u(x,0) = u^0(x), \quad x\in\Omega_0. \tag{2.11}$$

3. Space semidiscretization

For any $t\in[0,T]$ we denote by $\mathcal{T}_{h,t}$ a partition of the closure $\overline{\Omega}_t$ into a finite number of closed simplexes with disjoint interiors. Over a triangulation $\mathcal{T}_{h,t}$, for each positive integer k we define the broken Sobolev space

$$H^k(\Omega_t,\mathcal{T}_{h,t}) = \{v; v|_K\in H^k(K)\ \forall\, K\in\mathcal{T}_{h,t}\},$$

equipped with the seminorm

$$|v|_{H^k(\Omega_t,\mathcal{T}_{h,t})} = \Bigl(\sum_{K\in\mathcal{T}_{h,t}} |v|^2_{H^k(K)}\Bigr)^{1/2},$$

where $|\cdot|_{H^k(K)}$ denotes the seminorm in the space $H^k(K)$.

By $\mathcal{F}_{h,t}$ we denote the system of all faces of all elements $K\in\mathcal{T}_{h,t}$. It consists of the set of all inner faces $\mathcal{F}^I_{h,t}$ and the set of all boundary faces $\mathcal{F}^B_{h,t}$: $\mathcal{F}_{h,t} = \mathcal{F}^I_{h,t}\cup\mathcal{F}^B_{h,t}$.

Each $\Gamma \in \mathcal{F}_{h,t}$ will be associated with a unit normal vector $\mathbf{n}_\Gamma$. By $K_\Gamma^{(L)}$ and $K_\Gamma^{(R)} \in \mathcal{T}_{h,t}$ we denote the elements adjacent to the face $\Gamma \in \mathcal{F}_{h,t}$. Moreover, for $\Gamma \in \mathcal{F}_{h,t}^B$ the element adjacent to this face will be denoted by $K_\Gamma^{(L)}$. We shall use the convention that $\mathbf{n}_\Gamma$ is the outer normal to $\partial K_\Gamma^{(L)}$.

If $v \in H^1(\Omega_t, \mathcal{T}_{h,t})$ and $\Gamma \in \mathcal{F}_{h,t}$, then $v|_\Gamma^{(L)}, v|_\Gamma^{(R)}$ will denote the traces of v on Γ from the side of elements $K_\Gamma^{(L)}, K_\Gamma^{(R)}$ adjacent to Γ. For $\Gamma \in \mathcal{F}_{h,t}^I$ we set

$$\langle v \rangle_\Gamma = \frac{1}{2}(v|_\Gamma^{(L)} + v|_\Gamma^{(R)}), \quad [v]_\Gamma = v|_\Gamma^{(L)} - v|_\Gamma^{(R)}, \tag{3.1}$$

$$h(\Gamma) = \frac{h_{K_\Gamma^{(L)}} + h_{K_\Gamma^{(R)}}}{2} \quad \text{for } \Gamma \in \mathcal{F}_{h,t}^I, \quad h(\Gamma) = h_{K_\Gamma^{(L)}} \quad \text{for } \Gamma \in \mathcal{F}_{h,t}^B. \tag{3.2}$$

Now we introduce the space semidiscretization of problem (2.9)–(2.11). We assume that u is a sufficiently smooth solution of our problem. If we choose an arbitrary but fixed $t \in [0, T]$, multiply equation (2.9) by a test function $\varphi \in H^2(\Omega_t, \mathcal{T}_{h,t})$, integrate over any element K and finally sum over all elements $K \in \mathcal{T}_{h,t}$, then we get

$$\begin{aligned} &\sum_{K \in \mathcal{T}_{h,t}} \int_K D_t u \varphi \,\mathrm{d}x + \sum_{K \in \mathcal{T}_{h,t}} \int_K \sum_{s=1}^d \frac{\partial f_s(u)}{\partial x_s} \varphi \,\mathrm{d}x \\ &\quad - \sum_{K \in \mathcal{T}_{h,t}} \int_K \sum_{s=1}^d z_s \frac{\partial u}{\partial x_s} \varphi \,\mathrm{d}x - \sum_{K \in \mathcal{T}_{h,t}} \int_K \operatorname{div}(\beta(u) \nabla u) \varphi \,\mathrm{d}x = \sum_{K \in \mathcal{T}_{h,t}} \int_K g \varphi \,\mathrm{d}x. \end{aligned} \tag{3.3}$$

The individual terms in the above identity will be approximated with the aid of the following forms. If $u, \varphi \in H^2(\Omega_t, \mathcal{T}_{h,t})$, $\theta \in \mathbb{R}$ and $c_W > 0$, we set

$$\begin{aligned} a_h(u, \varphi, t) = &\sum_{K \in \mathcal{T}_{h,t}} \int_K \beta(u) \nabla u \cdot \nabla \varphi \,\mathrm{d}x \\ &- \sum_{\Gamma \in \mathcal{F}_{h,t}^I} \int_\Gamma (\langle \beta(u) \nabla u \rangle \cdot \mathbf{n}_\Gamma [\varphi] + \theta \langle \beta(u) \nabla \varphi \rangle \cdot \mathbf{n}_\Gamma [u]) \,\mathrm{d}S \\ &- \sum_{\Gamma \in \mathcal{F}_{h,t}^B} \int_\Gamma (\beta(u) \nabla u \cdot \mathbf{n}_\Gamma \varphi \\ &\qquad + \theta \beta(u) \nabla \varphi \cdot \mathbf{n}_\Gamma u - \theta \beta(u) \nabla \varphi \cdot \mathbf{n}_\Gamma u_D) \,\mathrm{d}S, \end{aligned} \tag{3.4}$$

$$J_h(u, \varphi, t) = c_W \sum_{\Gamma \in \mathcal{F}_{h,t}^I} h(\Gamma)^{-1} \int_\Gamma [u][\varphi] \,\mathrm{d}S + c_W \sum_{\Gamma \in \mathcal{F}_{h,t}^B} h(\Gamma)^{-1} \int_\Gamma u \varphi \,\mathrm{d}S, \tag{3.5}$$

$$J_h^B(u, \varphi, t) = c_W \sum_{\Gamma \in \mathcal{F}_{h,t}^B} h(\Gamma)^{-1} \int_\Gamma u \varphi \,\mathrm{d}S, \tag{3.6}$$

$$(3.7)\quad A_h(u,\varphi,t) = a_{h,t}(u,\varphi,t) + \beta_0 J_{h,t}(u,\varphi,t),$$

$$\begin{aligned}(3.8)\quad b_h(u,\varphi,t) = &-\sum_{K\in\mathcal{T}_{h,t}} \int_K \sum_{s=1}^{d} f_s(u)\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x \\ &+ \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma H(u_\Gamma^{(L)}, u_\Gamma^{(R)}, \mathbf{n}_\Gamma)[\varphi]\,\mathrm{d}S \\ &+ \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma H(u_\Gamma^{(L)}, u_\Gamma^{(L)}, \mathbf{n}_\Gamma)\varphi\,\mathrm{d}S,\end{aligned}$$

$$(3.9)\quad d_h(u,\varphi,t) = -\sum_{K\in\mathcal{T}_{h,t}} \int_K \sum_{s=1}^{d} z_s\frac{\partial u}{\partial x_s}\varphi\,\mathrm{d}x = -\sum_{K\in\mathcal{T}_{h,t}} \int_K (\boldsymbol{z}\cdot\nabla u)\varphi\,\mathrm{d}x,$$

$$(3.10)\quad l_h(\varphi,t) = \sum_{K\in\mathcal{T}_{h,t}} \int_K g\varphi\,\mathrm{d}x + \beta_0 c_W \sum_{\Gamma\in\mathcal{F}^B_{h,t}} h(\Gamma)^{-1}\int_\Gamma u_D\varphi\,\mathrm{d}S.$$

Further, if $\omega\subset\mathbb{R}^2$ is a measurable set and $\varphi,\psi\in L^2(\omega)$, we shall denote

$$(3.11)\qquad (\varphi,\psi)_\omega = \int_\omega \varphi\psi\,\mathrm{d}x, \quad \|\varphi\|_\omega = \left(\int_\omega |\varphi|^2\,\mathrm{d}x\right)^{1/2}.$$

Let us note that in integrals over faces we omit the subscript Γ. We consider $\theta = 1$, $\theta = 0$ and $\theta = -1$ and get the symmetric (SIPG), incomplete (IIPG) and nonsymmetric (NIPG) variants of the approximation of the diffusion terms, respectively.

