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THE GRADIENT SUPERCONVERGENCE OF THE FINITE VOLUME METHOD FOR A NONLINEAR ELLIPTIC PROBLEM OF NONMONOTONE TYPE

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Dedicated to Professor Ivo Babuška on his 90th birthday

Abstract. We study the superconvergence of the finite volume method for a nonlinear elliptic problem using linear trial functions. Under the condition of $C$-uniform meshes, we first establish a superclose weak estimate for the bilinear form of the finite volume method. Then, we prove that on the mesh point set $S$, the gradient approximation possesses the superconvergence: \[ \max_{P \in S} |(\nabla u - \nabla u_h)(P)| = O(h^2) |\ln h|^{3/2}, \] where $\nabla$ denotes the average gradient on elements containing vertex $P$. Furthermore, by using the interpolation post-processing technique, we also derive a global superconvergence estimate in the $H^1$-norm and establish an asymptotically exact a posteriori error estimator for the error $\|u - u_h\|_1$.

Keywords: finite volume method; nonlinear elliptic problem; local and global superconvergence in the $W^{1,\infty}$-norm; a posteriori error estimator

MSC 2010: 65M60, 65M15

1. Introduction

The finite volume method (FVM), also known as generalized difference method [13], [25], [26], box scheme [4], [29] or covolume method [16], [22], has been widely analyzed for various types of partial differential equations. The main benefit of this...
method is that it inherits some physical conservation laws of the original problem locally, which is very desirable in practical applications, for example, computational fluid mechanics. We refer to the monograph [26] for general presentation of the finite volume method, and to [16], [11], [15], [14], [19], [24], [30], [32], [34] and the references cited therein for details.

Superconvergence of the finite element method has long been an active research area in scientific computation, since it is of practical importance in enhancing the accuracy of numerical solutions [3], [1], [2], [8], [10], [9], [27], [31], [33], [35]. At present, many superconvergence results have also been obtained for the finite volume method. For linear elliptic problems, Ewing et al. [19] obtain the $H^1$ and $W^{1,\infty}$ superconvergence estimates between the FVM solution and the linear interpolation of the exact solution; Huang and Li [22] derive the $H^1$ and $W^{1,\infty}$ superconvergence estimates for the error between the FVM solution and the corresponding finite element (FEM) solution. Moreover, the superconvergence of the FVM solution in an average gradient norm has been also obtained. See, for example, [13], [26], [28]. For the linear elliptic and parabolic problems, Chou et al. [16] show the superconvergence estimates in the $L^p$-norm for the error between the FVM solution and the corresponding FEM solution and between their gradients. All the superconvergence results mentioned above are for linear elliptic problems.

In this paper, we consider the superconvergence of the FVM for the following nonlinear elliptic problem of nonmonotone type:

\[
\begin{aligned}
-\text{div}(a(x,u)\nabla u) &= f(x) & \text{in } \Omega, \\
u &= 0 & \text{on } \partial\Omega,
\end{aligned}
\]

where $\Omega$ is a convex bounded domain in $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$, the coefficient function $a(x,u) \geq \beta_1 > 0$ in $\Omega$. We do not assume the monotonicity condition for problem (1.1) (see (2.3)).

Some authors have studied the FVM for problem (1.1). Li [25] first obtained the error estimate in the $H^1$-norm. Chatzipantelidis et al. [12] establish the error estimates in the $H^{1-}, L^2-$, and $L^{\infty}$-norm. Bi [6] obtains the $H^1$ and $W^{1,\infty}$ superclose estimates for the error between the FVM solution and the corresponding FEM solution. Bi and Ginting [7] also analyze the two-grid FVM and derived the error estimates in a broken $H^1$-norm. Moreover, Bergam et al. [5] give a residual type a posteriori error estimate for the FVM solution.

To the authors’ best knowledge, there is no gradient superconvergence result available for the error between the FVM solution and the exact solution of nonlinear problem (1.1). Our main goal in this paper is to give some gradient superconvergence results for the linear finite volume approximation to problem (1.1). By treating the
FVM as a perturbation of the corresponding FEM, we first establish the superclose weak estimate for the bilinear form $a_h(\omega; \cdot, \cdot)$ of the FVM (see (2.8)),

\[(1.2) \quad |a_h(\omega; u - \Pi_h u, \Pi_h^* v_h)| \leq C h^2 \|u\|_{3,p} \|v_h\|_{1,q} \quad \forall v_h \in U_h, \ 1 < p \leq \infty,
\]

where $\omega \in W^{1,\infty}(\Omega)$, $1/p + 1/q = 1$, $\Pi_h$ is the usual linear interpolation operator, $\Pi_h^*$ is an interpolation operator from the trial function space $U_h$ to the test function space $V_h$. It is well known that such a superclose weak estimate plays an important role in the superconvergence analysis of FEM [33], [35]. By using this weak estimate and the Green’s function method [35], we can further derive

\[(1.3) \quad \|u_h - \Pi_h u\|_{1,\infty} \leq C(u) h^2 |\ln h|^{3/2}.
\]

Then, we consider the superconvergence at mesh points and prove the following superconvergence result:

\[(1.4) \quad \max_{P \in S} |(\nabla u - \nabla u_h)(P)| \leq C(u) h^2 |\ln h|^{3/2},
\]

where $S$ is the set of all interior mesh points, and $\nabla$ denotes the average gradient on elements containing point $P$. In order to obtain the global superconvergence, we introduce the interpolation post-processing technique [27] and prove that

\[(1.5) \quad \|u - Q_{2h} u_h\|_1 \leq C(u) h^2,
\]

where $Q_{2h}$ is the interpolation post-processing operator. Based on superconvergence estimate (1.5), an asymptotically exact a posteriori error estimate is also given for the error $\|u - u_h\|_1$.

This paper is organized as follows. In Section 2, we introduce the finite volume scheme and give some useful lemmas. In Section 3, we establish the superclose weak estimate and derive the superconvergence estimates for the error $u_h - \Pi_h u$ in the $H^1$- and $W^{1,\infty}$-norm. Section 4 is devoted to the discussion of mesh points and global superconvergence in the $H^1$-norm. Finally, in Section 5, some numerical experiments are provided to illustrate our theoretical analysis.

Throughout this paper, we adopt the notation $W^{m,p}(D)$ to stand for the usual Sobolev spaces on subdomain $D \subset \Omega$ equipped with the norm $\|\cdot\|_{m,p,D}$ and the semi-norm $|\cdot|_{m,p,D}$, and if $p = 2$, we set $W^{m,p}(D) = H^m(D)$, $\|\cdot\|_{m,p,D} = \|\cdot\|_{m,D}$. The inner product and the norm in space $L_2(D)$ are denoted by $(\cdot, \cdot)_D$ and $\|\cdot\|_D$, respectively. When $D = \Omega$, we omit the index $D$. We will use the letter $C$ to represent a generic positive constant, independent of the mesh size $h$. 

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2. Finite volume methods

2.1. Finite volume element approximation. Consider problem (1.1). As usual, we assume that \( a(x, u) \in W^{1,\infty}(\Omega \times \mathbb{R}) \) and there exist positive constants \( \beta_1, \beta_2, \) and \( \beta_3 \) such that

\[
\beta_1 \leq a(x, u) \leq \beta_2, \quad x = (x_1, x_2) \in \Omega, \quad u \in \mathbb{R},
\]

\[
|a'_{x_1}(x, u)| + |a'_{x_2}(x, u)| + |a'_u(x, u)| \leq \beta_3, \quad x \in \Omega, \quad u \in \mathbb{R}.
\]

For properly smooth \( a(x, u) \) and \( f(x) \), the unique existence of weak and classical solutions for problem (1.1) is proved in [17], [21], [20]. We assume that the solution of problem (1.1) possesses the smoothness and boundedness required in our analysis.

