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OSCILLATION CONDITIONS FOR DIFFERENCE EQUATIONS WITH SEVERAL VARIABLE ARGUMENTS

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Abstract. Consider the difference equation

$$\Delta x(n)+\sum_{i=1}^{m}p_i(n)x(\tau_i(n))=0,\quad n\geqslant 0\quad\Bigl[\nabla x(n)-\sum_{i=1}^{m}p_i(n)x(\sigma_i(n))=0,\quad n\geqslant 1\Bigr],$$

where $(p_i(n))$, $1\leqslant i\leqslant m$ are sequences of nonnegative real numbers, $\tau_i(n)$ $[\sigma_i(n)]$, $1\leqslant i\leqslant m$ are general retarded (advanced) arguments and Δ $[\nabla]$ denotes the forward (backward) difference operator $\Delta x(n)=x(n+1)-x(n)$ $[\nabla x(n)=x(n)-x(n-1)]$. New oscillation criteria are established when the well-known oscillation conditions

$$\limsup_{n\to\infty}\sum_{i=1}^{m}\sum_{j=\tau(n)}^{n}p_i(j)>1\quad\Bigl[\limsup_{n\to\infty}\sum_{i=1}^{m}\sum_{j=n}^{\sigma(n)}p_i(j)>1\Bigr]$$

and

$$\liminf_{n\to\infty}\sum_{i=1}^{m}\sum_{j=\tau_i(n)}^{n-1}p_i(j)>\frac{1}{\mathrm{e}}\quad\Bigl[\liminf_{n\to\infty}\sum_{i=1}^{m}\sum_{j=n+1}^{\sigma_i(n)}p_i(j)>\frac{1}{\mathrm{e}}\Bigr]$$

are not satisfied. Here $\tau(n)=\max\limits_{1\leqslant i\leqslant m}\tau_i(n)$ $[\sigma(n)=\min\limits_{1\leqslant i\leqslant m}\sigma_i(n)]$. The results obtained essentially improve known results in the literature. Examples illustrating the results are also given.

Keywords: difference equation; retarded argument; advanced argument; oscillatory solution; nonoscillatory solution

MSC 2010: 39A10, 39A21

1. Introduction

Consider the difference equation with several variable arguments of the form

$$\text{(E)} \qquad \Delta x(n) + \sum_{i=1}^{m} p_i(n)x(\tau_i(n)) = 0 \quad \Big[\nabla x(n) - \sum_{i=1}^{m} p_i(n)x(\sigma_i(n)) = 0\Big],$$

for every $n \in \mathbb{N}_0$ $[n \in \mathbb{N}]$, where $(p_i(n))$, $1 \leqslant i \leqslant m$ are sequences of nonnegative real numbers, $(\tau_i(n))$, $1 \leqslant i \leqslant m$ are sequences of integers such that

$$\text{(1.1)} \qquad \tau_i(n) \leqslant n-1, \quad n \in \mathbb{N}_0, \quad \text{and} \quad \lim_{n\to\infty} \tau_i(n) = \infty, \quad 1 \leqslant i \leqslant m,$$

$(\sigma_i(n))$, $1 \leqslant i \leqslant m$ are sequences of integers such that

$$\text{(1.2)} \qquad \sigma_i(n) \geqslant n+1, \quad n \in \mathbb{N}, \quad 1 \leqslant i \leqslant m,$$

Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$ and ∇ denotes the backward difference operator $\nabla x(n) = x(n) - x(n-1)$.

If $\tau_i(n) = n - k_i$ and $\sigma_i(n) = n + k_i$, where $k_i > 0$, $1 \leqslant i \leqslant m$, then equation (E) reduces to the difference equation with several constant arguments of the form

$$\text{(E}'\text{)} \qquad \Delta x(n) + \sum_{i=1}^{m} p_i(n)x(n-k_i) = 0 \quad \Big[\nabla x(n) - \sum_{i=1}^{m} p_i(n)x(n+k_i) = 0\Big].$$

Strong interest in equation (E) is motivated by the fact that it represents a discrete analogue of the differential equation with several variable arguments (see, for example, [7], [11] and the references cited therein)

$$x'(t) + \sum_{i=1}^{m} p_i(t)x(\tau_i(t)) = 0 \quad \Big[x'(t) - \sum_{i=1}^{m} p_i(t)x(\sigma_i(t)) = 0\Big],$$

for every $t \geqslant 0$ $[t \geqslant 1]$, where, for every $i \in \{1, \dots, m\}$, p_i is a nonnegative continuous real-valued function in the interval $[0, \infty)$, τ_i is a continuous real-valued function on $[0, \infty)$ such that $\tau_i(t) \leqslant t$, $t \geqslant 0$ and $\lim_{t\to\infty} \tau_i(t) = \infty$, and σ_i is a continuous real-valued function on $[1, \infty)$ such that $\sigma_i(t) \geqslant t$, $t \geqslant 1$.

By a *solution of the retarded difference equation* (E), we mean a sequence of real numbers $(x(n))_{n\geqslant -w}$ which satisfies (E) for all $n \geqslant 0$. Here, $w = -\min_{n\geqslant 0,\, 1\leqslant i\leqslant m} \tau_i(n)$.

It is clear that for each choice of real numbers $c_{-w}, c_{-w+1}, \dots, c_{-1}, c_0$, there exists a unique solution $(x(n))_{n\geqslant -w}$ of (E) which satisfies the initial conditions $x(-w) = c_{-w}$, $x(-w+1) = c_{-w+1}, \dots, x(-1) = c_{-1}$, $x(0) = c_0$.

By a *solution of the advanced difference equation* (E), we mean a sequence of real numbers $(x(n))_{n\geqslant 0}$ which satisfies (E) for all $n \geqslant 1$.

A solution $(x(n))_{n\geqslant -w}$ (or $(x(n))_{n\geqslant 0}$) of the difference equation (E) is called *oscillatory*, if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be *nonoscillatory*.

For the general theory of difference equations the reader is referred to the monographs [1], [13], [14].

In the last few decades, the asymptotic and oscillatory behavior of the solutions of difference equations has been extensively studied. See, for example, [2]–[12], [15]–[21] and the references cited therein. Most of these papers concern the special case of the equation (E′) with $m = 1$, while a small number of the papers deal with the general case of the equation (E) with $m = 1$, in which the arguments $(n-\tau_i(n))_{n\geqslant 0}$, $(\sigma_i(n)-n)_{n\geqslant 1}$, $1 \leqslant i \leqslant m$ are variable.