In (3.8), H is a numerical flux with the following properties:

(H1) $H(u,v,\mathbf{n})$ is defined in $\mathbb{R}^d\times B_1$, where $B_1 = \{\mathbf{n}\in\mathbb{R}^d;\ |\mathbf{n}| = 1\}$, and is Lipschitz-continuous with respect to u,v: there exists $L_H > 0$ such that $|H(u,v,\mathbf{n}) - H(u^*,v^*,\mathbf{n})| \leqslant L_H(|u-u^*| + |v-v^*|)$ for all $u,v,u^*,v^*\in\mathbb{R}$,

(H2) $H(u,v,\mathbf{n})$ is consistent: $H(u,v,\mathbf{n}) = \sum\limits_{s=1}^{d} f_s(u)n_s$, $u\in\mathbb{R}$, $\mathbf{n}\in B_1$,

(H3) $H(u,v,\mathbf{n})$ is conservative: $H(u,v,\mathbf{n}) = -H(v,u,-\mathbf{n})$, $u,v\in\mathbb{R}$, $\mathbf{n}\in B_1$.

4. Space-time discretization

In the time interval $[0,T]$ we construct a partition formed by time instants $0 = t_0 < t_1 < \ldots < t_M = T$ and set $I_m = (t_{m-1}, t_m)$, $\overline{I}_m = [t_{m-1}, t_m]$ and $\tau_m = t_m - t_{m-1}$ for $m = 1,\ldots,M$. Then we have

$$[0,T] = \bigcup_{m=1}^{M} \overline{I}_m \quad\text{and}\quad I_m\cap I_n = \emptyset \quad\text{for } m\neq n.$$

Further, we set $\tau = \max_{m=1,\ldots,M} \tau_m$. For a function φ defined in $\bigcup_{m=1}^{M} I_m$ we denote

$$\varphi_m^{\pm} = \varphi(t_m\pm) = \lim_{t\to t_m\pm} \varphi(t), \quad \{\varphi\}_m = \varphi(t_m+) - \varphi(t_m-).$$

Let $p, q \geqslant 1$ be integers. For any $t \in [0, T]$ we define the finite-dimensional space

$$S_{h,t}^p = \{v \in L^2(\Omega_t);\ v|_K \in P^p(K),\ K \in \mathcal{T}_{h,t},\ t \in [0,T]\}. \tag{4.1}$$

The approximate solution is sought in the space of piecewise polynomial functions in time and space

(4.2)

$$\begin{aligned} S_{h,\tau}^{p,q} = \{v \in L^2(Q_T);\ v = v(x,t),\ &\text{for each } X \in \Omega_0 \text{ and each } m = 1,\ldots,M \\ &\text{the function } t \in I_m \to v(\mathcal{A}_t(X),t) \text{ is a polynomial of degree } \leqslant q \text{ in } t, \\ &v(\cdot,t) \in S_{h,t}^p \text{ for all } t \in I_m\}, \end{aligned}$$

where $Q_T = \{(x,t);\ t \in (0,T),\ x \in \Omega_t\}$.

A function U is an approximate solution of problem (2.9)–(2.11), if $U \in S_{h,\tau}^{p,q}$ and

$$\begin{aligned} &\int_{I_m} \big((D_t U, \varphi)_{\Omega_t} + A_h(U,\varphi,t) + b_h(U,\varphi,t) + d_h(U,\varphi,t)\big)\,\mathrm{d}t \\ &\quad + (\{U\}_{m-1}, \varphi_{m-1}^{+})_{\Omega_{t_{m-1}}} = \int_{I_m} l_h(\varphi,t)\,\mathrm{d}t \quad \forall\, \varphi \in S_{h,\tau}^{p,q}, \quad m = 1,\ldots,M, \end{aligned} \tag{4.3}$$

$$U^0 \in S_{h,0}^p, \quad (U^0 - u^0, v_h) = 0 \quad \forall\, v_h \in S_{h,0}^p. \tag{4.4}$$

In what follows we are concerned with the case $q = 1$. This means that in the time discretization by the discontinuous Galerkin method we use piecewise linear approximations. We shall use properties (H1) and (H2) of the numerical flux H. (Assumption (H3) is important for proving the consistency of the method, but here it is not necessary.)

5. Analysis of the stability

In our further considerations for each $t \in [0,T]$ we introduce a system of triangulations $\{\mathcal{T}_{h,t}\}_{h\in(0,h_0)}$, where $h_0 > 0$. We assume that it is shape regular and locally quasiuniform. This means that there exist positive constants c_R and c_Q, independent of K, Γ, t and h such that for all $t \in [0,T]$

$$\frac{h_K}{\varrho_K} \leqslant c_R \quad \text{for all } K \in \mathcal{T}_{h,t}, \tag{5.1}$$

$$h_{K_\Gamma^{(L)}} \leqslant c_Q h_{K_\Gamma^{(R)}}, \quad h_{K_\Gamma^{(R)}} \leqslant c_Q h_{K_\Gamma^{(L)}} \quad \text{for all } \Gamma \in \mathcal{F}_{h,t}^I. \tag{5.2}$$

Under these assumptions, by [16] the multiplicative trace inequality and the inverse inequality hold: There exist constants $c_M, c_I > 0$ independent of v, h, t and K such that

$$\begin{gathered}\|v\|^2_{L^2(\partial K)} \leqslant c_M(\|v\|_{L^2(K)}|v|_{H^1(K)} + h_K^{-1}\|v\|^2_{L^2(K)}), \\ v \in H^1(K),\ K \in \mathcal{T}_{h,t},\ t \in [0,T],\ h \in (0,h_0),\end{gathered} \tag{5.3}$$

and

$$|v|_{H^1(K)} \leqslant c_I h_K^{-1}\|v\|_{L^2(K)},\ v \in P^p(K),\ K \in \mathcal{T}_{h,t},\ t \in [0,T],\ h \in (0,h_0). \tag{5.4}$$

Moreover, we assume that

$$\mathcal{T}_{h,t} = \{K_t = \mathcal{A}_t(K_0);\ K_0 \in \mathcal{T}_{h,0}\}. \tag{5.5}$$

This assumption is usually satisfied in practical computations, when the ALE mapping $\mathcal{A}_t$ is a continuous, piecewise affine mapping in $\overline{\Omega}_0$ for each $t \in [0,T]$.

In the space $H^1(\Omega, \mathcal{T}_{h,t})$ we define the norm

$$\|\varphi\|_{DG,t} = \biggl(\sum_{K\in\mathcal{T}_{h,t}} |\varphi|^2_{H^1(K)} + J_h(\varphi,\varphi,t)\biggr)^{1/2}. \tag{5.6}$$

Moreover, on $\partial\Omega$ we define the norm

$$\|u_D\|_{DGB,t} = \biggl(c_W \sum_{\Gamma\in\mathcal{F}^B_{h,t}} h^{-1}(\Gamma)\int_\Gamma |u_D|^2\,\mathrm{d}S\biggr)^{1/2} = (J_h^B(u_D,u_D,t))^{1/2}. \tag{5.7}$$

If we use $\varphi := U$ as a test function in (4.3), we get the basic identity

$$\begin{aligned}\int_{I_m} \bigl((D_tU,U)_{\Omega_t} + A_h(U,U,t) + b_h(U,U,t) + d_h(U,U,t)\bigr)\,\mathrm{d}t \\ + (\{U\}_{m-1}, U^+_{m-1})_{\Omega_{t_{m-1}}} = \int_{I_m} l_h(U,t)\,\mathrm{d}t.\end{aligned} \tag{5.8}$$

An important step is the proof of the coercivity of the diffusion and penalty terms.