Remark 2.1. In our analysis, the monotonicity condition

\[
(a(x, u)\nabla u - a(x, v)\nabla v, \nabla u - \nabla v) \geq \alpha_0 ||\nabla u - \nabla v||^2, \quad \alpha_0 \geq 0,
\]

is not needed for \( a(x, u) \), and since the coefficient \( a(x, u) \) depends on \( x \), the classical Kirchhoff transformation (see, for instance, [23]) cannot be used in our study.

Remark 2.2. In the case of domain \( \Omega \) with a smooth boundary \( \partial \Omega \), \( f \) being a \( \alpha \)-Hölder continuous function on \( \overline{\Omega} \) \( (0 < \alpha \leq 1) \), and \( a(x, u) \in C^{2}(\overline{\Omega} \times \mathbb{R}) \), it is shown in [17], [18] that the solution of problem (1.1) has the regularity \( u \in C^{2+\alpha}(\overline{\Omega}) \), and the solution is unique.

Let \( T_h = \bigcup\{K\} \) be a quasi-uniform triangulation of domain \( \Omega \) with mesh size \( h = \max h_K \), where \( h_K \) is the diameter of the triangle \( K \). The union of the triangles of \( T_h \) determines a polygonal domain \( \Omega_h \subset \Omega \) whose boundary vertices lie on \( \partial \Omega \). If \( \Omega \) itself is a polygonal domain, we may have \( \Omega_h = \Omega \). If \( \Omega \) is a domain with smooth boundary \( \partial \Omega \), we assume that triangulation \( T_h \) is such that for a positive constant \( C \), it holds that

\[
\text{dist}(x, \partial \Omega) \leq Ch^2 \quad \forall x \in \partial \Omega_h.
\]

Concerning triangulation \( T_h \), we construct the barycenter dual partition \( T^*_h \) by connecting the barycenter to the midpoints of edges of each \( K \in T_h \) by straight lines. Thus, for each nodal point \( P \) in \( T_h \), there exists a polygonal \( K^*_P \) surrounding \( P \), and \( K^*_P \in T^*_h \) is called the dual element or the control volume at point \( P \), see Figure 1.

For triangulations \( T_h \) and \( T^*_h \), we define the following trial function space \( U_h \) and test function space \( V_h \), respectively,

\[
U_h = \{u_h \in C^0(\overline{\Omega}) : u_h|_K \in P_1(K) \forall K \in T_h, u_h|_{\Omega \setminus \Omega_h} = 0\},
\]

\[
V_h = \{v_h \in L_2(\Omega) : v_h|_{K^*_P} = \text{constant} \forall P \in N_h, v_h|_{\Omega \setminus \Omega_h} = 0\},
\]
where \( P_1(K) \) is the set of all linear polynomials on \( K \) and \( N_h \) is the set of all nodal points of \( T_h \). Let \( \Pi_h^*: U_h \to V_h \) be the interpolation operator defined by

\[
\Pi_h^* u_h = \sum_{P \in N_h} u_h(P) \chi_P \quad \forall u_h \in U_h,
\]

where \( \chi_P \) is the characteristic function of the dual element \( K^*_p \).

A standard weak form for problem (1.1) is to find \( u \in H^1_0(\Omega) \) such that

\[
a(u; u, v) = (f, v) \quad \forall v \in H^1_0(\Omega),
\]

where

\[
a(\omega; u, v) = \int_\Omega a(x, \omega) \nabla u \cdot \nabla v \, dx, \quad (f, v) = \int_\Omega fv \, dx.
\]

This weak form is usually adopted for finite element approximations. However, for the FVM, we need a new weak form. Let \( u \) be the solution of problem (1.1) and \( v \in V_h \). Then, by using Green’s formula, we have

\[
- \int_{\partial K^*_p} \mathbf{n} \cdot (a(x, u) \nabla u) v \, ds = \int_{K^*_p} fv \, dx \quad \forall K^*_p \in T_h^*,
\]

where \( \mathbf{n} \) is the outward unit normal vector on the concerned boundary. Motivated by the weak form (2.7), we introduce the form

\[
a_h(\omega; u, v) = - \sum_{K^*_p \in T_h^*} \int_{\partial K^*_p} \mathbf{n} \cdot (a(x, \omega) \nabla u) v \, ds, \quad \omega, u \in H^1(\Omega), \ v \in V_h,
\]

and define the finite volume approximation to problem (1.1) by finding \( u_h \in U_h \) such that

\[
a_h(u_h; u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h.
\]
Note that for fixed \( \omega \), \( a_h(\omega; u, v) \) is linear in \( u \) and \( v \). Since \( \Pi^*_h \) is a one-to-one mapping from \( U_h \) onto red \( V_h \), the equivalent form of problem (2.9) is to find \( u_h \in U_h \) such that

\[
(2.10) \quad a_h(u_h; u_h, \Pi^*_h v_h) = (f, \Pi^*_h v_h) \quad \forall v_h \in U_h,
\]

which is the finite volume scheme to be used in our analysis. From (2.7) we know that scheme (2.10) is consistent, and the following error equation holds:

\[
(2.11) \quad a_h(u; u, \Pi^*_h v_h) - a_h(u_h; u_h, \Pi^*_h v_h) = 0 \quad \forall v_h \in U_h.
\]

2.2. Some lemmas. Let \( \Pi_h u \) be the usual linear interpolation approximation of a continuous function \( u \). In our analysis, the following approximation property, trace inequality, and the inverse inequality will be used frequently [33], [35]:

\[
(2.12) \quad \|u - \Pi_h u\|_{m,p,K} \leq C h_K^{2-m} \|u\|_{2,p,K}, \quad 0 \leq m \leq 2,
\]

\[
\quad u \in W^{2,p}(K), \quad 1 < p \leq \infty,
\]

\[
(2.13) \quad \|u\|_{0,p,\partial K} \leq C h_K^{-1/p}(\|u\|_{0,p,K} + h_K \|\nabla u\|_{0,p,K}),
\]

\[
\quad u \in W^{1,p}(K), \quad 1 \leq p \leq \infty,
\]

\[
(2.14) \quad \|u_h\|_{l,p,K} \leq C h_K^{m-l+2/p-2/q} \|u_h\|_{m,q,K}, \quad u_h \in P_1(K), \quad m \leq l,
\]

\[
\quad q \leq p, \quad 1 \leq p, \quad q \leq \infty,
\]

\[
(2.15) \quad \|u_h\|_{m,p,\partial K} \leq C h_K^{-1/p} \|u_h\|_{m,p,K}, \quad u_h \in P_1(K),
\]

\[
\quad m = 0, 1, \quad 1 \leq p \leq \infty.
\]

Remark 2.3. By decomposing the dual element \( K^*_p \) into several triangles, it is easy to see that the trace inequality (2.13) and the inverse inequality (2.14) also hold on the dual element \( K^*_p \in T^*_h \).

Furthermore, for the operator \( \Pi^*_h \), we have the following lemma.