In 1989 Erbe and Zhang [10], in 1999 Tang and Yu [19], and in 2001 Tang and Zhang [20] proved that either of the following conditions

$$\sum_{i=1}^{m}\Big(\liminf_{n\to\infty} p_i(n)\Big)\frac{(k_i+1)^{k_i+1}}{(k_i)^{k_i}} > 1, \tag{1.3}$$

$$\liminf_{n\to\infty}\sum_{i=1}^{m}\Big(\frac{k_i+1}{k_i}\Big)^{k_i+1}\sum_{j=n+1}^{n+k_i} p_i(j) > 1, \tag{1.4}$$

or

$$\limsup_{n\to\infty}\sum_{i=1}^{m}\sum_{j=n}^{n+k_i} p_i(j) > 1, \tag{1.5}$$

implies that all solutions of the retarded difference equation (E′) oscillate, while in 2002 Li and Zhu [15] proved that if

$$\liminf_{n\to\infty}\sum_{i=1}^{m}\Big(\frac{k_i+1}{k_i}\Big)^{k_i+1}\sum_{j=n-k_i}^{n-1} p_i(j) > 1, \tag{1.6}$$

then all solutions of the advanced difference equation (E′) oscillate.

Set

$$\tau(n) = \max_{1\leqslant i\leqslant m} \tau_i(n), \quad n \in \mathbb{N}_0, \tag{1.7}$$

$$\sigma(n) = \min_{1\leqslant i\leqslant m} \sigma_i(n), \quad n \in \mathbb{N}. \tag{1.8}$$

In 2005 Yan, Meng and Yan [21], and in 2006 Berezansky and Braverman [5] proved that if

$$\liminf_{n\to\infty}\sum_{j=\tau(n)}^{n-1}\sum_{i=1}^{m}p_i(j)\Big(\frac{n-\tau_i(j)+1}{n-\tau_i(j)}\Big)^{n-\tau_i(j)+1}>1, \tag{1.9}$$

or

$$\limsup_{n\to\infty}\sum_{i=1}^{m}p_i(n)>0 \quad\text{and}\quad \liminf_{n\to\infty}\sum_{i=1}^{m}\sum_{j=\tau(n)}^{n-1}p_i(j)>\frac{1}{\mathrm{e}}, \tag{1.10}$$

then all solutions of the retarded difference equation (E) oscillate.

Recently, Chatzarakis, Pinelas and Stavroulakis [9] proved that if

$$\limsup_{n\to\infty}\sum_{i=1}^{m}\sum_{j=\tau(n)}^{n}p_i(j)>1 \quad \Big[\limsup_{n\to\infty}\sum_{i=1}^{m}\sum_{j=n}^{\sigma(n)}p_i(j)>1\Big], \tag{1.11}$$

or, $\limsup\limits_{n\to\infty}\sum\limits_{i=1}^{m}p_i(n)>0$ and

$$\liminf_{n\to\infty}\sum_{i=1}^{m}\sum_{j=\tau_i(n)}^{n-1}p_i(j)>\frac{1}{\mathrm{e}} \quad \Big[\liminf_{n\to\infty}\sum_{i=1}^{m}\sum_{j=n+1}^{\sigma_i(n)}p_i(j)>\frac{1}{\mathrm{e}}\Big], \tag{1.12}$$

then all solutions of equation (E) oscillate.

Very recently, Chatzarakis et al. [7] established the following theorem.

Theorem 1.1 (See [7], Theorems 2.1 and 3.1)**.** *Assume that the sequences* $(\tau_i(n))$ $[(\sigma_i(n))]$, $1\leqslant i\leqslant m$ *are increasing,* (1.1), [(1.2)] *holds, and*

$$\alpha=\min\{\alpha_i\colon\ 1\leqslant i\leqslant m\}, \tag{1.13}$$

where

$$\alpha_i=\liminf_{n\to\infty}\sum_{j=\tau_i(n)}^{n-1}p_i(j) \quad \Big[\alpha_i=\liminf_{n\to\infty}\sum_{j=n+1}^{\sigma_i(n)}p_i(j)\Big]. \tag{1.14}$$

If $0<\alpha\leqslant 1/\mathrm{e}$, *and*

$$\limsup_{n\to\infty}\sum_{i=1}^{m}\sum_{j=\tau(n)}^{n}p_i(j),\ \limsup_{n\to\infty}\sum_{i=1}^{m}\sum_{j=n}^{\sigma(n)}p_i(j)>1-(1-\sqrt{1-\alpha})^2, \tag{1.15}$$

then all solutions of (E) *oscillate.*

If, additionally,

$$p_i(n) \geqslant 1 - \sqrt{1-\alpha} \quad \text{for all large } n, \quad 1 \leqslant i \leqslant m \tag{1.16}$$

and

$$\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n} p_i(j), \ \limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=n}^{\sigma(n)} p_i(j) > 1 - \alpha\left(\frac{1}{3\sqrt{1-\alpha}+\alpha-2} - 1\right), \tag{1.17}$$

then all solutions of (E) *oscillate.*

In this paper, our main objective is to improve the upper bound of the ratio $x(n+1)/x(\tau(n))$ $[x(n-1)/x(\sigma(n))]$ for possible nonoscillatory solutions $(x(n))_{n\geqslant -k}$ $[(x(n))_{n\geqslant 0}]$ of equation (E) and derive new oscillation conditions for all solutions of (E). Examples illustrating the results are also given.

2. Oscillation criteria

In this section, first a lemma is presented, which will be used in the proof of our main results. This lemma is an extension of Lemma 2.1 in [8] for the case of the difference equation (E) with several retarded or advanced arguments.

Lemma 2.1 (cf. [8]). *Assume that the sequences* $(\tau_i(n))$, $[(\sigma_i(n))]$, $1 \leqslant i \leqslant m$ *are increasing,* (1.1), [(1.2)] *holds,* $(\tau(n))$, $[(\sigma(n))]$ *is defined by* (1.7), [(1.8)] $(x(n))$ *is a nonoscillatory solution of* (E)*, and* α *is defined by* (1.13).

If $0 < \alpha \leqslant -1 + \sqrt{2}$, *then*

$$\liminf_{n\to\infty} \frac{x(n+1)}{x(\tau(n))}, \ \liminf_{n\to\infty} \frac{x(n-1)}{x(\sigma(n))} \geqslant \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}. \tag{2.1}$$

If, additionally,

$$p_i(n) \geqslant \frac{\alpha}{2} \quad \text{for all large } n, \ 1 \leqslant i \leqslant m, \tag{2.2}$$

then

$$\liminf_{n\to\infty} \frac{x(n+1)}{x(\tau(n))}, \ \liminf_{n\to\infty} \frac{x(n-1)}{x(\sigma(n))} \geqslant 2\frac{1-\sqrt{1-2\alpha-\alpha^2}}{2+\alpha} - \alpha. \tag{2.3}$$

P r o o f. The proof below refers to the retarded difference equation (E). The proof for the advanced difference equation (E) follows by a similar procedure and is omitted.