Theorem 1. *Let*

$$c_W \geqslant \frac{\beta_1^2}{\beta_0^2} c_M(c_I+1) \quad \textit{for } \theta=-1 \text{ (NIPG)}, \tag{5.9}$$

$$c_W \geqslant \frac{\beta_1^2}{\beta_0^2} c_M(c_I+1)(c_Q+1) \quad \textit{for } \theta=0 \text{ (IIPG)}, \tag{5.10}$$

$$c_W \geqslant \frac{16\beta_1^2}{\beta_0^2} c_M(c_I+1)(c_Q+1) \quad \textit{for } \theta=1 \text{ (SIPG)}. \tag{5.11}$$

Then

$$\int_{I_m} \big(a_h(U,U,t)+\beta_0 J_h(U,U,t)\big)\,\mathrm{d}t \geqslant \frac{\beta_0}{2}\int_{I_m} \|U\|_{DG,t}^2\,\mathrm{d}t - \frac{\beta_0}{2}\int_{I_m} \|u_D\|_{DGB,t}^2\,\mathrm{d}t. \tag{5.12}$$

P r o o f. 1) Let $\theta=-1$. Using assumption (2.5) and the definition of the $\|\cdot\|_{DG,t}$-norm, we have

$$a_h(U,U,t)+\beta_0 J_h(U,U,t) \geqslant \beta_0\|U\|_{DG,t}^2 - \sum_{\Gamma\in\mathcal{F}_{h,t}^B}\int_\Gamma \beta(U)\nabla U\cdot\mathbf{n}_\Gamma u_D\,\mathrm{d}S. \tag{5.13}$$

Now we have to estimate the last term on the right-hand side of (5.13). Using the properties of the function β and Young's inequality, for each $k_1,\delta>0$ we get

$$\begin{aligned}\sum_{\Gamma\in\mathcal{F}_{h,t}^B}\int_\Gamma |\beta(U)\nabla U\cdot\mathbf{n}_\Gamma u_D|\,\mathrm{d}S &\leqslant \beta_1 \sum_{\Gamma\in\mathcal{F}_{h,t}^B}\int_\Gamma |\nabla U|\,|u_D|\,\mathrm{d}S\\ &\leqslant \frac{\beta_1 k_1}{2\delta}\sum_{\Gamma\in\mathcal{F}_{h,t}^B}\int_\Gamma h_{K_\Gamma^{(L)}}^{-1}|u_D|^2\,\mathrm{d}S + \frac{\beta_1\delta}{2k_1}\sum_{\Gamma\in\mathcal{F}_{h,t}^B}\int_\Gamma h_{K_\Gamma^{(L)}}|\nabla U|^2\,\mathrm{d}S.\end{aligned}$$

If we set $\delta := \beta_0/\beta_1$ and use the definition of the form J_h^B, we obtain

$$\begin{aligned}\sum_{\Gamma\in\mathcal{F}_{h,t}^B}\int_\Gamma |\beta(U)\nabla U\cdot\mathbf{n}_\Gamma u_D|\,\mathrm{d}S\\ \leqslant \frac{\beta_1^2 k_1}{2\beta_0 c_W} J_h^B(u_D,u_D,t) + \frac{\beta_0}{2k_1}\sum_{\Gamma\in\mathcal{F}_{h,t}^B}\int_{\partial K_\Gamma^{(L)}} h_{K_\Gamma^{(L)}}|\nabla U|^2\,\mathrm{d}S.\end{aligned}$$

Now, we express the first term on the right-hand side with the aid of the definition of the $\|\cdot\|_{DGB,t}$-norm and to the second term we apply the multiplicative trace inequality (5.3) and the inverse inequality (5.4). We get

$$\sum_{\Gamma\in\mathcal{F}_{h,t}^B}\int_\Gamma |\beta(U)\nabla U\cdot\mathbf{n}_\Gamma u_D|\,\mathrm{d}S \\ \leqslant \frac{\beta_1^2 k_1}{2\beta_0 c_W}\|u_D\|_{DGB,t}^2+\frac{\beta_0}{2k_1}c_M(c_I+1)\sum_{K\in\mathcal{T}_{h,t}}\|\nabla U\|_{L^2(K)}^2.$$

If we use the inequality $\sum_{K\in\mathcal{T}_{h,t}}\|\nabla U\|_{L^2(K)}^2\leqslant\|U\|_{DG,t}^2$, which obviously follows from the definition of the $\|\cdot\|_{DG,t}$-norm, we get

$$\sum_{\Gamma\in\mathcal{F}_{h,t}^B}\int_\Gamma |\beta(U)\nabla U\cdot\mathbf{n}_\Gamma u_D|\,\mathrm{d}S\leqslant \frac{\beta_1^2 k_1}{2\beta_0 c_W}\|u_D\|_{DGB,t}^2+\frac{\beta_0}{2k_1}c_M(c_I+1)\|U\|_{DG,t}^2.$$

Substituting back to (5.13) and integrating over the interval I_m, we obtain

$$\int_{I_m}(a_h(U,U,t)+\beta_0 J_h(U,U,t))\,\mathrm{d}t \\ \geqslant \beta_0\Big(1-\frac{1}{2k_1}c_M(c_I+1)\Big)\int_{I_m}\|U\|_{DG,t}^2\,\mathrm{d}t-\frac{\beta_1^2 k_1}{2\beta_0 c_W}\int_{I_m}\|u_D\|_{DGB,t}^2\,\mathrm{d}t.$$

If we set $k_1=c_M(c_I+1)$ and use assumption (5.9), we finally get inequality (5.12), which we wanted to prove.

2) Let $\theta=0$. From assumption (2.5) and the definition of the $\|\cdot\|_{DG,t}$-norm, we get

$$a_h(U,U,t)+\beta_0 J_h(U,U,t) \\ \geqslant \beta_0\|U\|_{DG,t}^2-\beta_1\sum_{\Gamma\in\mathcal{F}_{h,t}^I}\int_\Gamma|\langle\nabla U\rangle\cdot\mathbf{n}_\Gamma[U]|\,\mathrm{d}S-\beta_1\sum_{\Gamma\in\mathcal{F}_{h,t}^B}\int_\Gamma|\nabla U\cdot\mathbf{n}_\Gamma U|\,\mathrm{d}S \\ \geqslant \beta_0\|U\|_{DG,t}^2-\beta_1\Big(\sum_{\Gamma\in\mathcal{F}_{h,t}^I}\int_\Gamma\frac{|\nabla U_\Gamma^{(L)}|+|\nabla U_\Gamma^{(R)}|}{2}|[U]|\,\mathrm{d}S+\sum_{\Gamma\in\mathcal{F}_{h,t}^B}\int_\Gamma|\nabla U|\,|U|\,\mathrm{d}S\Big).$$

Now applying Young's inequality with $\delta>0$ separately to the first and the second term above in round brackets and using the inequality $(a+b)^2\leqslant 2(a^2+b^2)$ valid for

$a, b \in \mathbb{R}$, we obtain

$$
\begin{aligned}
(5.14)\quad & \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma \frac{|\nabla U^{(L)}| + |\nabla U^{(R)}_\Gamma|}{2} |[U]| \,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma |\nabla U|\,|U| \,\mathrm{d}S \\
&\leqslant \frac{1}{2} \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma \frac{h(\Gamma)}{\delta c_W} \frac{(|\nabla U^{(L)}_\Gamma| + |\nabla U^{(R)}_\Gamma|)^2}{4} \,\mathrm{d}S + \frac{1}{2} \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma \frac{\delta c_W}{h(\Gamma)} |[U]|^2 \,\mathrm{d}S \\
&\quad + \frac{1}{2} \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma \frac{h(\Gamma)}{\delta c_W} |\nabla U^{(L)}_\Gamma|^2 \,\mathrm{d}S + \frac{1}{2} \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma \frac{\delta c_W}{h(\Gamma)} |U|^2 \,\mathrm{d}S \\
&\leqslant \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma \frac{h_{K^{(L)}_\Gamma} + h_{K^{(R)}_\Gamma}}{2\delta c_W} \frac{|\nabla U^{(L)}_\Gamma|^2 + |\nabla U^{(R)}_\Gamma|^2}{4} \,\mathrm{d}S \\
&\quad + \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma \frac{h_{K^{(L)}_\Gamma}}{2\delta c_W} |\nabla U^{(L)}_\Gamma|^2 \,\mathrm{d}S + \frac{\delta}{2} J_h(U,U,t).
\end{aligned}
$$

Using the quasiuniformity of the system of triangulations, we get

$$
\begin{aligned}
(5.15)\quad & \frac{1}{8\delta c_W} \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma (h_{K^{(L)}_\Gamma} + h_{K^{(R)}_\Gamma})(|\nabla U^{(L)}_\Gamma|^2 + |\nabla U^{(R)}_\Gamma|^2) \,\mathrm{d}S \\
&\quad + \frac{1}{2\delta c_W} \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma h_{K^{(L)}_\Gamma} |\nabla U^{(L)}_\Gamma|^2 \,\mathrm{d}S + \frac{\delta}{2} J_h(U,U,t) \\
&\leqslant \frac{c_Q+1}{8\delta c_W} \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma (h_{K^{(L)}_\Gamma} |\nabla U^{(L)}_\Gamma|^2 + h_{K^{(R)}_\Gamma} |\nabla U^{(R)}_\Gamma|^2) \,\mathrm{d}S \\
&\quad + \frac{1}{2\delta c_W} \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma h_{K^{(L)}_\Gamma} |\nabla U^{(L)}_\Gamma|^2 \,\mathrm{d}S + \frac{\delta}{2} J_h(U,U,t) \\
&\leqslant \frac{c_Q+1}{2\delta c_W} \sum_{K\in\mathcal{T}_{h,t}} \int_{\partial K} h_K |\nabla U|^2 \,\mathrm{d}S + \frac{\delta}{2} J_h(U,U,t).
\end{aligned}
$$