Lemma 2.1. For \( K \in T_h \), let \( \tau \subset \partial K \) be an edge of \( K \). Then, for \( v_h \in U_h \), \( 1 \leq q \leq \infty \), we have

\[
(2.16) \quad \int_K (v_h - \Pi^*_h v_h) = 0, \quad \int_\tau (v_h - \Pi^*_h v_h) \, ds = 0,
\]

\[
(2.17) \quad \|v_h - \Pi^*_h v_h\|_{0,q,K} \leq C h_K \|\nabla v_h\|_{0,q,K},
\]

\[
(2.18) \quad \|v_h - \Pi^*_h v_h\|_{0,q,\partial K} \leq C h_K^{1-1/q} \|\nabla v_h\|_{0,q,K}.
\]
Proof. Noting that $v_h$ is a linear function on $K$, formula (2.16) can be derived by a direct calculation. For (2.17), let $P$ be a vertex of $K$ and $K^* = K_p^* \cap K$ a third of $K$, then $\Pi_{h}^* v_h = v_h(P)$ on $K^*$. By using the inverse inequality, we have

$$
\|v_h - \Pi_{h}^* v_h\|_{0,q,K^*} = \|v_h - v_h(P)\|_{0,q,K^*} \leq h_{K^*} |K^*|^{1/q} \|\nabla v_h\|_{0,\infty,K^*} \leq C h_{K^*} \|\nabla v_h\|_{0,q,K^*},
$$

which gives (2.17). Now let $\tau^* = \partial K \cap \partial K^*$. Then, from (2.13) and (2.17) we obtain that

$$
\|v_h - \Pi_{h}^* v_h\|_{0,q,\tau^*} \leq C h_{K^*}^{-1/q} (\|v_h - \Pi_{h}^* v_h\|_{0,q,K^*} + h_{K^*} \|\nabla v_h\|_{0,q,K^*}) \leq C h_{K^*}^{-1/q} \|\nabla v_h\|_{0,q,K^*}.
$$

Thus, the proof is completed. \[\square\]

Similarly, we prove the following approximation properties.

**Lemma 2.2.** Let $K_p^* \in T_h^*$ and $v_h \in U_h$. Then we have

\begin{align}
(2.19) \quad &\|v_h - \Pi_{h}^* v_h\|_{0,q,K_p^*} \leq C h \|\nabla v_h\|_{0,q,K_p^*}, \quad 1 \leq q \leq \infty, \\
(2.20) \quad &\|v_h - \Pi_{h}^* v_h\|_{0,q,\partial K_p^*} \leq C h_{K_p^*}^{-1/q} \|\nabla v_h\|_{0,q,K_p^*}, \quad 1 \leq q \leq \infty.
\end{align}

The basic approach of our analysis is to consider the FVM as a perturbation of the FEM [19], [34], so we need to give the difference between $a_h(\omega; u_h, \Pi_{h}^* v_h)$ and $a(\omega; u_h, v_h)$.

In what follows, we will omit the variable $x$ in $a(x, u)$, except when its arising is necessary.

**Lemma 2.3.** For any $\omega \in H^1(\Omega)$, $w \in U_h + H^2(\Omega)$, $v_h \in U_h$, we have

\begin{align}
(2.21) \quad &a_h(\omega; w, \Pi_{h}^* v_h) - a(\omega; w, v_h) = \sum_{K \in T_h} \int_{\partial K} n \cdot (a(\omega)\nabla w)(\Pi_{h}^* v_h - v_h) \, ds \\
&\quad - \sum_{K \in T_h} \int_K \text{div}(a(\omega)\nabla w)(\Pi_{h}^* v_h - v_h) \, dx.
\end{align}

**Proof.** By Green’s formula, we have

$$
\int_K a(\omega)\nabla w \cdot \nabla v_h \, dx = - \int_K \text{div}(a(\omega)\nabla w)v_h \, dx + \int_{\partial K} n \cdot (a(\omega)\nabla w)v_h \, ds,
$$
and (see Figure 1)
\[
\sum_{K \in T_h} \int_K \text{div}(a(\omega)\nabla w) \Pi_h^* v_h \, dx = \sum_{K \in T_h} \sum_{K_p \in T_h} \int_{K_p \cap K} \text{div}(a(\omega)\nabla w) \Pi_h^* v_h \, dx
\]
\[
= \sum_{K \in T_h} \int_{\partial K} \mathbf{n} \cdot (a(\omega)\nabla w) \Pi_h^* v_h \, ds + \sum_{K_p \in T_h} \int_{\partial K_p^*} \mathbf{n} \cdot (a(\omega)\nabla w) \Pi_h^* v_h \, ds.
\]
Substituting the above two identities into the definitions of \(a(\omega; w, v_h)\) and \(a_h(\omega; w, \Pi_h^* v_h)\), the proof is completed.

The following lemma shows that the finite volume form \(a_h(\omega; u, \Pi_h^* v)\) is an \(h\)-perturbation of the finite element form \(a(\omega; u, v)\).

**Lemma 2.4** ([12]). There exists a positive constant \(C\) such that for \(\omega_h, u_h, v_h \in U_h\),
\[
|a(\omega_h; u_h, v_h) - a_h(\omega_h; u_h, \Pi_h^* v_h)| \leq C h \|\nabla \omega_h \cdot \nabla u_h\|_{0,p} \|v_h\|_{0,q},
\]
where \(1 \leq p, q \leq \infty, 1/p + 1/q = 1\).

**Lemma 2.5.** Let \(\omega_h \in U_h\) and \(\|\nabla \omega_h\|_{0,p} \leq M, p > 2\). Then for \(h\) small enough we have that
\[
a_h(\omega_h; u_h, \Pi_h^* u_h) \geq C \|\nabla u_h\|^2 \quad \forall u_h \in U_h.
\]

**Proof.** It follows from Lemma 2.4 and the inverse inequality that
\[
|a(\omega_h; u_h, u_h) - a_h(\omega_h; u_h, \Pi_h^* u_h)| \leq C h \|\nabla \omega_h \cdot \nabla u_h\| \|\nabla u_h\|
\]
\[
\leq C h |\nabla \omega|_{0,\infty} \|\nabla u_h\|^2 \leq C h^{1-2/p} \|\nabla \omega_h\|_{0,p} \|\nabla u_h\|^2 \leq C h^{1-2/p} M \|\nabla u_h\|^2.
\]
Next, from the condition (2.1) we can see that
\[
a(\omega_h; u_h, u_h) \geq \beta_1 \|\nabla u_h\|^2 \quad \forall \omega_h, u_h \in U_h.
\]
Thus, we obtain
\[
a_h(\omega_h; u_h, \Pi_h^* u_h) = a(\omega_h; u_h, u_h) + a_h(\omega_h; u_h, \Pi_h^* u_h) - a(\omega_h; u_h, u_h)
\]
\[
\geq (\beta_1 - C h^{1-2/p} M) \|\nabla u_h\|^2.
\]
This gives the conclusion of Lemma 2.5.
A direct calculation yields

\[
(2.23) \quad \text{div}(a(x, \omega) \nabla w) = (a'_{x_1}, a'_{x_2}) \cdot \nabla w + a'_{\omega} \nabla w \cdot \nabla w + a \text{ div } \nabla w.
\]

Particularly, for \( u_h \in U_h \), since \( \nabla u_h \) is a constant on \( K \), we have

\[
(2.24) \quad \text{div}(a(x, \omega) \nabla u_h) = \nabla a(x, \omega) \cdot \nabla u_h, \quad u_h \in U_h.
\]

An analysis of the FVM for solving (1.1) with \( a(x, u) = a(u) \) has been given in [12]. We note that the techniques in [12], together with the application of identity (2.24), can be used to establish the existence and uniqueness, and the \( H^1 \)- and \( L_2 \)-norm error estimates of the FVM solution to problem (1.1) with coefficient \( a(x, u) \). We collect these results in the following two lemmas without proof.