Define for $n \leqslant t < n+1,\ n \in \mathbb{N}_0,\ 1 \leqslant i \leqslant m$

$$q_i(t) = p_i(n) \quad \text{and} \quad \varphi_i(t) = \tau_i(n).$$

Clearly, $q_i,\ \varphi_i,\ 1 \leqslant i \leqslant m$ are nonnegative real-valued functions on the interval $[0, \infty)$, which are continuous on each of the intervals $(n, n+1)$ for $n = 0, 1, \ldots$ We can immediately see that

$$\varphi_i(t) < t \quad \text{for all } t \geqslant 0, \quad \text{and} \quad \lim_{t\to\infty} \varphi_i(t) = \infty, \quad 1 \leqslant i \leqslant m$$

and the functions φ_i are increasing on $[0, \infty)$.

Suppose that

$$\varphi(t) = \max_{1\leqslant i\leqslant m} \varphi_i(t) = \max_{1\leqslant i\leqslant m} \tau_i(n) = \tau(n) \quad \text{for } n \leqslant t < n+1. \tag{2.4}$$

(Clearly, the function φ is increasing.)

Let $(x(n))_{n\geqslant -w}$ be a solution of the retarded difference equation (E). We define

$$y(t) = x(n) + (\Delta x(n))(t-n), \quad n \leqslant t < n+1,\ n = -w, -w+1, \ldots$$

It is obvious that $y(n) = x(n)$ for all $n \geqslant -w$. Moreover, it is easy to verify that the real-valued function y is continuous on the interval $[-w, \infty)$. Also, we see that y is continuously differentiable on each of the intervals $(n, n+1)$ for $n = -w, -w+1, \ldots$ with

$$y'(t) = \Delta x(n) \quad \text{for} \quad n < t < n+1,\ n = -w, -w+1, \ldots$$

Furthermore, as $(x(n))_{n\geqslant -w}$ satisfies (E) for all $n \geqslant 0$, we can easily conclude that the function y satisfies

$$y'(t) + \sum_{i=1}^{m} q_i(t) y(\varphi_i(t)) = 0 \quad \text{for } n < t < n+1,\ n = 0, 1, \ldots \tag{2.5}$$

Since the solution $(x(n))_{n\geqslant -w}$ of (E) is nonoscillatory, it is either eventually positive or eventually negative. As $(-(x(n)))_{n\geqslant -w}$ is also a solution of (E), we may restrict ourselves only to the case where $x(n) > 0$ for all large n. Let $n_1 \geqslant -w$ be an integer such that $x(n) > 0$ for all $n \geqslant n_1$. Then, there exists $n_2 \geqslant n_1$ such that $x(\tau_i(n)) > 0$ for all $n \geqslant n_2$, $1 \leqslant i \leqslant m$. In view of this, equation (E) becomes

$$\Delta x(n) = -\sum_{i=1}^{m} p_i(n) x(\tau_i(n)) \leqslant 0, \quad n \geqslant n_2,$$

which means that the sequence $(x(n))$ is eventually decreasing. Furthermore, it is not difficult to conclude that the function y is positive on the interval $[n_1, \infty)$ and that y is decreasing on $[n_2, \infty)$.

Consider an arbitrary real number ε with $0 < \varepsilon < \alpha_i$, where α_i is defined by (1.14). Then we can choose an integer $n_0 > n_2$ such that $\tau_i(n) \geqslant n_2$ for $n \geqslant n_0$, and

$$\sum_{j=\tau_i(n)}^{n-1} p_i(j) > \alpha_i - \varepsilon \geqslant \alpha - \varepsilon, \quad n \geqslant n_0, \ 1 \leqslant i \leqslant m.$$

For any point $t \geqslant n_0$, there exists an integer $n \geqslant n_0$ such that $n \leqslant t < n+1$, and consequently

$$\int_{\varphi_i(t)}^{t} q_i(s)\,\mathrm{d}s = \int_{\tau_i(n)}^{t} q_i(s)\,\mathrm{d}s \geqslant \int_{\tau_i(n)}^{n} q_i(s)\,\mathrm{d}s = \sum_{j=\tau_i(n)}^{n-1} p_i(j) > \alpha_i - \varepsilon,$$

or

$$\int_{\varphi_i(t)}^{t} q_i(s)\,\mathrm{d}s > \alpha - \varepsilon, \quad t \geqslant n_0, \ 1 \leqslant i \leqslant m. \tag{2.6}$$

Now we shall show that for each point $t \geqslant n_0$, there exists a $t^* > t$ such that $\varphi_i(t^*) < t$, and

$$\int_{t}^{t^*} q_i(s)\,\mathrm{d}s = \alpha - \varepsilon. \tag{2.7}$$

Indeed, let us consider an arbitrary point $t \geqslant n_0$. Set

$$f_i(\varrho) = \int_{t}^{\varrho} q_i(s)\,\mathrm{d}s \quad \text{for } \varrho \geqslant t.$$

We see that $f_i(t) = 0$. Moreover, it is not difficult to show that (2.6) guarantees that $\int_0^\infty q_i(s)\,\mathrm{d}s = \infty$ and, in particular, $\int_t^\infty q_i(s)\,\mathrm{d}s = \infty$, i.e., $\lim_{\varrho\to\infty} f_i(\varrho) = \infty$. Thus, as the function f_i is continuous on the interval $[t, \infty)$, there always exists a point $t^* > t$ such that $f_i(t^*) = \alpha - \varepsilon$, i.e., such that (2.7) is satisfied. Using (2.6) (for the point t^*) as well as (2.7), we obtain

$$\int_{\varphi_i(t^*)}^{t} q_i(s)\,\mathrm{d}s = \int_{\varphi_i(t^*)}^{t^*} q_i(s)\,\mathrm{d}s - \int_{t}^{t^*} q_i(s)\,\mathrm{d}s > (\alpha - \varepsilon) - (\alpha - \varepsilon) = 0,$$

which means that $\varphi_i(t^*) < t$.