In the last inequality we have used that $c_Q > 0$ and

$$
\frac{c_Q+1}{8\delta c_W} \leqslant \frac{c_Q+1}{2\delta c_W}, \quad \frac{1}{2\delta c_W} \leqslant \frac{c_Q+1}{2\delta c_W}.
$$

The multiplicative trace inequality and the inverse inequality imply that

$$
\begin{aligned}
(5.16)\quad & \int_{\partial K} h_K |\nabla U|^2 \,\mathrm{d}S = h_K \|\nabla U\|^2_{L^2(\partial K)} \\
&\qquad \leqslant c_M(1+c_I) \|\nabla U\|^2_{L^2(K)} = c_M(1+c_I) |U|^2_{H^1(K)}.
\end{aligned}
$$

Now, summarizing (5.14)–(5.16) yields

$$
\begin{aligned}
(5.17)\quad & a_h(U,U,t)+\beta_0 J_h(U,U,t)\\
& \geqslant \beta_0\|U\|^2_{DG,t}-\frac{\beta_1 c_M(1+c_I)(c_Q+1)}{2\delta c_W}\sum_{K\in\mathcal{T}_{h,t}}|U|^2_{H^1(K)}-\frac{\beta_1\delta}{2}J_{h,t}(U,U,t).
\end{aligned}
$$

If we set $\delta=\frac{\beta_0}{\beta_1}$, we find that

$$
\begin{aligned}
(5.18)\qquad & a_h(U,U,t)+\beta_0 J_h(U,U,t)\geqslant \beta_0\|U\|^2_{DG,m}\\
& -\frac{2\beta_1^2 c_M(1+c_I)(c_Q+1)}{2\beta_0 c_W}\sum_{K\in\mathcal{T}_{h,t}}|U|^2_{H^1(K)}-\frac{\beta_0}{2}J_h(U,U,t).
\end{aligned}
$$

Using assumption (5.10) for the constant c_W and the definition of the $\|\cdot\|_{DG,t}$-norm, we have

$$
(5.19)\qquad a_h(U,U,t)+\beta_0 J_h(U,U,t)\geqslant \frac{\beta_0}{2}\|U\|^2_{DG,t}.
$$

Integrating both sides over the interval I_m, we finally get

$$
(5.20)\qquad \int_{I_m}(a_h(U,U,t)+\beta_0 J_h(U,U,t))\,\mathrm{d}t\geqslant\frac{\beta_0}{2}\int_{I_m}\|U\|^2_{DG,t}\,\mathrm{d}t.
$$

3) Let $\theta=1$. From assumption (2.5) and the definition of the $\|\cdot\|_{DG,t}$-norm, we get

$$
\begin{aligned}
(5.21)\qquad & a_h(U,U,t)+\beta_0 J_h(U,U,t)\geqslant \beta_0\|U\|^2_{DG,t}\\
&-2\beta_1\sum_{\Gamma\in\mathcal{F}^I_{h,t}}\int_\Gamma|\langle\nabla U\rangle\cdot\mathbf{n}_\Gamma[U]|\,\mathrm{d}S\\
&-2\beta_1\sum_{\Gamma\in\mathcal{F}^B_{h,t}}\int_\Gamma|\nabla U\cdot\mathbf{n}_\Gamma U|\,\mathrm{d}S-\beta_1\sum_{\Gamma\in\mathcal{F}^B_{h,t}}\int_\Gamma|\nabla U\cdot\mathbf{n}_\Gamma u_D|\,\mathrm{d}S\\
&\geqslant\beta_0\|U\|^2_{DG,t}\\
&-2\beta_1\Big(\sum_{\Gamma\in\mathcal{F}^I_{h,t}}\int_\Gamma\frac{|\nabla U^{(L)}_\Gamma|+|\nabla U^{(R)}_\Gamma|}{2}|[U]|\,\mathrm{d}S+\sum_{\Gamma\in\mathcal{F}^B_{h,t}}\int_\Gamma|\nabla U|\,|U|\,\mathrm{d}S\Big)\\
&-\beta_1\sum_{\Gamma\in\mathcal{F}^B_{h,t}}\int_\Gamma|\nabla U|\,|u_D|\,\mathrm{d}S.
\end{aligned}
$$

The expression in the round brackets has already been estimated in the proof of the previous part, see estimates (5.14)–(5.16). We have

$$\tag{5.22} \begin{aligned} &\sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma \frac{|\nabla U^{(L)}_\Gamma| + |\nabla U^{(R)}_\Gamma|}{2} |[U]| \,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma |\nabla U|\,|U| \,\mathrm{d}S \\ &\quad\leqslant \frac{c_Q+1}{2\delta c_W} \sum_{K\in\mathcal{T}_{h,t}} \int_{\partial K} h_K |\nabla U|^2 \,\mathrm{d}x + \frac{\delta}{2} J_h(U,U,t) \\ &\quad\leqslant c_M(1+c_I)\frac{c_Q+1}{2\delta c_W} \sum_{K\in\mathcal{T}_{h,t}} |U|^2_{H^1(K)} + \frac{\delta}{2} J_h(U,U,t). \end{aligned}$$

It follows from (5.21)–(5.22) that

$$\tag{5.23} \begin{aligned} &a_h(U,U,t) + \beta_0 J_h(U,U,t) \\ &\quad\geqslant \beta_0 \|U\|^2_{DG,t} - \frac{\beta_1 c_M(1+c_I)(c_Q+1)}{\delta c_W} \sum_{K\in\mathcal{T}_{h,t}} |U|^2_{H^1(K)} - \beta_1\delta J_h(U,U,t) \\ &\qquad - \beta_1 \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma |\nabla U|\,|u_D| \,\mathrm{d}S. \end{aligned}$$

The last term on the right-hand side can be estimated similarly to the proof of part 1). For each $k_1 > 0$ we get

$$\tag{5.24} \begin{aligned} &\beta_1 \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma |\nabla U|\,|u_D| \,\mathrm{d}S \\ &\quad\leqslant \beta_1 k_1 \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma h^{-1}_{K^{(L)}_\Gamma} |u_D|^2 \,\mathrm{d}S + \frac{\beta_1}{k_1} \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma h_{K^{(L)}_\Gamma} |\nabla U|^2 \,\mathrm{d}S \\ &\quad\leqslant \frac{\beta_1 k_1}{c_W} J^B_h(u_D,u_D,t) + \frac{\beta_1}{k_1} \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma h_{K^{(L)}_\Gamma} |\nabla U|^2 \,\mathrm{d}S \\ &\quad\leqslant \frac{\beta_1 k_1}{c_W} \|u_D\|^2_{DGB,t} + \frac{\beta_1}{k_1} c_M(c_I+1) \sum_{K\in\mathcal{T}^B_{h,t}} \|\nabla U\|^2_{L^2(K)} \\ &\quad\leqslant \frac{\beta_1 k_1}{c_W} \|u_D\|^2_{DGB,t} + \frac{\beta_1}{k_1} c_M(c_I+1) \|U\|^2_{DG,t}. \end{aligned}$$

Substituting back to (5.23), we obtain

$$\tag{5.25} \begin{aligned} &a_h(U,U,t) + \beta_0 J_h(U,U,t) \\ &\quad\geqslant \beta_0 \|U\|^2_{DG,t} - \frac{\beta_1 c_M(c_I+1)(c_Q+1)}{\delta c_W} \sum_{K\in\mathcal{T}_{h,t}} |U|^2_{H^1(K)} - \beta_1\delta J_h(U,U,t) \\ &\qquad - \frac{\beta_1 k_1}{c_W} \|u_D\|^2_{DGB,t} - \frac{\beta_1}{k_1} c_M(c_I+1) \|U\|^2_{DG,t}. \end{aligned}$$

If we set $\delta := \beta_0/4\beta_1$ and $k_1 := 4\beta_1\beta_0^{-1}c_M(c_I+1)$, we find that

$$\begin{aligned}(5.26)\quad & a_h(U,U,t)+\beta_0 J_h(U,U,t)\\ &\geqslant \beta_0\|U\|_{DG,t}^2 - \frac{4\beta_1^2 c_M(c_I+1)(c_Q+1)}{\beta_0 c_W}\sum_{K\in\mathcal{T}_{h,t}}|U|_{H^1(K)}^2 - \frac{\beta_0}{4}J_h(U,U,t)\\ &\quad - \frac{4\beta_1^2}{\beta_0 c_W}c_M(c_I+1)\|u_D\|_{DGB,t}^2 - \frac{\beta_0}{4}\|U\|_{DG,t}^2.\end{aligned}$$