**Lemma 2.6** ([12]). Let \( M > 0 \) be such that \( \| f \| \leq M \alpha^{-1} \). Then, for \( h \) small enough, problem (2.10) has a solution \( u_h \) in the ball

\[
B_M = \{ v_h \in U_h : \| \nabla v_h \|_{0,p} \leq M, \ 2 < p \leq 2 + \delta \},
\]

where \( \alpha \) and \( \delta \) appear in the inf-sup condition [12]: there exist constants \( \alpha = \alpha(a(u), \Omega) \) and \( \delta = \delta(a(u), \Omega) \) such that for \( u_h \in U_h, \omega_h \in B_M \),

\[
\| \nabla u_h \|_{0,p} \leq \alpha \sup_{0 \neq v_h \in U_h} \frac{a_h(\omega_h; u_h, \Pi_h v_h)}{\| \nabla v_h \|_{0,q}}, \quad 2 < p \leq 2 + \delta, \ 1/p + 1/q = 1.
\]

Furthermore, if \( a'_{\omega} \) is Lipschitz continuous, then for sufficiently small data \( f \) and \( h \), the solution \( u_h \) is unique.

**Lemma 2.7** ([12]). Let \( u \) and \( u_h \) be the solutions of problems (1.1) and (2.10), respectively. Then, if \( \gamma = \alpha \beta \gamma M < 1 \), we have for \( h \) small enough that

\[
(2.25) \quad \| u - u_h \|_1 \leq Ch, \ u \in H^2(\Omega), \ f \in L_2(\Omega),
\]

\[
(2.26) \quad \| u - u_h \| \leq Ch^2, \ u \in W^{2,p}(\Omega), \ f \in H^1(\Omega), \ p > 2,
\]

where \( C = C(u, f) \) represents a positive constant which only depends on \( u \) and \( f \).

The \( H^1 \)- and \( L_2 \)-norm error estimates of the FVM solution in Lemma 2.7 will be used in our superconvergence analysis. An application of the \( H^1 \)-error estimate is to bound \( u_h \) in the \( W^{1,\infty} \)-norm.
Lemma 2.8. Let \( u \in W^{2,p}(\Omega), \ p > 2 \) and \( u_h \in U_h \) be the solutions of problems (1.1) and (2.10), respectively. Then, for \( h \) small enough, we have

\[
|u_h|_{1,\infty} \leqslant C(u).
\]

Proof. According to the inverse inequality, we have

\[
|u_h|_{1,\infty} \leqslant |u_h - \Pi_h u|_{1,\infty} + |\Pi_h u|_{1,\infty} \leqslant Ch^{-1}||u_h - \Pi_h u||_1 + C|u|_{1,\infty}.
\]

Note that \( |u|_{1,\infty} \leqslant C|u|_{2,p}, \ p > 2 \). In addition, from Lemma 2.7 we know

\[
||u_h - \Pi_h u||_1 \leqslant ||u_h - u||_1 + ||u - \Pi_h u||_1 \leqslant C(u)h + Ch||u||_2.
\]

Hence, substituting this estimate into (2.27), the proof is completed.

\[\Box\]

3. Superclose estimate of the interpolation function

In this section, we will give some superclose estimates for the interpolation function, which are very useful in our superconvergence analysis.

3.1. The interpolation weak estimate. We first establish the interpolation weak estimate. To this end, we need to strengthen the triangulation condition.

Definition 3.1. Let \( T_h \) be a quasi-uniform triangulation. Then \( T_h \) is called \( C \)-uniform if any two adjacent triangle elements of \( T_h \) form an approximate parallelogram in the sense that (see Figure 2)

\[
|P_1P_2 + P_3P_4| + |P_2P_3 + P_4P_1| = O(h^2).
\]

\[\text{Figure 2. An approximate parallelogram.}\]
It is well known that the interpolation weak estimate plays an important role in the superconvergence analysis of FEM [33], [35]. For FEM defined on $C$-uniform triangle meshes, the following interpolation weak estimate has been established (see, for example, [33], [35]): For any given $\omega \in W^{1,\infty}(\Omega)$,

$$
|a(\omega; u-\Pi_h u,v)| \leq C(\omega)h^2\|u\|_{3,p}\|v\|_{1,q} \quad \forall v \in U_h, \ 1 < p \leq \infty, \ 1/p + 1/q = 1,
$$

where $C(\omega)$ is a positive constant which only depends on $\|\omega\|_{1,\infty}$.

Now we give a similar weak estimate for the FVM.

**Theorem 3.1.** Let triangulation $T_h$ be $C$-uniform, and $u \in W^{3,p}(\Omega)$, $\omega \in W^{1,\infty}(\Omega)$. Then, we have

$$
|a_h(\omega; u-\Pi_h u, \Pi_h^* v)| \leq C(\omega)h^2\|u\|_{3,p}\|v\|_{1,q}
$$

$$
\forall v \in U_h, \ 1 < p \leq \infty, \ 1/p + 1/q = 1,
$$

where $C(\omega)$ is a positive constant which depends on $\|\omega\|_{1,\infty}$.

**Proof.** Let $a_M$ be the value of $a(x,\omega(x))$ at the midpoint of edge $\tau \subset \partial K$. Obviously, we have

$$
|a(x,\omega) - a_M| \leq h_K|a(x,\omega)|_{1,\infty} \leq C(\omega)h_K, \quad x \in \tau.
$$

From Lemma 2.3, we obtain for $v \in U_h$ that

$$
a_h(\omega; u-\Pi_h u, \Pi_h^* v) - a(\omega; u-\Pi_h u, v)
= \sum_{K \in T_h} \int_{\partial K} n \cdot (a(\omega) - a_M)\nabla(u-\Pi_h u)(\Pi_h^* v - v) \, ds
+ \sum_{K \in T_h} \int_{\partial K} n \cdot (a_M \nabla(u-\Pi_h u))(\Pi_h^* v - v) \, ds
+ \sum_{K \in T_h} (-\text{div}(a(\omega)\nabla(u-\Pi_h u)), \Pi_h^* v - v)_K
= E_1 + E_2 + E_3.
$$

We need to estimate $E_1 \sim E_3$. Using (2.12)–(2.13) and (2.18), we first obtain

$$
|E_1| \leq \sum_{K \in T_h} h_K|a(x,\omega)|_{1,\infty}\|\nabla(u-\Pi_h u)\|_{0,p,\partial K}\|v-\Pi_h^* v\|_{0,q,\partial K}
\leq C(\omega)h^2\|u\|_{2,p}\|v\|_{1,q}.
$$
Next, we write (noting that \((\Pi_h^* v - v)|_{\partial \Omega_h} = 0\))

\[
E_2 = \sum_{K \in T_h} \sum_{\tau \subset \partial K \setminus \partial \Omega_h} \int_{\tau} \mathbf{n} \cdot (a_M \nabla (u - \Pi_h u))(\Pi_h^* v - v) \, ds.
\]