Now, we choose an integer $n_3 > n_0$ such that $\tau_i(n) \geqslant n_0$ for all $n \geqslant n_3$. Let us consider an arbitrary point $t \geqslant n_3$. Then there exists a $t^* > t$ such that $\varphi_i(t^*) < t$, and (2.7) holds. From (2.5) it follows that

$$y(t) = y(t^*) + \sum_{i=1}^{m} \int_t^{t^*} q_i(s) y(\varphi_i(s))\,\mathrm{d}s. \tag{2.8}$$

Let s be any point with $t \leqslant s \leqslant t^*$. As the function φ is increasing on $[0,\infty)$, we have $n_0 \leqslant \varphi(t) \leqslant \varphi(s) \leqslant \varphi(t^*) < t$, and $n_2 \leqslant \varphi(u) \leqslant \varphi(t)$ for every u with $\varphi(s) \leqslant u \leqslant t$. Thus, by taking into account the fact that the function y is decreasing on $[n_2,\infty)$, from (2.5) we obtain

$$\begin{aligned} y(\varphi(s)) &= y(t) + \sum_{i=1}^{m} \int_{\varphi(s)}^{t} q_i(u) y(\varphi_i(u))\,\mathrm{d}u \\ &\geqslant y(t) + \sum_{i=1}^{m} \int_{\varphi(s)}^{t} q_i(u) y(\varphi(u))\,\mathrm{d}u \geqslant y(t) + \sum_{i=1}^{m} \Bigl(\int_{\varphi(s)}^{t} q_i(u)\,\mathrm{d}u\Bigr) y(\varphi(t)) \\ &= y(t) + \sum_{i=1}^{m} \Bigl(\int_{\varphi(s)}^{s} q_i(u)\,\mathrm{d}u - \int_t^s q_i(u)\,\mathrm{d}u\Bigr) y(\varphi(t)). \end{aligned}$$

So, by applying (2.6) (for the point s), we get

$$y(\varphi(s)) > y(t) + \Bigl(m(\alpha-\varepsilon) - \sum_{i=1}^{m} \int_t^s q_i(u)\,\mathrm{d}u\Bigr) y(\varphi(t)). \tag{2.9}$$

As this inequality holds true for all s with $t \leqslant s \leqslant t^*$, combining (2.8) and (2.9) we have

$$\begin{aligned} y(t) &= y(t^*) + \sum_{i=1}^{m} \int_t^{t^*} q_i(s) y(\varphi_i(s))\,\mathrm{d}s \geqslant y(t^*) + \sum_{i=1}^{m} \int_t^{t^*} q_i(s) y(\varphi(s))\,\mathrm{d}s \\ &> y(t^*) + \sum_{i=1}^{m} \int_t^{t^*} q_i(s) \Bigl(y(t) + \Bigl(m(\alpha-\varepsilon) - \sum_{i=1}^{m} \int_t^s q_i(u)\,\mathrm{d}u\Bigr) y(\varphi(t))\Bigr)\,\mathrm{d}s \\ &= y(t^*) + \Bigl(\sum_{i=1}^{m} \int_t^{t^*} q_i(s)\,\mathrm{d}s\Bigr) y(t) + \Bigl\{ m(\alpha-\varepsilon) \sum_{i=1}^{m} \int_t^{t^*} q_i(s)\,\mathrm{d}s \\ &\quad - \sum_{i=1}^{m}\sum_{i=1}^{m} \int_t^{t^*} q_i(s) \Bigl(\int_t^s q_i(u)\,\mathrm{d}u\Bigr)\,\mathrm{d}s \Bigr\} y(\varphi(t)) \end{aligned} \tag{2.10}$$

or

$$\begin{aligned} y(t) &> y(t^*) + m(\alpha-\varepsilon) y(t) \\ &\quad + \Bigl\{ m^2(\alpha-\varepsilon)^2 - \sum_{i=1}^{m}\sum_{i=1}^{m} \int_t^{t^*} q_i(s) \Bigl(\int_t^s q_i(u)\,\mathrm{d}u\Bigr)\,\mathrm{d}s \Bigr\} y(\varphi(t)). \end{aligned}$$

Noting the known formula

$$\int_t^{t^*} q_i(s)\Big(\int_t^s q_i(u)\,\mathrm{d}u\Big)\,\mathrm{d}s = \int_t^{t^*} q_i(u)\Big(\int_u^{t^*} q_i(s)\,\mathrm{d}s\Big)\,\mathrm{d}u$$

or

$$\int_t^{t^*} q_i(s)\Big(\int_t^s q_i(u)\,\mathrm{d}u\Big)\,\mathrm{d}s = \int_t^{t^*} q_i(s)\Big(\int_s^{t^*} q_i(u)\,\mathrm{d}u\Big)\,\mathrm{d}s,$$

we have

$$\begin{aligned}(2.11)\quad &\int_t^{t^*} q(s)\Big(\int_t^s q(u)\,\mathrm{d}u\Big)\,\mathrm{d}s \\ &= \frac{1}{2}\Big\{\int_t^{t^*} q(s)\Big(\int_t^s q(u)\,\mathrm{d}u\Big)\,\mathrm{d}s + \int_t^{t^*} q(s)\Big(\int_s^{t^*} q(u)\,\mathrm{d}u\Big)\,\mathrm{d}s\Big\} \\ &= \frac{1}{2}\int_t^{t^*} q(s)\Big(\int_t^s q(u)\,\mathrm{d}u + \int_s^{t^*} q(u)\,\mathrm{d}u\Big)\,\mathrm{d}s \\ &= \frac{1}{2}\int_t^{t^*} q(s)\Big(\int_t^{t^*} q(u)\,\mathrm{d}u\Big)\,\mathrm{d}s = \frac{1}{2}\Big(\int_t^{t^*} q(s)\,\mathrm{d}s\Big)^2 = \frac{1}{2}(\alpha-\varepsilon)^2.\end{aligned}$$

Combining (2.10) and (2.11) we have

$$y(t) > y(t^*) + m(\alpha-\varepsilon)y(t) + \Big(m^2(\alpha-\varepsilon)^2 - \frac{m^2}{2}(\alpha-\varepsilon)^2\Big)y(\varphi(t)).$$

Since $m \geqslant 1$, the last inequality guarantees that

$$y(t) > y(t^*) + (\alpha-\varepsilon)y(t) + \frac{1}{2}(\alpha-\varepsilon)^2 y(\varphi(t)). \tag{2.12}$$

Therefore

$$y(t) > \frac{(\alpha-\varepsilon)^2}{2(1-(\alpha-\varepsilon))}y(\varphi(t)) = \lambda_1 y(\varphi(t)), \quad t \geqslant n_3, \tag{2.13}$$

where $\lambda_1 = (\alpha-\varepsilon)^2/(2(1-(\alpha-\varepsilon)))$.

Let us again consider an arbitrary point $t \geqslant n_3$. Then there exists a $t^* > t$ such that $\varphi(t^*) < t$, and (2.7) holds. Then (2.12) is also fulfilled. Moreover, in view of (2.13) (for the point t^*), we have

$$y(t^*) > \lambda_1 y(\varphi(t^*)) \geqslant \lambda_1 y(t)$$

and hence (2.12) yields

$$y(t) > \lambda_1 y(t) + (\alpha-\varepsilon)y(t) + \frac{1}{2}(\alpha-\varepsilon)^2 y(\varphi(t))$$

or

$$(1-(\alpha-\varepsilon)-\lambda_1)y(t) > \frac{1}{2}(\alpha-\varepsilon)^2 y(\varphi(t)).$$

This implies, in particular, that $1-(\alpha-\varepsilon)-\lambda_1 > 0$. Consequently,

$$y(t) > \frac{(\alpha-\varepsilon)^2}{2(1-(\alpha-\varepsilon)-\lambda_1)} y(\varphi(t)) = \lambda_2 y(\varphi(t)), \quad t \geqslant n_3,$$

where $\lambda_2 = (\alpha-\varepsilon)^2/(2(1-(\alpha-\varepsilon)-\lambda_1))$.