Using assumption (5.11) for the constant c_W implies that

$$\begin{aligned}(5.27)\quad & a_h(U,U,t)+\beta_0 J_h(U,U,t)\\ &\geqslant \beta_0\|U\|_{DG,t}^2 - \frac{\beta_0}{4}\sum_{K\in\mathcal{T}_{h,t}}|U|_{H^1(K)}^2 - \frac{\beta_0}{4}J_h(U,U,t)\\ &\quad - \frac{\beta_0}{4(c_Q+1)}\|u_D\|_{DGB,t}^2 - \frac{\beta_0}{4}\|U\|_{DG,t}^2\\ &\geqslant \frac{\beta_0}{2}\|U\|_{DG,t}^2 - \frac{\beta_0}{2}\|u_D\|_{DGB,t}^2.\end{aligned}$$

Finally, using the definition of the $\|\cdot\|_{DG,t}$-norm and integrating over the interval I_m, we get (5.12). $\square$

Estimating the convective terms:

Theorem 2. *For each $k_2>0$ there exists a constant $c_b>0$ such that for the approximate solution U of problem* (2.9)–(2.11) *we have the inequality*

$$(5.28)\qquad \int_{I_m}|b_h(U,U,t)|\,\mathrm{d}t \leqslant \frac{\beta_0}{k_2}\int_{I_m}\|U\|_{DG,t}^2\,\mathrm{d}t + c_b\int_{I_m}\|U\|_{\Omega_t}^2\,\mathrm{d}t.$$

(The constant c_b depends on k_2, namely, $c_b=c_1^2k_2/\beta_0$, where $c_1>0$ is independent of k_2.)

Proof. By (3.8),

$$\begin{aligned}(5.29)\quad b_h(U,U,t) = &\underbrace{-\sum_{K\in\mathcal{T}_{h,t}}\int_K\sum_{s=1}^d f_s(U)\frac{\partial U}{\partial x_s}\,\mathrm{d}x}_{:=\sigma_1}\\ &\underbrace{+\sum_{\Gamma\in\mathcal{F}_{h,t}^I}\int_\Gamma H(U_\Gamma^{(L)},U_\Gamma^{(R)},\mathbf{n}_\Gamma)[U]\,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}_{h,t}^B}\int_\Gamma H(U_\Gamma^{(L)},U_\Gamma^{(L)},\mathbf{n}_\Gamma)U\,\mathrm{d}S.}_{:=\sigma_2}\end{aligned}$$

Then from the Lipschitz-continuity of the functions f_s, $s=1,\ldots,d$, with the modul

$L_f > 0$, the assumption that $f_s(0) = 0$ and the Cauchy inequality, we obtain

$$
\begin{aligned}
(5.30)\qquad |\sigma_1| &\leqslant \sum_{K\in\mathcal{T}_{h,t}} \int_K \sum_{s=1}^{d} |f_s(U) - f_s(0)| \left|\frac{\partial U}{\partial x_s}\right| \mathrm{d}x \\
&\leqslant L_f \sum_{K\in\mathcal{T}_{h,t}} \int_K \sum_{s=1}^{d} |U| \left|\frac{\partial U}{\partial x_s}\right| \mathrm{d}x \leqslant L_f\sqrt{d}\, \|U\|_{L^2(\Omega_t)} |U|_{H^1(\Omega_t,\mathcal{T}_{h,t})}.
\end{aligned}
$$

Now we shall estimate σ_2. From the relation $f_s(0) = 0$, $s = 1,\dots,d$, and the consistency of property (H2) of the numerical flux H we have $H(0,0,\mathbf{n}_\Gamma) = 0$. Then we can use the Lipschitz-continuity of H and get

$$
|\sigma_2| \leqslant L_H \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma (|U_\Gamma^{(L)}| + |U_\Gamma^{(R)}|)|[U]|\,\mathrm{d}S + L_H \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma (|U_\Gamma^{(L)}| + |U_\Gamma^{(L)}|)|U_\Gamma^{(L)}|\,\mathrm{d}S.
$$

Using that $U_\Gamma^{(R)} = U_\Gamma^{(L)}$ for $\Gamma \in \mathcal{F}^B_{h,t}$, the Cauchy inequality, and the relation $h(\Gamma) \leqslant \frac{1}{2}(c_Q + 1)h_K$ if $\Gamma \subset \partial K$, we obtain

$$
\begin{aligned}
(5.31)\quad |\sigma_2| &\leqslant L_H \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma (|U_\Gamma^{(L)}| + |U_\Gamma^{(R)}|)|U_\Gamma^{(L)}|\,\mathrm{d}S \\
&\quad + L_H \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma (|U_\Gamma^{(L)}| + |U_\Gamma^{(R)}|)|U_\Gamma^{(L)}|\,\mathrm{d}S \\
&\leqslant \frac{L_H}{\sqrt{c_W}} \biggl(c_W \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma \frac{|U_\Gamma^{(L)}|^2}{h(\Gamma)}\,\mathrm{d}S + c_W \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma \frac{(U_\Gamma^{(L)})^2}{h(\Gamma)}\,\mathrm{d}S \biggr)^{1/2} \\
&\quad \times \biggl(\sum_{\Gamma\in\mathcal{F}_{h,t}} h(\Gamma) \int_\Gamma (|U_\Gamma^{(L)}| + |U_\Gamma^{(R)}|)^2\,\mathrm{d}S \biggr)^{1/2} \\
&\leqslant \frac{L_H}{\sqrt{c_W}} J_h(U,U,t)^{1/2} \biggl(\sum_{\Gamma\in\mathcal{F}_{h,t}} 2h(\Gamma) \int_\Gamma |U_\Gamma^{(L)}|^2 + |U_\Gamma^{(R)}|^2\,\mathrm{d}S \biggr)^{1/2} \\
&\leqslant L_H \sqrt{\frac{c_Q+1}{c_W}} J_h(U,U,t)^{1/2} \\
&\quad \times \biggl(\sum_{\Gamma\in\mathcal{F}_{h,t}} h_{K_\Gamma^{(L)}} \int_{\partial K_\Gamma^{(L)}\cap\Gamma} |U_\Gamma^{(L)}|^2\,\mathrm{d}S + h_{K_\Gamma^{(R)}} \int_{\partial K_\Gamma^{(R)}\cap\Gamma} |U_\Gamma^{(R)}|^2\,\mathrm{d}S \biggr)^{1/2} \\
&\leqslant L_H \sqrt{\frac{c_Q+1}{c_W}} J_h(U,U,t)^{1/2} \biggl(\sum_{K\in\mathcal{T}_{h,t}} \int_{\partial K} h_K |U|^2\,\mathrm{d}S \biggr)^{1/2} \\
&= L_H \sqrt{\frac{c_Q+1}{c_W}} J_h(U,U,t)^{1/2} \biggl(\sum_{K\in\mathcal{T}_{h,t}} h_K \|U\|^2_{L^2(\partial K)} \biggr)^{1/2}.
\end{aligned}
$$

Substituting (5.30) and (5.31) into (5.29), using the Cauchy inequality and the definition of the $\|\cdot\|_{DG,t}$-norm, we find that

$$
\begin{aligned}
|b_h(U,U,t)| &\leqslant L_f\sqrt{d}\,\|U\|_{\Omega_t}\,|U|_{H^1(\Omega_t,\mathcal{T}_{h,t})} \\
&\qquad + L_H\sqrt{\frac{c_Q+1}{c_W}}J_h(U,U,t)^{1/2}\Big(\sum_{K\in\mathcal{T}_{h,t}} h_K\|U\|^2_{L^2(\partial K)}\Big)^{1/2} \\
&\leqslant \Big(L_f^2 d\|U\|^2_{\Omega_t} + L_H^2\frac{c_Q+1}{c_W}\sum_{K\in\mathcal{T}_{h,t}} h_K\|U\|^2_{L^2(\partial K)}\Big)^{1/2} \\
&\qquad \times (|U|^2_{H^1(\Omega_t,\mathcal{T}_{h,t})} + J_h(U,U,t))^{1/2} \\
&\leqslant c\|U\|_{DG,t}\Big(\|U\|_{\Omega_t} + \Big(\sum_{K\in\mathcal{T}_{h,t}} h_K\|U\|^2_{L^2(\partial K)}\Big)^{1/2}\Big),
\end{aligned}
$$

where $c = \big(\max\{L_f^2\,d, L_H^2(c_Q+1)/c_W\}\big)^{1/2}$. Furthermore, the multiplicative trace inequality and the inverse inequality imply that

$$
\begin{aligned}
\sum_{K\in\mathcal{T}_{h,t}} h_K\|U\|^2_{L^2(\partial K)} &\leqslant c_M\sum_{K\in\mathcal{T}_{h,t}} h_K(\|U\|_{L^2(K)}|U|_{H^1(K)} + h_K^{-1}\|U\|^2_{L^2(K)}) \\
&\leqslant c_M(c_I+1)\sum_{K\in\mathcal{T}_{h,t}}\|U\|^2_{L^2(K)} = c_M(c_I+1)\|U\|^2_{\Omega_t}.
\end{aligned}
$$