Let \(\tau\) be an interior edge, that is, a common side of two adjacent elements \(K\) and \(K'\).
Since \(\mathbf{n}|_{\tau \cap \partial K} = -\mathbf{n}|_{\tau \cap \partial K'}\) and \(a_M \nabla u(\Pi_h^* v - v)\) is continuous across edge \(\tau\) (except for the midpoint), it follows from (2.16) and \(a_M \nabla \Pi_h u\) being constant that

\[
E_2 = \sum_{K \in T_h} \sum_{\tau \subset \partial K \setminus \partial \Omega_h} \int_{\tau} -\mathbf{n} \cdot (a_M \nabla \Pi_h u)(\Pi_h^* v - v) \, ds = 0.
\]

Finally, to estimate \(E_3\), let \(w^c\) be the piecewise constant approximation of function \(w\) on \(T_h\), which satisfies

\[
\|w - w^c\|_{0,p,K} \leq C h_K \|w\|_{1,p,K}, \quad 1 \leq p \leq \infty.
\]

Moreover, from (2.23) and (2.16) we find that

\[
E_3 = \sum_{K \in T_h} \left( -\nabla a(\omega) \cdot \nabla (u - \Pi_h u) - a(\omega) \text{div} \nabla (u - \Pi_h u), \Pi_h^* v - v \right)_K
\]

\[
= -\sum_{K \in T_h} \left( \nabla a(\omega) \cdot \nabla (u - \Pi_h u), \Pi_h^* v - v \right)_K
- \sum_{K \in T_h} \left( a(\omega) \text{div} \nabla u - (a(\omega) \text{div} \nabla u)^c, \Pi_h^* v - v \right)_K
\leq C|a(\omega)|_{1,\infty} \sum_{K \in T_h} \bar{h}_K (\|u\|_{2,p,K} + \|u\|_{3,p,K}) \|v - \Pi_h^* v\|_{0,q,K}
\leq C(\omega) h^2 \|u\|_{3,p} \|v\|_{1,q}.
\]

Substituting estimates \(E_1 \sim E_3\) into (3.4), we complete the proof by using (3.2). \(\Box\)

### 3.2. Superclose estimates for \(\Pi_h u - u_h\).

By means of the interpolation weak estimate in Theorem 3.1, we can obtain some important superclose results for \(\Pi_h u - u_h\) in the \(H^1\)- and \(W^{1,\infty}\)-norm.

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Thus, it follows from (3.8)–(3.9), the trace inequality, (2.20), and Lemma 2.7 that

\[
\|\Pi_h u - u_h\|_1 \leq C(u)h^2\|u\|_3.
\]

\textbf{Proof.} Let } \theta = u_h - \Pi_h u. \text{ Then, from Lemma 2.5 and the error equation (2.11) we can obtain that}

\[
C\|u_h - \Pi_h u\|_1^2 \leq a_h(u_h; u_h - \Pi_h u, \Pi_h^*(u_h - \Pi_h u))
\]

\[
= a_h(u_h; u_h - u, \Pi_h^*\theta) + a_h(u_h; u - \Pi_h u, \Pi_h^*\theta)
\]

\[
= a_h(u_h; u, \Pi_h^*\theta) - a_h(u_h; u, \Pi_h^*\theta) + a_h(u_h; u - \Pi_h u, \Pi_h^*\theta)
\]

\[
= a_h(u; u, \Pi_h^*\theta) - a_h(u_h; u, \Pi_h^*\theta) + a_h(u_h; u - \Pi_h u, \Pi_h^*\theta) = F_1 + F_2.
\]

To estimate } \(F_1 = a_h(u; u, \Pi_h^*\theta) - a_h(u_h; u, \Pi_h^*\theta)\), we need the following results:

\[
|a(u) - a(u_h)| = \left| \int_0^1 \frac{d}{dt}a(tu + (1 - t)u_h) \ dt \right|
\]

\[
= \left| \int_0^1 a'(t)u(tu + (1 - t)u_h) \ dt(u - u_h) \right| \leq |a'_u| \infty |u - u_h|.
\]

In addition, } \(a(u) - a(u_h))\nabla u \text{ and } \theta \text{ are continuous across } \partial K_p^*, \text{ which implies that}

\[
\sum_{K_p^* \in T_h} \int_{\partial K_p^*} \mathbf{n} \cdot (a(u) - a(u_h))\nabla u \theta \ ds = 0 \quad \forall \theta \in U_h.
\]

Thus, it follows from (3.8)–(3.9), the trace inequality, (2.20), and Lemma 2.7 that

\[
F_1 = - \sum_{K_p^* \in T_h} \int_{\partial K_p^*} \mathbf{n} \cdot (a(u) - a(u_h))\nabla u \Pi_h^*\theta \ ds
\]

\[
= - \sum_{K_p^* \in T_h} \int_{\partial K_p^*} \mathbf{n} \cdot (a(u) - a(u_h))\nabla u (\Pi_h^*\theta - \theta) \ ds
\]

\[
\leq |a'_u| \infty |\nabla u| \infty \sum_{K_p^* \in T_h} \|u - u_h\|_{0, \partial K_p^*} \|\Pi_h\theta - \theta\|_{0, \partial K_p^*}
\]

\[
\leq C \sum_{K_p^* \in T_h} (\|u - u_h\|_{0, K_p^*} + h\|\nabla (u - u_h)\|_{0, K_p^*})\|\theta\|_{1, K_p^*}
\]

\[
\leq C(\|u - u_h\| + h\|\nabla (u - u_h)\|)\|\theta\|_1 \leq C(u)h^2\|\theta\|_1.
\]

For } \(F_2\), from } \(\|u_h\|_{1, \infty} \leq C(u)\) \text{ and the weak estimate (3.3) we know that}

\[
F_2 = a_h(u_h; u - \Pi_h u, \Pi_h^*\theta) \leq C(u)h^2\|u\|_3\|\theta\|_1.
\]

Substituting estimates } \(F_1\) \text{ and } \(F_2\) \text{ into (3.7), the proof is completed.}
A direct result of Theorem 3.2 are the following optimal error estimates in the $L_p$- and $W^{1,p}$-norm.

**Theorem 3.3.** Let triangulation $T_h$ be $C$-uniform, and $u$ and $u_h$ be the solutions of problems (1.1) and (2.10), respectively. Then, we have

\begin{align}
\|u - u_h\|_{0,p} & \leq C_p(u)h^2, \quad u \in H^3(\Omega), \ 1 < p < \infty, \\
\|u - u_h\|_{0,\infty} & \leq C(u)|\ln h|^{1/2}h^2, \quad u \in H^3(\Omega) \cap W^{2,\infty}(\Omega), \\
\|u - u_h\|_{1,p} & \leq C(u)h, \quad u \in H^3(\Omega) \cap W^{2,p}(\Omega), \ 1 < p \leq \infty.
\end{align}

**Proof.** Using the embedding theory and (3.6), we obtain

\begin{align*}
\|u - u_h\|_{0,p} & \leq \|u - \Pi_h u\|_{0,p} + \|\Pi_h u - u_h\|_{0,p} \\
& \leq \|u - \Pi_h u\|_{0,p} + C_p\|\Pi_h u - u_h\|_1 \\
& \leq C_p h^2(\|u\|_{2,p} + \|u\|_3), \quad 1 < p < \infty.
\end{align*}