Following the above procedure, we can inductively construct a sequence of positive real numbers $(\lambda_\nu)_{\nu\geqslant 1}$ with

$$1-(\alpha-\varepsilon)-\lambda_\nu > 0, \quad \nu = 1,2,\dots$$

and

$$\lambda_{\nu+1} = \frac{(\alpha-\varepsilon)^2}{2(1-(\alpha-\varepsilon)-\lambda_\nu)}, \quad \nu = 1,2,\dots$$

such that

$$y(t) > \lambda_\nu y(\varphi(t)), \quad t \geqslant n_3,\ \nu = 1,2,\dots \tag{2.14}$$

As $\lambda_1 > 0$, we obtain

$$\lambda_2 = \frac{(\alpha-\varepsilon)^2}{2(1-(\alpha-\varepsilon)-\lambda_1)} > \frac{(\alpha-\varepsilon)^2}{2(1-(\alpha-\varepsilon))} = \lambda_1,$$

i.e., $\lambda_2 > \lambda_1$. By an easy induction, one can immediately see that the sequence $(\lambda_\nu)_{\nu\geqslant 1}$ is strictly increasing. Furthermore, by taking into account the fact that the function y is decreasing on $[n_2,\infty)$ and using (2.14) (for $t = n_3$), we get

$$y(n_3) > \lambda_\nu y(\varphi(n_3)) \geqslant \lambda_\nu y(n_3), \quad \nu = 1,2,\dots$$

Therefore, for each integer $\nu \geqslant 1$, we have $\lambda_\nu < 1$. This ensures that the sequence $(\lambda_\nu)_{\nu\geqslant 1}$ is bounded. Since $(\lambda_\nu)_{\nu\geqslant 1}$ is a strictly increasing and bounded sequence of positive real numbers, it follows that $\lim\limits_{\nu\to\infty} \lambda_\nu$ exists as a positive real number.

Set $\Lambda = \lim\limits_{\nu\to\infty} \lambda_\nu$. Then (2.14) gives

$$y(t) \geqslant \Lambda y(\varphi(t)), \quad t \geqslant n_3. \tag{2.15}$$

Because of the definition of $(\lambda_\nu)_{\nu\geqslant 1}$, it holds that

$$\Lambda = \frac{(\alpha-\varepsilon)^2}{2(1-(\alpha-\varepsilon)-\Lambda)},$$

i.e.,

$$\Lambda = \frac{1-(\alpha-\varepsilon) \pm \sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2}}{2}.$$

Therefore

$$\Lambda \geqslant \frac{1-(\alpha-\varepsilon) - \sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2}}{2}$$

and consequently (2.15) yields

$$y(t) \geqslant \frac{1-(\alpha-\varepsilon) - \sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2}}{2} y(\varphi(t)), \quad t \geqslant n_3. \tag{2.16}$$

Let n be an arbitrary integer with $n \geqslant n_3$. Then, by (2.15),

$$y(t) \geqslant \frac{1-(\alpha-\varepsilon) - \sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2}}{2} y(\varphi(t)) \quad \text{for } n \leqslant t < n+1.$$

But, $y(\varphi(t)) = y(\tau(n)) = x(\tau(n))$ for $n \leqslant t < n+1$. So,

$$y(t) \geqslant \frac{1-(\alpha-\varepsilon) - \sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2}}{2} x(\tau(n)) \quad \text{for } n \leqslant t < n+1$$

and therefore

$$\lim_{t\to(n+1)-0} y(t) \geqslant \frac{1-(\alpha-\varepsilon) - \sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2}}{2} x(\tau(n)).$$

Note that $\lim\limits_{t\to(n+1)-0} y(t) = y(n+1) = x(n+1)$. We have thus proved that

$$x(n+1) \geqslant \frac{1-(\alpha-\varepsilon) - \sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2}}{2} x(\tau(n)), \quad n \geqslant n_3. \tag{2.17}$$

Finally, we see that (2.17) is written as

$$\frac{x(n+1)}{x(\tau(n))} \geqslant \frac{1-(\alpha-\varepsilon) - \sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2}}{2}, \quad n \geqslant n_3$$

and consequently

$$\liminf_{n\to\infty} \frac{x(n+1)}{x(\tau(n))} \geqslant \frac{1-(\alpha-\varepsilon) - \sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2}}{2}.$$

The last inequality holds true for all real numbers ε with $0 < \varepsilon < \alpha$. Hence, we obtain (2.1).

Next, we consider the particular case where (2.2) holds. Then

$$p_i(n) > \frac{\alpha-\varepsilon}{2} \quad \text{for all large } n,\ 1 \leqslant i \leqslant m.$$

In view of (2.1), it is clear that $x(n+1) > \frac{1}{2}(1-(\alpha-\varepsilon)-\sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2})x(\tau(n))$. Thus, from (E) we have

$$\begin{aligned} x(n) &= x(n+1) + \sum_{i=1}^{m} p_i(n)x(\tau_i(n)) \geqslant x(n+1) + \sum_{i=1}^{m} p_i(n)x(\tau(n)) \\ &> \frac{1-(\alpha-\varepsilon)-\sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2}}{2}x(\tau(n)) + \frac{\alpha-\varepsilon}{2}x(\tau(n)), \end{aligned}$$

or

$$x(n) > \frac{1-\sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2}}{2}x(\tau(n)). \tag{2.18}$$

Summing up (E) from $\tau(n)$ to $n-1$, and using the fact that the function x is decreasing and the function τ (as defined by (1.7)) is increasing, we have

$$\begin{aligned} x(\tau(n)) &= x(n) + \sum_{i=1}^{m}\sum_{j=\tau(n)}^{n-1} p_i(j)x(\tau_i(j)) \geqslant x(n) + \sum_{i=1}^{m}\sum_{j=\tau(n)}^{n-1} p_i(j)x(\tau(j)) \\ &\geqslant x(n) + x(\tau(n-1))\sum_{i=1}^{m}\sum_{j=\tau(n)}^{n-1} p_i(j), \end{aligned}$$

which, in view of (1.14) and (1.13), gives

$$x(\tau(n)) \geqslant x(n) + (\alpha-\varepsilon)x(\tau(n-1)). \tag{2.19}$$

Combining inequalities (2.18) and (2.19), we obtain

$$x(n) > \frac{1-\sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2}}{2}(x(n) + (\alpha-\varepsilon)x(\tau(n-1))),$$

or

$$\frac{x(n)}{x(\tau(n-1))} > 2\frac{1-\sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2}}{2+(\alpha-\varepsilon)} - (\alpha-\varepsilon),$$

and, for large n, we have

$$\frac{x(n+1)}{x(\tau(n))} > 2\frac{1-\sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2}}{2+(\alpha-\varepsilon)} - (\alpha-\varepsilon).$$

Hence,

$$\liminf_{n\to\infty}\frac{x(n+1)}{x(\tau(n))} \geqslant 2\frac{1-\sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2}}{2+(\alpha-\varepsilon)} - (\alpha-\varepsilon),$$

which, for arbitrarily small values of ε, implies (2.3).