Hence, from this relation and Young's inequality we get

$$
\begin{aligned}
|b_h(U,U,t)| &\leqslant c\|U\|_{DG,t}\Big(\|U\|_{\Omega_t} + \Big(\sum_{K\in\mathcal{T}_{h,t}} h_K\|U\|^2_{L^2(\partial K)}\Big)^{1/2}\Big) \\
&\leqslant c_1\|U\|_{DG,t}\|U\|_{\Omega_t} \leqslant \frac{\beta_0}{k_2}\|U\|^2_{DG,t} + c_1^2\frac{k_2}{\beta_0}\|U\|^2_{\Omega_t} = \frac{\beta_0}{k_2}\|U\|^2_{DG,t} + c_b\|U\|^2_{\Omega_t},
\end{aligned}
$$

where $c_1 = c\big(1+\sqrt{c_M(c_I+1)}\big)$, $k_2 > 0$ and $c_b = c_1^2 k_2/\beta_0$. Integrating over the interval I_m, we finally have (5.28). □

Theorem 3. *There exists a constant $c_d > 0$ such that for the approximate solution U of problem* (2.9)–(2.11) *we have the inequality*

$$
\int_{I_m}|d_h(U,U,t)|\,\mathrm{d}t \leqslant \frac{\beta_0}{2}\int_{I_m}\|U\|^2_{DG,t}\,\mathrm{d}t + \frac{c_d}{2\beta_0}\|U\|^2_{\Omega_t}\,\mathrm{d}t. \tag{5.32}
$$

P r o o f. By (3.9), (2.7) and the Cauchy and Young's inequalities,

$$\begin{aligned}\int_{I_m} |d_h(U,U,t)| \,\mathrm{d}t &\leqslant c_z \int_{I_m} \sum_{K\in\mathcal{T}_{h,t}} \int_K \sum_{s=1}^{d} |U| \Big|\frac{\partial U}{\partial x_s}\Big| \,\mathrm{d}x \,\mathrm{d}t \\ &\leqslant c_z \int_{I_m} \|U\|_{\Omega_t} |U|_{H^1(\Omega_t,\mathcal{T}_{h,t})} \,\mathrm{d}t \\ &\leqslant c_z \int_{I_m} \|U\|_{\Omega_t} \|U\|_{DG,t} \,\mathrm{d}t \leqslant \frac{\beta_0}{2} \int_{I_m} \|U\|_{DG,t}^2 \,\mathrm{d}t + \frac{c_z^2}{2\beta_0} \|U\|_{\Omega_t}^2 \,\mathrm{d}t,\end{aligned}$$

which is (5.32) with $c_d = c_z^2$. □

Estimating the right-hand side form:

Theorem 4. *For the approximate solution U of problem* (2.9)–(2.11) *and any $k_3 > 0$ we have*

$$\begin{aligned}\int_{I_m} |l_h(U,t)| \,\mathrm{d}t \leqslant \frac{1}{2} \int_{I_m} (\|g\|_{\Omega_t}^2 + \|U\|_{\Omega_t}^2) \,\mathrm{d}t \\ + \beta_0 k_3 \int_{I_m} \|u_D\|_{DGB,t}^2 \,\mathrm{d}t + \frac{\beta_0}{k_3} \int_{I_m} \|U\|_{DG,t}^2 \,\mathrm{d}t.\end{aligned} \tag{5.33}$$

P r o o f. It follows from (3.10) that

$$|l_h(U,t)| = \Big|(g,U)_{\Omega_t} + \beta_0 c_W \sum_{\Gamma\in\mathcal{F}_{h,t}^B} h(\Gamma)^{-1} \int_\Gamma u_D U \,\mathrm{d}S\Big|.$$

After using the Cauchy inequality for the first term on the right-hand side and applying Young's inequality with $k_3 > 0$ to the second term, we find that

$$\begin{aligned}&\Big|(g,U)_{\Omega_t} + \beta_0 c_W \sum_{\Gamma\in\mathcal{F}_{h,t}^B} h(\Gamma)^{-1} \int_\Gamma u_D U \,\mathrm{d}S\Big| \\ &\qquad\leqslant \frac{1}{2}(\|g\|_{\Omega_t}^2 + \|U\|_{\Omega_t}^2) + \beta_0 k_3 \underbrace{c_W \sum_{\Gamma\in\mathcal{F}_{h,t}^B} h_{K_\Gamma^{(L)}}^{-1} \int_\Gamma |u_D|^2 \,\mathrm{d}S}_{=\|u_D\|_{DGB,t}^2} \\ &\qquad\quad + \frac{\beta_0}{k_3} \underbrace{c_W \sum_{\Gamma\in\mathcal{F}_{h,t}^B} h_{K_\Gamma^{(L)}}^{-1} \int_\Gamma |U|^2 \,\mathrm{d}S}_{\leqslant J_h(U,U,t)\leqslant\|U\|_{DG,t}^2}.\end{aligned}$$

Hence,

$$|l_h(U,t)| \leqslant \frac{1}{2}(\|g\|^2_{\Omega_t} + \|U\|^2_{\Omega_t}) + \beta_0 k_3 \|u_D\|^2_{DGB,t} + \frac{\beta_0}{k_3}\|U\|^2_{DG,t},$$

from which we get (5.33) by integrating both sides over the interval I_m. □

In what follows, we are concerned with the derivation of inequalities based on estimating the expression $\int_{I_m} (D_t U, U)_{\Omega_t}\,\mathrm{d}t$.

Lemma 1. *There exist constants $c_1, c_2 > 0$ independent of h, τ, m, M and U such that*

$$\begin{aligned}(5.34)\qquad &\|U_m^-\|^2_{\Omega_{t_m}} - \|U_{m-1}^-\|^2_{\Omega_{t_{m-1}}} + \|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}} + \frac{\beta_0}{2}\int_{I_m} \|U\|^2_{DG,t}\,\mathrm{d}t \\ &\leqslant c_1\Big(\int_{I_m} \|g\|^2_{\Omega_t}\,\mathrm{d}t + \int_{I_m} \|u_D\|^2_{DGB,t}\,\mathrm{d}t\Big) + c_2\int_{I_m}\|U\|^2_{\Omega_t}\,\mathrm{d}t.\end{aligned}$$

Moreover, for any $\delta_1 > 0$ we have

$$\begin{aligned}(5.35)\qquad &\|U_m^+\|^2_{\Omega_{t_m}} + \|U_{m-1}^-\|^2_{\Omega_{t_{m-1}}} + \frac{\beta_0}{2}\int_{I_m} \|U\|^2_{DG,t}\,\mathrm{d}t \\ &\leqslant c_1\Big(\int_{I_m} \|g\|^2_{\Omega_t}\,\mathrm{d}t + \int_{I_m} \|u_D\|^2_{DGB,t}\,\mathrm{d}t\Big) + c_2\int_{I_m}\|U\|^2_{\Omega_t}\,\mathrm{d}t \\ &\quad + \frac{1}{\delta_1}\|U_{m-1}^-\|^2_{\Omega_{t_{m-1}}} + \delta_1\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}}.\end{aligned}$$

P r o o f. We have

$$(5.36)\qquad \int_{I_m} (D_t U, U)_{\Omega_t}\,\mathrm{d}t = \sum_{K\in\mathcal{T}_{h,t}} \int_{I_m} (D_t U, U)_K\,\mathrm{d}t.$$

By virtue of assumption (5.5), the Reynolds transport theorem (see, e.g., [22] or [1]) and relation (2.8), we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_K U^2(x,t)\,\mathrm{d}x \\ &\quad = \int_K \Big(\frac{U^2(x,t)}{\partial t} + \boldsymbol{z}(x,t)\cdot\nabla(U^2(x,t)) + U^2(x,t)\,\mathrm{div}\,\boldsymbol{z}(x,t)\Big)\,\mathrm{d}x \\ &\quad = \int_K \Big(2U(x,t)\Big(\frac{U(x,t)}{\partial t} + \boldsymbol{z}(x,t)\cdot\nabla U(x,t)\Big) + U^2(x,t)\,\mathrm{div}\,\boldsymbol{z}(x,t)\Big)\,\mathrm{d}x \\ &\quad = 2(D_t U, U)_K + (U^2, \mathrm{div}\,\boldsymbol{z})_K.\end{aligned}$$