Furthermore, from the discrete embedding inequality in finite element space [35], we have

\[ \|v_h\|_{0,\infty} \leq C|\ln h|^{1/2}\|v_h\|_1 \quad \forall v_h \in U_h, \]

which, together with (3.6), yields that

\begin{align*}
\|u - u_h\|_{0,\infty} & \leq \|u - \Pi_h u\|_{0,\infty} + \|\Pi_h u - u_h\|_{0,\infty} \\
& \leq \|u - \Pi_h u\|_{0,\infty} + C|\ln h|^{1/2}\|\Pi_h u - u_h\|_1 \\
& \leq C(u)h^2(\|u\|_{2,\infty} + |\ln h|^{1/2}\|u\|_3).
\end{align*}

In addition, the inverse inequality implies that

\begin{align*}
\|u - u_h\|_{1,p} & \leq \|u - \Pi_h u\|_{1,p} + \|\Pi_h u - u_h\|_{1,\infty} \\
& \leq \|u - \Pi_h u\|_{1,p} + Ch^{-1}\|\Pi_h u - u_h\|_1 \\
& \leq C(u)(h\|u\|_{2,p} + h\|u\|_3).
\end{align*}

The proof is completed. \qed

**Remark 3.1.** Bi [6] gives the following $L_\infty$-norm error estimate:

\[ \|u - u_h\|_{0,\infty} \leq C(u)|\ln h|h^2. \]

Obviously, our result (3.11) is better than the above one.
Below we consider the superconvergence estimate in space $W^{1,\infty}(\Omega)$. To this end, we need to introduce the regularized Green function [33], [35].

For any given $z \in \Omega$, let $\delta_h^z \in U_h$ be the regularized $\delta$-function which satisfies

$$
(\delta_h^z, v_h) = v_h(z), \quad z \in \Omega \quad \forall \, v_h \in U_h.
$$

For any appointed direction $L$, define the direction derivative

$$
\partial_z v(z) = \lim_{\Delta z \to 0, \Delta z \parallel L} \frac{v(z + \Delta z) - v(z)}{|\Delta z|},
$$

where the notation $\Delta z \parallel L$ means that the increment $\Delta z$ is parallel to the direction $L$. Then, for given $\omega \in W^{1,\infty}(\Omega)$, there exists a regularized Green function of derivative type $\partial_z G^z(x) \in H^1_0(\Omega) \cap H^2(\Omega)$ such that

$$
a(\omega; v, \partial_z G^z) = (\partial_z \delta_h^z, v) \quad \forall \, v \in H^1_0(\Omega).
$$

Let $\partial_z G_h^z \in U_h$ be the finite element approximation of $\partial_z G^z$ such that

$$
a(\omega; v_h, \partial_z G^z - \partial_z G_h^z) = 0 \quad \forall \, v_h \in U_h.
$$

Clearly, we have

$$
(3.13) \quad a(\omega; v_h, \partial_z G_h^z) = a(\omega; v_h, \partial_z G^z) = (\partial_z \delta_h^z, v_h) = \partial_z v_h(z) \quad \forall \, v_h \in U_h.
$$

The following boundedness estimates have been given in [33], [35]:

$$
(3.14) \quad \|\partial_z G_h^z\|_1 \leq Ch^{-1}|\ln h|^{1/2}; \quad \|\partial_z G_h^z\|_{1,1} \leq C|\ln h|,
$$

where $C$ is a positive constant independent of $z \in \Omega$.

**Theorem 3.4.** Let triangulation $T_h$ be C-uniform, and $u$ and $u_h$ be the solutions of problems (1.1) and (2.10), respectively, and $u \in W^{3,\infty}(\Omega)$. Then, we have

$$
(3.15) \quad \|\Pi_h u - u_h\|_{1,\infty} \leq C h^2 |\ln h|^{3/2} \|u\|_{3,\infty}.
$$

**Proof.** Let $g_h^z = \partial_z G_h^z$. From (3.13) and Lemma 2.3 we have

$$
(3.16) \quad \partial_z (u_h - \Pi_h u)(z) = a(u_h; u_h - \Pi_h u, g_h^z)
$$

$$
= a(u_h; u_h - \Pi_h u, g_h^z) - a_h(\Pi_h u, u_h - \Pi_h u, \Pi_h g_h^z) + a_h(u_h; u_h - \Pi_h u, \Pi_h g_h^z) + \sum_{K \in T_h} \int_{\partial K} \mathbf{n} \cdot (a(u_h) \nabla (u_h - \Pi_h u))(g_h^z - \Pi_h g_h^z) \, ds
$$

$$
+ \sum_{K \in T_h} (-\text{div}(a(u_h) \nabla (u_h - \Pi_h u)), g_h^z - \Pi_h g_h^z) K
$$

$$
+ a_h(u_h; u_h - \Pi_h u, \Pi_h g_h^z) = S_1 + S_2 + S_3.
$$

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Thus, it follows from (2.16), (2.13), (2.18), Theorem 3.2, and (3.14) that
\[ S_1 = \sum_{K \in T_h} \int_{\partial K} n \cdot ((a(u_h) - a^c(u_h))(u_h - \Pi_h u))(g_h^* - \Pi_h g_h^*) \, ds \]
\[ \leq C \sum_{K \in T_h} h_K |a(u_h)|_{1,\infty} \|\nabla (u_h - \Pi_h u)\|_{0,\partial K} \|g_h^* - \Pi_h g_h^*\|_{0,\partial K} \]
\[ \leq C h^{1/2} |\nabla u_h|_{\infty} \|\nabla (u_h - \Pi_h u)\|_{1,\infty} \|g_h^*\|_1 \leq C h^2 |\ln h|^{1/2} \|u\|_3. \]

Moreover, from (2.24), (2.17), Theorem 3.2, and (3.14) we know
\[ S_2 \leq C \sum_{K \in T_h} |a(u_h)|_{1,\infty} \|u_h - \Pi_h u\|_{1,K} \|g_h^* - \Pi_h g_h^*\|_{0,K} \]
\[ \leq C(u) h^2 \|u\|_3 \|g_h^*\|_1 \leq C(u) h^2 |\ln h|^{1/2} \|u\|_3. \]

Furthermore, according to the error equation (2.11), we may write
\[ (3.17) \quad S_3 = a_h(u_h; u_h - \Pi_h u, \Pi_h g_h^*) \]
\[ = a_h(u_h; u_h - u, \Pi_h g_h^*) + a_h(u_h; u - \Pi_h u, \Pi_h g_h^*) \]
\[ = a_h(u_h; u_h, \Pi_h g_h^*) - a_h(u_h; u, \Pi_h g_h^*) + a_h(u_h; u - \Pi_h u, \Pi_h g_h^*) \]
\[ = a_h(u; u, \Pi_h g_h^*) - a_h(u_h; u, \Pi_h g_h^*) + a_h(u_h; u - \Pi_h u, \Pi_h g_h^*) \]
\[ = S_{31} + S_{32}. \]