The proof of the lemma is complete. □

Our main result is the following theorem.

Theorem 2.1. *Assume that the sequences* $(\tau_i(n))$ $[(\sigma_i(n))]$, $1 \leqslant i \leqslant m$ *are increasing,* (1.1), [(1.2)] *holds,* $(\tau(n))$ $[(\sigma(n))]$ *is defined by* (1.7) [(1.8)], *and define* α *by* (1.13).

If $0 < \alpha \leqslant 1/\mathrm{e}$, *and*

$$\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n} p_i(j) > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} \tag{2.20}$$

$$\left[\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=n}^{\sigma(n)} p_i(j) > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}\right],$$

then all solutions of (E) *oscillate.*

If, additionally, (2.2) *holds and*

$$\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n} p_i(j) > 1 - \left(2\frac{1-\sqrt{1-2\alpha-\alpha^2}}{2+\alpha} - \alpha\right) \tag{2.21}$$

$$\left[\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=n}^{\sigma(n)} p_i(j) > 1 - \left(2\frac{1-\sqrt{1-2\alpha-\alpha^2}}{2+\alpha} - \alpha\right)\right],$$

then all solutions of (E) *oscillate.*

P r o o f. The proof below refers to the retarded difference equation (E). The proof for the advanced difference equation (E) follows by a similar procedure and is omitted.

Assume, for the sake of contradiction, that $(x(n))_{n\geqslant -w}$ is a nonoscillatory solution of (E). Then it is either eventually positive or eventually negative. As $(-x(n))_{n\geqslant -w}$ is also a solution of (E), we may restrict ourselves only to the case where $x(n) > 0$ for all large n. Let $n_1 \geqslant -w$ be an integer such that $x(n) > 0$ for all $n \geqslant n_1$. Then, there exists $n_2 \geqslant n_1$ such that $x(\tau_i(n)) > 0$, for all $n \geqslant n_2$, $1 \leqslant i \leqslant m$. In view of this, equation (E) becomes

$$\Delta x(n) = -\sum_{i=1}^{m} p_i(n) x(\tau_i(n)) \leqslant 0, \quad n \geqslant n_2,$$

which means that the sequence $(x(n))$ is eventually decreasing.

Summing up (E) from $\tau(n)$ to n, and using the fact that the function x is decreasing and the function τ (as defined by (1.7)) is increasing, we obtain, for every $n \geqslant n_2$

$$x(\tau(n)) = x(n+1) + \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n} p_i(j) x(\tau_i(j)) \geqslant x(n+1) + x(\tau(n)) \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n} p_i(j).$$

Consequently,

$$\sum_{i=1}^{m}\sum_{j=\tau(n)}^{n} p_i(j) \leqslant 1 - \frac{x(n+1)}{x(\tau(n))}, \quad n \geqslant n_2,$$

which gives

$$\limsup_{n\to\infty}\sum_{i=1}^{m}\sum_{j=\tau(n)}^{n} p_i(j) \leqslant 1 - \liminf_{n\to\infty}\frac{x(n+1)}{x(\tau(n))}. \tag{2.22}$$

First, assume that $0 < \alpha \leqslant 1/\mathrm{e}$ (clearly, $\alpha < -1+\sqrt{2}$). Then by Lemma 2.1, inequality (2.1) is fulfilled, and so (2.22) leads to

$$\limsup_{n\to\infty}\sum_{i=1}^{m}\sum_{j=\tau(n)}^{n} p_i(j) \leqslant 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2},$$

which contradicts condition (2.20).

Next, let us suppose that (2.2) holds. Then Lemma 2.1 ensures that (2.3) is satisfied. Thus, from (2.22), it follows that

$$\limsup_{n\to\infty}\sum_{i=1}^{m}\sum_{j=\tau(n)}^{n} p_i(j) \leqslant 1 - \left(2\frac{1-\sqrt{1-2\alpha-\alpha^2}}{2+\alpha} - \alpha\right),$$

which contradicts condition (2.21).

The proof of the theorem is complete. $\square$

R e m a r k 2.1. It is easy to see that

$$2\frac{1-\sqrt{1-2\alpha-\alpha^2}}{2+\alpha} - \alpha > \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}$$
$$> \alpha\left(\frac{1}{3\sqrt{1-\alpha}+\alpha-2} - 1\right) > (1-\sqrt{1-\alpha})^2.$$

Therefore, when (2.2) holds, then condition (2.21) is weaker than conditions (2.20), (1.17) and (1.15).

R e m a r k 2.2. When $\alpha \to 0$, then all the above mentioned conditions (2.21), (2.20), (1.17) and (1.15) reduce to

$$\limsup_{n\to\infty}\sum_{i=1}^{m}\sum_{j=\tau(n)}^{n} p_i(j) > 1 \quad \left[\limsup_{n\to\infty}\sum_{i=1}^{m}\sum_{j=n}^{\sigma(n)} p_i(j) > 1\right],$$

that is, to condition (1.11). However, the improvement is clear when

$$\alpha \to \frac{1}{\mathrm{e}} \simeq 0.367879441.$$

For illustrative purposes we give the values of the lower bound on the above conditions when $\alpha = 0.367879441$:

$$\begin{aligned}
&(1.15)\colon\ 0.957999636,\\
&(1.17)\colon\ 0.879366479,\\
&(2.20)\colon\ 0.863457014,\\
&(2.21)\colon\ 0.826495955.
\end{aligned}$$

That is, our conditions (2.20) and (2.21) essentially improve (1.11), (1.15) and (1.17).

3. Examples

We illustrate the significance of our results by the following examples.