Integration over I_m and summing over $K \in \mathcal{T}_{h,t}$ together with assumption (2.7) imply that

$$
\begin{aligned}
(5.37)\quad \int_{I_m} (D_t U, U)_{\Omega_t}\,\mathrm{d}t &= \frac{1}{2}\int_{I_m}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} U^2\,\mathrm{d}x\Big)\,\mathrm{d}t - \frac{1}{2}\int_{I_m} (U^2, \operatorname{div} z)_{\Omega_t}\,\mathrm{d}t \\
&= \frac{1}{2}\|U_m^-\|^2_{\Omega_{t_m}} - \frac{1}{2}\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - \frac{1}{2}\int_{I_m} (U^2, \operatorname{div} z)_{\Omega_t}\,\mathrm{d}t \\
&\geqslant \frac{1}{2}\|U_m^-\|^2_{\Omega_{t_m}} - \frac{1}{2}\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - \frac{1}{2}c_z\int_{I_m} \|U\|^2_{\Omega_t}\,\mathrm{d}t.
\end{aligned}
$$

By a simple manipulation we find that

$$
(\{U\}_{m-1}, U_{m-1}^+)_{\Omega_{t_{m-1}}} = \frac{1}{2}(\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} + \|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}} - \|U_{m-1}^-\|^2_{\Omega_{t_{m-1}}}).
$$

Now we have already estimated all terms in our basic identity (5.8). Using all these estimates above, after some manipulation we get (5.34).

Another useful relation reads

$$
\begin{aligned}
(5.38)\quad &\int_{I_m} (D_t U, U)_{\Omega_t}\,\mathrm{d}t + (\{U\}_{m-1}, U_{m-1}^+)_{\Omega_{t_{m-1}}} \\
&= \frac{1}{2}\|U_m^-\|^2_{\Omega_{t_m}} - \frac{1}{2}\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - \frac{1}{2}\int_{I_m} (U^2, \operatorname{div} z)_{\Omega_t}\,\mathrm{d}t \\
&\quad + \|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - (U_{m-1}^-, U_{m-1}^+)_{\Omega_{t_{m-1}}} \\
&\geqslant \frac{1}{2}\Big(\|U_m^-\|^2_{\Omega_{t_m}} + \frac{1}{2}\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}}\Big) - (U_{m-1}^-, U_{m-1}^+)_{\Omega_{t_{m-1}}} - \frac{1}{2}c_z\int_{I_m} \|U\|^2_{\Omega_t}\,\mathrm{d}t.
\end{aligned}
$$

Then, analogously as above, using (5.38) and Young's inequality for the expression $(U_{m-1}^-, U_{m-1}^+)_{\Omega_{t_{m-1}}}$, we get estimate (5.35). $\square$

As we see, it is necessary to estimate the term $\int_{I_m} \|U\|^2_{\Omega_t}\,\mathrm{d}t$. We start with proving some useful inequalities. As was mentioned above, we consider the case $q = 1$.

Lemma 2. *There exist constants L_1 and M_1 such that*

$$
(5.39)\qquad \|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} + \|U_m^-\|^2_{\Omega_{t_m}} \geqslant \frac{L_1}{\tau_m}\int_{I_m} \|U\|^2_{\Omega_t}\,\mathrm{d}t,
$$

$$
(5.40)\qquad \|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} \leqslant \frac{M_1}{\tau_m}\int_{I_m} \|U\|^2_{\Omega_t}\,\mathrm{d}t.
$$

P r o o f. Let $\phi \in P^1(0,1)$ be a polynomial depending on $\vartheta \in (0,1)$ of degree at most one. Since the expressions

$$
\Big(\sum_{l=0}^{1} \phi^2(l)\Big)^{1/2}, \quad \Big(\int_0^1 \phi^2\,\mathrm{d}\phi\Big)^{1/2}
$$

are equivalent norms in the finite dimensional space $P^1(0,1)$, there exist constants $\widetilde{L}_1, \widetilde{M}_1 > 0$ such that

$$\widetilde{L}_1 \int_0^1 \phi^2 \,\mathrm{d}\vartheta \leqslant \sum_{l=0}^{1} \phi^2(l) \leqslant \widetilde{M}_1 \int_0^1 \phi^2 \,\mathrm{d}\vartheta.$$

Putting $\vartheta = (t - t_{m-1})/\tau_m$ for $t \in I_m$ and using the substitution theorem, we find that

$$p^2(t_{m-1}) + p^2(t_m) \geqslant \frac{\widetilde{L}_1}{\tau_m} \int_{I_m} p^2 \,\mathrm{d}t, \tag{5.41}$$

$$p^2(t_{m-1}) \leqslant \frac{\widetilde{M}_1}{\tau_m} \int_{I_m} p^2 \,\mathrm{d}t \tag{5.42}$$

for all $p \in P^1(I_m)$. We set

$$\begin{aligned} \widetilde{U}_{m-1} &:= U^+_{m-1} \circ \mathcal{A}_{t_{m-1}} \colon \ \Omega_0 \to \mathbb{R}, \\ \widetilde{U}_m &:= U^-_m \circ \mathcal{A}_{t_m} \colon \ \Omega_0 \to \mathbb{R}, \\ \widetilde{U}(t) &:= U(t) \circ \mathcal{A}_t \colon \ \Omega_0 \to \mathbb{R}. \end{aligned}$$

Then $\widetilde{U}_{m-1} = \widetilde{U}(t_{m-1})$, $\widetilde{U}_m = \widetilde{U}(t_m)$ and $\widetilde{U}(X,\cdot) \in P^1(I_m)$ for $X \in \Omega_0$.

For all $X \in \Omega_0$, using (5.41), we get

$$|\widetilde{U}(X, t_{m-1})|^2 + |\widetilde{U}(X, t_m)|^2 \geqslant \frac{\widetilde{L}_1}{\tau_m} \int_{I_m} |\widetilde{U}(X,t)|^2 \,\mathrm{d}t. \tag{5.43}$$

Let us use the notation

$$J(X,t) = \det \frac{D\mathcal{A}_t(X)}{DX}$$

for the Jacobian determinant of the mapping $\mathcal{A}_t$. Then, by virtue of the regularity of the mapping $\mathcal{A}_t$, we have

$$0 < C_J^- \leqslant |J(X,t)| \leqslant C_J^+ \quad \forall\, X \in \Omega_0,\ t \in \overline{I}_m,\ m = 1, \ldots, M, \tag{5.44}$$

where C_J^-, C_J^+ are constants independent of X, t, and m.

Now, using (5.43) and (5.44), we get

$$\begin{aligned} &|\widetilde{U}(X, t_{m-1})|^2 |J(X, t_{m-1})| + |\widetilde{U}(X, t_m)|^2 |J(X, t_m)| \\ &\qquad \geqslant C_J^- (|\widetilde{U}(X, t_{m-1})|^2 + |\widetilde{U}(X, t_m)|^2) \\ &\qquad \geqslant C_J^- \frac{\widetilde{L}_1}{\tau_m} \int_{I_m} |\widetilde{U}(X,t)|^2 \,\mathrm{d}t \\ &\qquad \geqslant \frac{C_J^-}{C_J^+} \frac{\widetilde{L}_1}{\tau_m} \int_{I_m} |\widetilde{U}(X,t)|^2 |J(X,t)| \,\mathrm{d}t. \end{aligned}$$

Integrating over the domain Ω_0, setting $L_1 = \widetilde{L}_1 \frac{C_J^-}{C_J^+}$ and using the Fubini theorem, we find that

$$\begin{aligned}\int_{\Omega_0} &(|\widetilde{U}(X,t_{m-1})|^2|J(X,t_{m-1})| + |\widetilde{U}(X,t_m)|^2|J(X,t_m)|)\,\mathrm{d}X \\ &\geqslant \frac{L_1}{\tau_m}\int_{\Omega_0}\Big(\int_{I_m}|\widetilde{U}(X,t)|^2|J(X,t)|\,\mathrm{d}t\Big)\,\mathrm{d}X \\ &= \frac{L_1}{\tau_m}\int_{I_m}\Big(\int_{\Omega_0}|\widetilde{U}(X,t)|^2|J(X,t)|\,\mathrm{d}X\Big)\,\mathrm{d}t.\end{aligned}$$

Finally, the substitution theorem gives

$$\int_{\Omega_{t_{m-1}}}|U(x,t_{m-1}^+)|^2\,\mathrm{d}x + \int_{\Omega_{t_m}}|U(x,t_m^-)|^2\,\mathrm{d}x \geqslant \frac{L_1}{\tau_m}\int_{I_m}\Big(\int_{\Omega_t}|U(x,t)|^2\,\mathrm{d}x\Big)\,\mathrm{d}t,$$

which is (5.39). Inequality (5.40) can be proved analogously with the aid of (5.42). $\square$

Now we can prove the theorem about estimation of the term $\int_{I_m}\|U\|^2_{\Omega_t}\,\mathrm{d}t$.