From (3.8), (3.9), (2.20), and Theorem 3.3 we find that
\[ S_{31} = a_h(u; u, \Pi_h g_h^*) - a_h(u_h; u, \Pi_h g_h^*) \]
\[ = - \sum_{K_\infty \in T_h} \int_{\partial K_\infty} n \cdot (a(u) - a(u_h)) \nabla u \Pi_h^* g_h^* \, ds \]
\[ \leq |a'_u|_{\infty} |\nabla u|_{\infty} \|u - u_h\|_{0,\infty} \sum_{K_\infty \in T_h} \|\Pi_h^* g_h^* - g_h^*\|_{0,1,\partial K_\infty} \]
\[ \leq C(u) \|u - u_h\|_{0,\infty} \sum_{K_\infty \in T_h} \|g_h^*\|_{1,K_\infty} \]
\[ \leq C(u) h^2 |\ln h|^{1/2} \|g_h^*\|_{1,1} \leq C(u) h^2 |\ln h|^3/2. \]

For $S_{32}$, the weak estimate (3.3) implies that
\[ S_{32} = a_h(u_h; u - \Pi_h u, \Pi_h \theta) \leq C(u) h^2 \|u\|_{3,\infty} \|g_h^*\|_{1,1} \leq C(u) h^2 |\ln h|. \]
Hence,
\[ S_3 = S_{31} + S_{32} \leq C(u)h^2 |\ln h|^{3/2}. \]
Substituting estimates \( S_1 \sim S_3 \) into (3.16), we complete the proof. □

4. Mesh points and global superconvergence in \( W^{1,\infty} \)-norm

In this section, we first give the gradient superconvergence on mesh points. Then, by using the interpolation post-processing technique, we further derive a global superconvergence result, and establish an asymptotically exact a posteriori error estimator in the \( W^{1,\infty} \)-norm.

4.1. Gradient superconvergence on mesh points. Let \( P_0 \) be an interior nodal point surrounded by elements \( K_1, \ldots, K_m \), \( K_i = \triangle P_0P_1P_{i+1} \). Denote by \( N_i \) the midpoint of a common edge \( P_0P_{i+1} \) of two adjacent elements \( K_{i-1} \) and \( K_i \) (see Figure 3 for the \( m = 6 \) case). It is well known that if the triangle meshes are \( C \)-uniform, the midpoint \( N_i \) is the optimal stress point [35], that is (set \( K_0 = K_m \)),

\[
|\nabla u(N_i) - \frac{1}{2}(\nabla \Pi_h u|_{K_{i-1}} + \nabla \Pi_h u|_{K_i})| \leq Ch^2 \|u\|_{3,\infty,K_{i-1}\cup K_i}, \quad i = 1, \ldots, m.
\]

![Figure 3. C-uniform triangle elements around node P0.](image-url)

In the existing literature, the set of optimal stress points on triangle meshes only has the midpoints of the interior edges. See, for example, [26], [35]. Below we prove that any interior mesh point \( P \) is also an optimal stress point if \( C \)-uniform meshes are used.
Lemma 4.1. Assume that the triangulation $T_h$ is C-uniform. Then

$$\left| \frac{1}{m} \sum_{i=1}^{m} (N_i - P_0) \right| \leq Ch^2. \tag{4.2}$$

Proof. Let $P_{i_1} = P_{i_2}$ if $i_1 = i_2 \pmod{m}$, and vector $\overrightarrow{P_0P_i} = P_i - P_0$. Since $N_i = \frac{1}{2}(P_0 + P_i)$, we have

$$\frac{1}{m} \sum_{i=1}^{m} (N_i - P_0) = \frac{1}{2m} \sum_{i=1}^{m} (P_i - P_0) = \frac{1}{2m} \sum_{i=1}^{m} \overrightarrow{P_0P_i}. \tag{4.3}$$

By the vector operation rule and C-uniform condition (3.1), we can obtain (see Figure 3 for $m = 6$) that

$$\sum_{i=1}^{m} \overrightarrow{P_{i+1}P_{i+2}} = \overrightarrow{P_2P_3} + \overrightarrow{P_3P_4} + \overrightarrow{P_4P_5} + \overrightarrow{P_5P_6} + \overrightarrow{P_6P_1} + \overrightarrow{P_1P_2} = \overrightarrow{0},$$

and $|\overrightarrow{P_0P_i} + \overrightarrow{P_{i+1}P_{i+2}}| \leq Ch^2, \ i = 1, 2, \ldots, m$.

Hence,

$$\left| \sum_{i=1}^{m} \overrightarrow{P_0P_i} \right| = \left| \sum_{i=1}^{m} (\overrightarrow{P_0P_i} + \overrightarrow{P_{i+1}P_{i+2}}) \right| \leq Ch^2. \tag{4.4}$$

Now, Lemma 4.1 follows from (4.3) and (4.4).

Denote by $\overline{\nabla \Pi_h u(P)}$ the arithmetic mean of the gradient $\nabla \Pi_h u$ on elements containing mesh point $P$.

Theorem 4.1. Assume that the triangulation $T_h$ is C-uniform, and $u \in W^{3,\infty}(\Omega)$. Then, any interior mesh point $P$ is also an optimal stress point, that is,

$$|\nabla u(P) - \overline{\nabla \Pi_h u(P)}| \leq Ch^2 \|u\|_{3,\infty,E}, \tag{4.5}$$

where $E$ is the union of elements containing point $P$.

Proof. Let $P_0$ be an interior mesh point (see Figure 3). First, from (4.1) we have (setting $K_0 = K_m$) that

$$\left| \frac{1}{m} \sum_{i=1}^{m} \nabla u(N_i) - \nabla \Pi_h u(P_0) \right| = \left| \frac{1}{m} \sum_{i=1}^{m} \nabla u(N_i) - \frac{1}{m} \sum_{i=1}^{m} \nabla \Pi_h u|_{K_i} \right|$$

$$= \frac{1}{m} \left| \sum_{i=1}^{m} \nabla u(N_i) - \frac{1}{2} \left( \sum_{i=1}^{m} \nabla \Pi_h u|_{K_{i-1}} + \sum_{i=1}^{m} \nabla \Pi_h u|_{K_i} \right) \right|$$

$$\leq \frac{1}{m} \sum_{i=1}^{m} \left| \nabla u(N_i) - \frac{1}{2} (\nabla \Pi_h u|_{K_{i-1}} + \nabla \Pi_h u|_{K_i}) \right| \leq Ch^2 \|u\|_{3,\infty,E}.$$
Next, let \( \partial u = \partial_x u \) or \( \partial_y u \). Then, by means of the Taylor expansion, we obtain
\[
\partial u(N_i) = \partial u(P_0) + \nabla \partial u(P_0) \cdot (N_i - P_0) + R_i,
\]
where the remainder term \( |R_i| \leq C h^2 \| u \|_{3, \infty, K_{i-1} \cup K_i} \). Hence,
\[
\frac{1}{m} \sum_{i=1}^{m} (\partial u(N_i) - \partial u(P_0)) = \frac{1}{m} \nabla \partial u(P_0) \cdot \sum_{i=1}^{m} (N_i - P_0) + \frac{1}{m} \sum_{i=1}^{m} R_i.
\]
So, from Lemma 4.1 we have that
\[
\left| \frac{1}{m} \sum_{i=1}^{m} \nabla u(N_i) - \nabla u(P_0) \right| \leq C h^2 \| u \|_{3, \infty, E},
\]
which, together with (4.6), completes the proof. \( \square \)

Now, combining Theorems 3.4 and 4.1, we immediately obtain the following superconvergence result.