Example 3.1. Consider the difference equation with three retarded arguments

$$\Delta x(n) + p_1(n)x(n-1) + p_2(n)x(n-2) + p_3(n)x(n-3) = 0, \quad n \geqslant 0, \tag{3.1}$$

where

$$\begin{gathered}
p_1(2n) = \frac{7}{100}, \quad p_1(2n+1) = \frac{4}{10},\\
p_2(3n) = p_2(3n+1) = \frac{5}{100}, \quad p_2(3n+2) = \frac{35}{100},\\
p_3(4n) = p_3(4n+1) = p_3(4n+2) = \frac{3}{100}, \quad p_3(4n+3) = \frac{98}{1000}.
\end{gathered}$$

Here $m = 3$, $\tau_1(n) = n-1$, $\tau_2(n) = n-2$, $\tau_3(n) = n-3$ and $\tau(n) = n-1$. It is easy to see that

$$\alpha_1 = \liminf_{n\to\infty} \sum_{j=n-1}^{n-1} p_1(j) = \frac{7}{100} = 0.07,$$

$$\alpha_2 = \liminf_{n\to\infty} \sum_{j=n-2}^{n-1} p_2(j) = 2\cdot\frac{5}{100} = 0.1,$$

$$\alpha_3 = \liminf_{n\to\infty} \sum_{j=n-3}^{n-1} p_3(j) = 3\cdot\frac{3}{100} = 0.09.$$

Thus

$$\alpha = \min\{\alpha_i\colon\ 1 \leqslant i \leqslant 3\} = \min\{0.07, 0.1, 0.09\} = 0.07 < \frac{1}{\mathrm{e}}.$$

Also,

$$\limsup_{n\to\infty} \sum_{i=1}^{3} \sum_{j=n-1}^{n} p_i(j) = \limsup_{n\to\infty} \Big(\sum_{j=n-1}^{n} p_1(j) + \sum_{j=n-1}^{n} p_2(j) + \sum_{j=n-1}^{n} p_3(j) \Big)$$
$$= \frac{7}{100} + \frac{4}{10} + \frac{5}{100} + \frac{35}{100} + \frac{3}{100} + \frac{98}{1000} = 0.998.$$

Observe that

$$0.998 > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} \simeq 0.997358086,$$

that is, condition (2.20) of Theorem 2.1 is satisfied and therefore all solutions of equation (3.1) oscillate.

Observe, however, that

$$0.998 < 1,$$
$$0.998 < 1 - (1-\sqrt{1-\alpha})^2 \simeq 0.998730152,$$
$$\liminf_{n\to\infty} \sum_{i=1}^{3} \sum_{j=\tau(n)}^{n-1} p_i(j) = \liminf_{n\to\infty} \Big(\sum_{j=n-1}^{n-1} p_1(j) + \sum_{j=n-1}^{n-1} p_2(j) + \sum_{j=n-1}^{n-1} p_3(j) \Big)$$
$$= \frac{7}{100} + \frac{5}{100} + \frac{3}{100} = 0.15 < \frac{1}{\mathrm{e}},$$
$$\liminf_{n\to\infty} \sum_{i=1}^{3} \sum_{j=n-k_i}^{n-1} p_i(j) = \liminf_{n\to\infty} \Big(\sum_{j=n-1}^{n-1} p_1(j) + \sum_{j=n-2}^{n-1} p_2(j) + \sum_{j=n-3}^{n-1} p_3(j) \Big)$$
$$= \frac{7}{100} + 2\cdot\frac{5}{100} + 3\cdot\frac{3}{100} = 0.26 < \frac{1}{\mathrm{e}},$$
$$\liminf_{n\to\infty} \sum_{i=1}^{3} \Big(\frac{k_i+1}{k_i}\Big)^{k_i+1} \sum_{j=n+1}^{n+k_i} p_i(j)$$
$$= \liminf_{n\to\infty} \bigg(\Big(\frac{2}{1}\Big)^2 \sum_{j=n+1}^{n+1} p_1(j) + \Big(\frac{3}{2}\Big)^3 \sum_{j=n+1}^{n+2} p_2(j) + \Big(\frac{4}{3}\Big)^4 \sum_{j=n+1}^{n+3} p_2(j) \bigg)$$
$$= 2^2\cdot\frac{7}{100} + \Big(\frac{3}{2}\Big)^3\cdot 2\cdot\frac{5}{100} + \Big(\frac{4}{3}\Big)^4\cdot 3\cdot\frac{3}{100} = 0.901944444 < 1,$$
$$\sum_{i=1}^{3} \Big(\liminf_{n\to\infty} p_i(n)\Big) \frac{(k_i+1)^{k_i+1}}{(k_i)^{k_i}} = \frac{7}{100}\cdot\frac{2^2}{1^1} + \frac{5}{100}\cdot\frac{3^3}{2^2} + \frac{3}{100}\cdot\frac{4^4}{3^3}$$
$$= 0.901944444 < 1,$$

$$\liminf_{n\to\infty}\sum_{j=\tau(n)}^{n-1}\sum_{i=1}^{3}p_i(j)\Big(\frac{n-\tau_i(j)+1}{n-\tau_i(j)}\Big)^{n-\tau_i(j)+1}$$
$$=\Big(\frac{2}{1}\Big)^2\cdot\frac{7}{100}+\Big(\frac{3}{2}\Big)^3\cdot\frac{5}{100}+\Big(\frac{4}{3}\Big)^4\cdot\frac{3}{100}=0.543564814<1,$$
and therefore none of the conditions (1.11), (1.15), (1.10), (1.12), (1.4), (1.3) and (1.9) is satisfied.

E x a m p l e 3.2. Consider the difference equation with two retarded arguments

$$\Delta x(n)+p_1(n)x(n-2)+p_2(n)x(n-1)=0,\quad n\geqslant 0, \tag{3.2}$$

where

$$p_1(3n)=p_1(3n+1)=\frac{1}{10},\quad p_1(3n+2)=\frac{1}{2},\quad n\geqslant 0,$$
$$p_2(2n)=\frac{7}{100},\quad p_2(2n+1)=\frac{3273}{10000},\quad n\geqslant 0.$$

Here $m=2$, $\tau_1(n)=n-2$, $\tau_2(n)=n-1$ and $\tau(n)=n-1$. It is easy to see that

$$\alpha_1=\liminf_{n\to\infty}\sum_{j=n-2}^{n-1}p_1(j)=2\cdot\frac{1}{10}=0.2,$$
$$\alpha_2=\liminf_{n\to\infty}\sum_{j=n-1}^{n-1}p_2(j)=\frac{7}{100}=0.07.$$

Thus

$$\alpha=\min\{\alpha_i\colon\ 1\leqslant i\leqslant 2\}=\min\{0.2,0.07\}=0.07<\frac{1}{\mathrm{e}}.$$

Furthermore, it is clear that

$$p_i(n)>\frac{\alpha}{2}=0.035\quad\text{for all large } n,\quad 1\leqslant i\leqslant 2,$$
$$p_i(n)>1-\sqrt{1-\alpha}\simeq 0.035634923\quad\text{for all large } n,\quad 1\leqslant i\leqslant 2.$$

Also,

$$\limsup_{n\to\infty}\sum_{i=1}^{2}\sum_{j=n-1}^{n}p_i(j)=\limsup_{n\to\infty}\Big(\sum_{j=n-1}^{n}p_1(j)+\sum_{j=n-1}^{n}p_2(j)\Big)$$
$$=\frac{1}{10}+\frac{1}{2}+\frac{7}{100}+\frac{3273}{10000}=0.9973.$$

Observe that

$$0.9973>1-\Big(2\frac{1-\sqrt{1-2\alpha-\alpha^2}}{2+\alpha}-\alpha\Big)\simeq 0.997262002,$$

that is, conditions (2.2) and (2.21) of Theorem 2.1 are satisfied and therefore all solutions of equation (3.2) oscillate.