Theorem 5. *Under the assumption $q=1$ there exists a constant c^* (depending on c_2 and L_1) such that*

$$\int_{I_m}\|U\|^2_{\Omega_t}\,\mathrm{d}t \leqslant \frac{2c_1}{L_1}\tau_m\int_{I_m}(\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t})\,\mathrm{d}t + \frac{8M_1}{L_1^2}\tau_m\|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}} \tag{5.45}$$

holds, if

$$0 < \tau_m \leqslant c^*. \tag{5.46}$$

Here c_1 and c_2 are the constants from Lemma 1.

P r o o f. From (5.35), (5.39), and (5.40) we get

$$\begin{aligned}\Big(\frac{L_1}{\tau_m} - c_2\Big)&\int_{I_m}\|U\|^2_{\Omega_t}\,\mathrm{d}t + \frac{\beta_0}{2}\int_{I_m}\|U\|^2_{DG,t}\,\mathrm{d}t \\ &\leqslant c_1\int_{I_m}(\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t})\,\mathrm{d}t + \frac{1}{\delta_1}\|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}} + \delta_1\frac{M_1}{\tau_m}\int_{I_m}\|U\|^2_{\Omega_t}\,\mathrm{d}t,\end{aligned} \tag{5.47}$$

which can be written in the form

$$\begin{aligned}(L_1 - \delta_1 M_1 - c_2\tau_m)&\int_{I_m}\|U\|^2_{\Omega_t}\,\mathrm{d}t + \frac{\beta_0}{2}\int_{I_m}\|U\|^2_{DG,t}\,\mathrm{d}t \\ &\leqslant c_1\tau_m\int_{I_m}(\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t})\,\mathrm{d}t + \frac{\tau_m}{\delta_1}\|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}}.\end{aligned} \tag{5.48}$$

Let $L_1 - \delta_1 M_1 = \frac{3}{4} L_1$ and $c_2 \tau_m \leqslant \frac{1}{4} L_1$. If we set $\delta_1 = L_1/4M_1$ and assume that

$$0 < \tau_m \leqslant c^* := \frac{L_1}{4c_2},$$

using (5.48) we find that

$$\frac{L_1}{2} \int_{I_m} \|U\|^2_{\Omega_t} \,\mathrm{d}t + \frac{\beta_0}{2} \int_{I_m} \|U\|^2_{DG,t} \,\mathrm{d}t \leqslant c_1 \tau_m \int_{I_m} (\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t}) \,\mathrm{d}t + \frac{4M_1}{L_1} \tau_m \|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}},$$

from which we immediately get (5.45). □

To prove our main theorem on the stability, we shall apply the discrete Gronwall lemma.

Lemma 3 (Discrete Gronwall lemma). *Let x_m, a_m, b_m and c_m, where $m = 1, 2, \ldots$, be non-negative sequences and let the sequence a_m be non-decreasing. If*

$$x_0 + c_0 \leqslant a_0,$$
$$x_m + c_m \leqslant a_m + \sum_{j=0}^{m-1} b_j x_j \quad \text{for } m \geqslant 1,$$

then we have

$$x_m + c_m \leqslant a_m \prod_{j=0}^{m-1} (1 + b_j) \quad \text{for } m \geqslant 0.$$

The proof can be carried out by induction.

Finally, we come to our main result on the unconditional stability of the STDGM.

Theorem 6. *Let $q = 1$ and $0 < \tau_m \leqslant c^*$. Then there exists a constant $c > 0$ such that*

$$\text{(5.49)} \qquad \|U^-_m\|^2_{\Omega_{t_m}} + \sum_{j=1}^{m} \|\{U_{j-1}\}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2} \sum_{j=1}^{m} \int_{I_j} \|U\|^2_{DG,j} \,\mathrm{d}t \leqslant c\Big(\|U^-_0\|^2_{\Omega_{t_0}} + \sum_{j=1}^{m} \int_{I_j} R_j \,\mathrm{d}t \Big), \quad m = 1, \ldots, M,\ h \in (0, h_0),$$

where

$$R_j = c_1 \Big(1 + \frac{2c_2}{L_1} \tau_j\Big) (\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t}).$$

P r o o f. Writing j instead of m in (5.34) and using (5.45), we obtain

$$
\begin{aligned}
(5.50)\quad & \|U_j^-\|^2_{\Omega_{t_j}} - \|U_{j-1}^-\|^2_{\Omega_{t_{j-1}}} + \|\{U\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\int_{I_j} \|U\|^2_{DG,j}\,\mathrm{d}t \\
& \leqslant c_1\Big(1 + \frac{2c_2}{L_1}\tau_m\Big)\int_{I_j} (\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t})\,\mathrm{d}t + c_2\frac{8M_1}{L_1^2}\tau_j\|U_{j-1}^-\|^2_{\Omega_{t_{j-1}}} \\
& = \int_{I_j} R_j\,\mathrm{d}t + c_2\frac{8M_1}{L_1^2}\tau_j\|U_{j-1}^-\|^2_{\Omega_{t_{j-1}}}.
\end{aligned}
$$

Let $m \geqslant 1$. Summing (5.50) over all $j = 1,\dots,m$, we get

$$
\begin{aligned}
& \|U_m^-\|^2 + \sum_{j=1}^{m} \|\{U\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\sum_{j=1}^{m}\int_{I_j}\|U\|^2_{DG,j}\,\mathrm{d}t \\
& \qquad \leqslant \|U_0^-\|^2_{\Omega_0} + c_2\frac{8M_1}{L_1^2}\sum_{j=0}^{m-1}\tau_{j+1}\|U_j^-\|^2_{\Omega_{t_j}} + \sum_{j=1}^{m}\int_{I_j} R_j\,\mathrm{d}t.
\end{aligned}
$$

Using the discrete Gronwall inequality setting

$$
\begin{aligned}
x_0 &= a_0 = \|U_0^-\|^2_{\Omega_{t_0}}, \quad c_0 = 0, \\
x_m &= \|U_m^-\|^2_{\Omega_{t_m}}, \\
c_m &= \sum_{j=1}^{m} \|\{U_{j-1}\}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\sum_{j=1}^{m}\int_{I_j}\|U\|^2_{DG,j}\,\mathrm{d}t, \\
a_m &= \|U_0^-\|^2_{\Omega_{t_0}} + \sum_{j=1}^{m}\int_{I_m} R_j\,\mathrm{d}t, \\
b_j &= c_2\frac{8M_1}{L_1^2}\tau_{j+1}, \quad j = 0,1,\dots,m,
\end{aligned}
$$

yields

$$
\begin{aligned}
(5.51)\qquad & \|U_m^-\|^2 + \sum_{j=1}^{m} \|\{U_{j-1}\}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\sum_{j=1}^{m}\int_{I_j}\|U\|^2_{DG,j}\,\mathrm{d}t \\
& \leqslant \bigg(\|U_0^-\|^2 + \sum_{j=1}^{m}\int_{I_j} R_j\,\mathrm{d}t\bigg)\prod_{j=0}^{m-1}\Big(1 + c_2\frac{8M_1}{L_1^2}\tau_{j+1}\Big).
\end{aligned}
$$

Finally, (5.51) and the inequality $1 + \sigma < \exp(\sigma)$ valid for any $\sigma > 0$ immediately yield (5.49) with the constant $c := \exp(c_2 \cdot 8M_1L_1^{-2}T)$. $\square$

6. Conclusion

The subject of the paper is the stability analysis of the space-time discontinuous Galerkin method for the numerical solution of an initial-boundary value problem in a time-dependent domain. A parabolic equation with nonlinear convection and diffusion, equipped with initial and Dirichlet boundary conditions, is formulated by the ALE method. The space discretization is carried out by the SIPG, IIPG, and NIPG versions of the discontinuous Galerkin method using piecewise polynomial approximations of degree $p \geqslant 1$. In time the discontinuous Galerkin piecewise linear discretization is used. The main result is the proof of unconditional stability of the method.

The subject of a further research will be the stability analysis for higher-order time discontinuous Galerkin discretization and the derivation of error estimates.

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