**Theorem 4.2.** Assume that the triangulation \( T_h \) is C-uniform, \( u \) and \( u_h \) are the solutions of problems (1.1) and (2.10), respectively, and \( u \in W^{3, \infty}(\Omega) \). Then, we have
\[
\max_{P \in S} |\nabla u(P) - \nabla u_h(P)| \leq C h^2 \ln h^{3/2} \| u \|_{3, \infty},
\]
where \( S \) is the set of all interior mesh points.

### 4.2. Global superconvergence in \( H^1 \)-norm

In order to derive the global superconvergence approximation to solution \( u \), we need to introduce the interpolation post-processing technique proposed by Lin et al. in [27].

Let \( T_{2h} \) be a coarser mesh triangulation of domain \( \Omega \) with the mesh size \( 2h \). The triangulation \( T_h \) is a refined triangulation of \( T_{2h} \) obtained by connecting the midpoints of all edges of elements in \( T_{2h} \). We assume that \( T_{2h} \) is C-uniform. It is easy to see that the triangulation \( T_h \) is C-uniform if \( T_{2h} \) is. Let \( Q_{2h} \) be the piecewise quadratic interpolation operator defined on \( T_{2h} \) by
\[
Q_{2h} u(z_i) = u(z_i), \quad i = 1, 2, \ldots
\]
where \( \{z_i\} \) are the vertices and the midpoints of edges of all elements in \( T_{2h} \). Note that each interpolation node of \( Q_{2h} \) is also the vertex of an element in \( T_h \).
Lemma 4.2 ([27]). The interpolation operator $Q_{2h}$ satisfies
\[ Q_{2h}u = Q_{2h}\Pi_h u, \quad u \in H^2(\Omega), \]
\[ \|u - Q_{2h}u\|_1 \leq Ch^2\|u\|_3, \]
\[ \|Q_{2h}v_h\|_1 \leq C\|v_h\|_1 \quad \forall v_h \in U_h. \]

Let $u_h$ be the FVM solution. Then, the interpolation post-processed solution is defined by $Q_{2h}u_h$.

**Theorem 4.3.** Let the triangulation $T_{2h}$ be $C$-uniform, $u$ and $u_h$ be the solutions of problems (1.1) and (2.10), respectively, and $u \in H^3(\Omega)$. Then, we have the following superconvergence estimate:
\[ \|u - Q_{2h}u_h\|_1 \leq Ch^2\|u\|_3. \]

**Proof.** From Lemma 4.2 we have
\[ \|u - Q_{2h}u_h\|_1 = \|u - Q_{2h}u + Q_{2h}\Pi_h u - Q_{2h}u_h\|_1 \leq \|u - Q_{2h}u\|_1 + \|Q_{2h}(\Pi_h u - u_h)\|_1 \leq Ch^2\|u\|_3 + C\|\Pi_h u - u_h\|_1. \]

The proof is completed by using Theorem 3.2. \hfill \Box

It is very important for a numerical method to have a computable a posteriori error bound, so that we can assess and enhance the accuracy of the numerical solution by an adaptive algorithm in practical applications. By virtue of the superconvergence result of Theorem 4.3, we can further derive an asymptotically exact a posteriori error estimator for the error $\|u - u_h\|_1$.

Define the error estimator $E(u_h) = \|u_h - Q_{2h}u_h\|_1$. Obviously, $E(u_h)$ is a computable quantity in terms of the FVM solution $u_h$.

**Theorem 4.4.** Assume that the conditions of Theorem 4.5 hold. Then, $E(u_h)$ is an asymptotically exact a posteriori error estimator for the error $\|u - u_h\|_1$, namely
\[ \lim_{h \to 0} \frac{\|u_h - Q_{2h}u_h\|_1}{\|u - u_h\|_1} = 1. \]

**Proof.** Using the triangle inequality, we obtain
\[ \|u - u_h\|_1 - \|u - Q_{2h}u_h\|_1 \leq E(u_h) \leq \|u - u_h\|_1 + \|u - Q_{2h}u_h\|_1, \]
or

\begin{equation}
1 - \frac{\|u - Q_{2h}u_h\|_1}{\|u - u_h\|_1} \leq \frac{E(u_h)}{\|u - u_h\|_1} \leq 1 + \frac{\|u - Q_{2h}u_h\|_1}{\|u - u_h\|_1}.
\end{equation}

From Theorem 4.3, we know that $\|u - Q_{2h}u_h\|_1 = O(h^2)$. However, for the FVM solution, generally speaking, we only have $\|u - u_h\|_1 = O(h)$. Thus, letting $h \to 0$ in (4.9), the proof is completed.

\section{5. Numerical example}

In this section, we will present a numerical example to illustrate the theoretical analysis.

Let us consider problem (1.1) with the data:

$$a(x, u) = 1 + \frac{x_1 x_2}{1 + u^2}, \quad u(x) = x_1^{3.5} \ln x_1 \cdot x_2^{3.5} \ln x_2,$$

and the source term $f = -\text{div}(a(x, u) \nabla u)$. For simplicity, we take $\Omega = [0, 1]^2$.

In the numerical experiment, we first partition the domain $\Omega$ into square meshes with mesh size $h = 1/N$, and then we obtain the $C$-uniform triangle meshes by perturbing randomly the inner nodes of the square meshes within $h^2$ and dividing each derived quadrilateral into two triangles (see Figure 4). The refined meshes are obtained by successively halving the mesh size $h$. The finite volume equation (2.10) is reduced to a nonlinear system of algebraic equations $A(U)U = F$, where $U$ is a vector whose entries are the values of $u_h$ at the mesh points. We have used a fixed point type iteration to solve this system, that is, we solve the linearization system $A(U^{(k-1)})U^{(k)} = F$, where $U^{(k-1)}$ is the previous iteration vector. The fixed point iteration continues until a tolerance of $|U^{(k)} - U^{(k-1)}| < 10^{-6}$ or $|A(U^{(k-1)})U^{(k)} - F| < 10^{-6}$ is reached.

Figure 4. $C$-uniform triangle meshes.
Let $e_h = \max_{P \in S} |\nabla u(P) - \nabla u_h(P)|$ (or $e_h = \|u - Q_{2h}u_h\|_1$) be the computational error with mesh size $h$. The numerical convergence rate is computed by using the formula $r = \ln(e_h/e_{h/2})/\ln 2$. Denote by $\sigma = \|u_h - Q_{2h}u_h\|_1/\|u - u_h\|_1$ the efficiency index of an a posteriori error estimator $\|u_h - Q_{2h}u_h\|_1$. Table I gives the numerical results with successively halved mesh size $h$. We see that for approximations $\nabla u_h$ and $Q_{2h}u_h$, a convergence rate of $O(h^2)$-order is achieved as the theoretical prediction and the a posteriori error estimator is efficient, that is, $\sigma \approx 1$, as $h \to 0$.

| mesh $h$ | max $|\nabla u(P) - \nabla u_h(P)|$ | $\|u - Q_{2h}u_h\|_1$ | estimator $\sigma$-index |
|---------|-------------------------------|-----------------|------------------|
| 1/8     | 0.6635                         | 0.3421          | 3.82             |
| 1/16    | 1.6796e-1                      | 8.6479e-2       | 1.984            | 2.21             |
| 1/32    | 4.2428e-2                      | 2.1815e-2       | 1.987            | 2.12             |
| 1/64    | 1.0710e-2                      | 0.5488e-2       | 1.991            | 1.13             |
| 1/128   | 2.7041e-3                      | 1.3801e-3       | 1.992            | 1.11             |

Table I. The convergence rate and the estimator.

References


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