Observe, however, that

$$
\begin{gathered}
0.9973 < 1, \\
\liminf_{n\to\infty} \sum_{i=1}^{3} \sum_{j=\tau(n)}^{n-1} p_i(j) = \liminf_{n\to\infty} \Big(\sum_{j=n-1}^{n-1} p_1(j) + \sum_{j=n-1}^{n-1} p_2(j) \Big) \\
= \frac{1}{10} + \frac{7}{100} = 0.17 < \frac{1}{\mathrm{e}}, \\
\liminf_{n\to\infty} \sum_{i=1}^{2} \sum_{j=\tau_i(n)}^{n-1} p_i(j) = \liminf_{n\to\infty} \Big(\sum_{j=n-2}^{n-1} p_1(j) + \sum_{j=n-1}^{n-1} p_2(j) \Big) \\
= 2\frac{1}{10} + \frac{7}{100} = 0.27 < \frac{1}{\mathrm{e}}, \\
0.9973 < 1 - (1 - \sqrt{1-\alpha})^2 \simeq 0.998730152, \\
0.9973 < 1 - \alpha \Big(\frac{1}{3\sqrt{1-\alpha} + \alpha - 2} - 1 \Big) \simeq 0.997317675, \\
\liminf_{n\to\infty} \sum_{i=1}^{2} \Big(\frac{k_i+1}{k_i} \Big)^{k_i+1} \sum_{j=n-k_i}^{n-1} p_i(j) = \liminf_{n\to\infty} \Big(\Big(\frac{3}{2}\Big)^3 2\frac{1}{10} + 2^2 \frac{7}{100} \Big) = 0.955 < 1, \\
0.9973 < 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2} \simeq 0.997358086, \\
\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} \sum_{i=1}^{2} p_i(j) \Big(\frac{n - \tau_i(j) + 1}{n - \tau_i(j)} \Big)^{n-\tau_i(j)+1} \\
= \Big(\frac{3}{2}\Big)^3 \frac{1}{10} + \Big(\frac{2}{1}\Big)^2 \frac{7}{100} = 0.61754 < 1,
\end{gathered}
$$

and therefore none of the conditions (1.11), (1.10), (1.12), (1.15), (1.17), (1.6), (2.20) and (1.9) is satisfied.

Example 3.3. Consider the advanced difference equation

$$
\nabla x(n) - p_1(n)x(n+2) - p_2(n)x(n+1) = 0, \quad n \geqslant 1 \tag{3.3}
$$

where

$$
\begin{gathered}
p_1(3n) = p_1(3n+1) = \frac{1}{10}, \quad p_1(3n+2) = \frac{1}{2}, \quad n \geqslant 1 \\
p_2(2n) = \frac{8}{100}, \quad p_2(2n+1) = \frac{3164}{10000}, \quad n \geqslant 1.
\end{gathered}
$$

Here $m = 2$, $\sigma_1(n) = n+2$, $\sigma_2(n) = n+1$ and $\sigma(n) = n+1$. It is easy to see that

$$\alpha_1 = \liminf_{n\to\infty} \sum_{j=n+1}^{n+2} p_1(j) = 2 \cdot \frac{1}{10} = 0.2,$$

$$\alpha_2 = \liminf_{n\to\infty} \sum_{j=n+1}^{n+1} p_2(j) = \frac{8}{100} = 0.08.$$

Thus

$$\alpha = \min\{\alpha_i \colon\ 1 \leqslant i \leqslant 2\} = \min\{0.2, 0.08\} = 0.08 < \frac{1}{\mathrm{e}}.$$

Furthermore, it is clear that

$$p_i(n) > \frac{\alpha}{2} = 0.04 \quad \text{for all large } n, \quad 1 \leqslant i \leqslant 2,$$

$$p_i(n) > 1 - \sqrt{1-\alpha} \simeq 0.040833695 \quad \text{for all large } n, \quad 1 \leqslant i \leqslant 2.$$

Also,

$$\limsup_{n\to\infty} \sum_{i=1}^{2} \sum_{j=n}^{\sigma(n)} p_i(j) = \limsup_{n\to\infty} \Big(\sum_{j=n}^{n+1} p_1(j) + \sum_{j=n}^{n+1} p_2(j) \Big)$$

$$= \frac{1}{10} + \frac{1}{2} + \frac{8}{100} + \frac{3164}{10000} = 0.9964.$$

Observe that

$$0.9964 > 1 - \left(2\frac{1-\sqrt{1-2\alpha-\alpha^2}}{2+\alpha} - \alpha \right) \simeq 0.996362477,$$

that is, conditions (2.2) and (2.21) of Theorem 2.1 are satisfied and therefore all solutions of equation (3.3) oscillate.

Observe, however, that

$$0.9964 < 1,$$

$$\liminf_{n\to\infty} \sum_{i=1}^{2} \sum_{j=n+1}^{n+k_i} p_i(j) = \liminf_{n\to\infty} \Big(\sum_{j=n+1}^{n+2} p_1(j) + \sum_{j=n+1}^{n+1} p_2(j) \Big)$$

$$= 0.2 + 0.08 = 0.28 < \frac{1}{\mathrm{e}},$$

$$0.9964 < 1 - (1-\sqrt{1-\alpha})^2 \simeq 0.998332609,$$

$$0.9964 < 1 - \alpha \left(\frac{1}{3\sqrt{1-\alpha}+\alpha-2} - 1 \right) \simeq 0.996448991,$$

$$\liminf_{n\to\infty} \sum_{i=1}^{2} \Big(\frac{k_i+1}{k_i} \Big)^{k_i+1} \sum_{j=n-k_i}^{n-1} p_i(j) = \liminf_{n\to\infty} \Big(\Big(\frac{3}{2}\Big)^3 2\frac{1}{10} + 2^2 \frac{8}{100} \Big) = 0.995 < 1,$$

$$0.9964 < 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} \simeq 0.996508488,$$

and therefore none of the conditions (1.11), (1.12), (1.15), (1.17), (1.6) and (2.20) is satisfied.

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