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A REPRESENTATION THEOREM FOR TENSE $n \times m$-VALUED ŁUKASIEWICZ-MOISIL ALGEBRAS

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Abstract. In 2000, Figallo and Sanza introduced $n \times m$-valued Łukasiewicz-Moisil algebras which are both particular cases of matrix Łukasiewicz algebras and a generalization of n-valued Łukasiewicz-Moisil algebras. Here we initiate an investigation into the class $\mathbf{tLM}_{n\times m}$ of tense $n \times m$-valued Łukasiewicz-Moisil algebras (or tense $\mathrm{LM}_{n\times m}$-algebras), namely $n\times m$-valued Łukasiewicz-Moisil algebras endowed with two unary operations called tense operators. These algebras constitute a generalization of tense Łukasiewicz-Moisil algebras (or tense LM_n-algebras). Our most important result is a representation theorem for tense $\mathrm{LM}_{n\times m}$-algebras. Also, as a corollary of this theorem, we obtain the representation theorem given by Georgescu and Diaconescu in 2007, for tense LM_n-algebras.

Keywords: n-valued Łukasiewicz-Moisil algebra; tense n-valued Łukasiewicz-Moisil algebra; $n \times m$-valued Łukasiewicz-Moisil algebra

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1. Introduction

Classical tense logic is an extension of classical logic obtained by adding to the bivalent logic the tense operators G (*it is always going to be the case that*) and H (*it has always been the case that*). Taking into account that tense Boolean algebras constitute the algebraic basis for the bivalent tense logic (see [4]), Diaconescu and Georgescu introduced in [10] tense MV-algebras and tense Łukasiewicz-Moisil algebras as algebraic structures for some many-valued tense logics. In the last years, these two classes of algebras have become very interesting for several authors (see [2], [5]–[9], [11]–[14]). In particular, in [8], [9] Chiriţă introduced tense θ-valued Łukasiewicz-Moisil algebras and proved an important representation theorem which

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allowed to show the completeness of the tense θ-valued Moisil logic (see [9]). In [10], the authors formulated an open problem about the representation of tense MV-algebras; this problem was solved in [3], [20] for semisimple tense MV-algebras. Also, in [2], tense basic algebras were studied, which is an interesting generalization of tense MV-algebras.

On the other hand, in 1975 Suchoń [25] defined matrix Łukasiewicz algebras so generalizing n-valued Łukasiewicz algebras without negation [19]. In 2000, Figallo and Sanza [17] introduced $n \times m$-valued Łukasiewicz algebras with negation which are both a particular case of matrix Łukasiewicz algebras and a generalization of n-valued Łukasiewicz-Moisil algebras [1]. It is worth noting that unlike what happens in n-valued Łukasiewicz-Moisil algebras, generally the De Morgan reducts of $n \times m$-valued Łukasiewicz algebras with negation are not Kleene algebras. Furthermore, in [22] an important example which legitimated the study of this new class of algebras was provided. Following the terminology established in [1], these algebras were called $n \times m$-valued Łukasiewicz-Moisil algebras (or $\mathrm{LM}_{n\times m}$-algebras for short).

In the present paper, we introduce and investigate tense $n \times m$-valued Łukasiewicz-Moisil algebras which constitute a generalization of tense Łukasiewicz-Moisil algebras [10]. Our most important result is a representation theorem for tense $\mathrm{LM}_{n\times m}$-algebras. Also, as a corollary of this theorem, we obtain the representation theorem given by Georgescu and Diaconescu in [10] for tense LM_n-algebras.

2. Preliminaries

2.1. Tense Boolean algebras. Tense Boolean algebras are the algebraic structures for tense logic. In this subsection we will recall some basic definitions and results on the representation of tense Boolean algebras (see [4], [18]).

Definition 2.1. An algebra $(\mathcal{B}, G, H)$ is a tense Boolean algebra if

$$\mathcal{B} = \langle B, \wedge, \vee, \neg, 0_B, 1_B \rangle$$

is a Boolen algebra and G and H are two unary operations on B such that

(tb1) $G(1_B) = 1_B$ and $H(1_B) = 1_B$;
(tb2) $G(x \wedge y) = G(x) \wedge G(y)$ and $H(x \wedge y) = H(x) \wedge H(y)$;
(tb3) $G(x) \vee y = 1_B$ if only if $x \vee H(y) = 1_B$.

Let $\mathcal{B} = \langle B, \wedge, \vee, \neg, 0_B, 1_B \rangle$ be a Boolean algebra. In the following we will denote by id_B, O_B and I_B the functions $\mathrm{id}_B, O_B, I_B \colon B \to B$, defined by $\mathrm{id}_B(x) = x$, $O_B(x) = 0_B$ and $I_B(x) = 1_B$ for all $x \in B$. We also denote by $\mathbf{2}$ the two-element Boolean algebra.

R e m a r k 2.1. Let $\mathcal{B} = \langle B, \wedge, \vee, \neg, 0_B, 1_B \rangle$ be a Boolean algebra. Then $(\mathcal{B}, I_B, I_B)$ is a tense Boolean algebra.

R e m a r k 2.2. Let $(\mathbf{2}, G, H)$ be a tense Boolean algebra. Then $G = H = \mathrm{id}_{\mathbf{2}}$ or $G = H = I_{\mathbf{2}}$.

Proposition 2.1. *Let $\mathcal{B} = \langle B, \wedge, \vee, \neg, 0_B, 1_B \rangle$ be a Boolean algebra and G, H two unary operations on B that satisfy conditions* (tb1) *and* (tb2). *Then the condition* (tb3) *is equivalent to*

(tb3)′ $x \leqslant GP(x)$ *and* $x \leqslant HF(x)$, *where* $F, P\colon B \to B$ *are the unary operations defined by* $F(x) = \neg G(\neg x)$ *and* $P(x) = \neg H(\neg x)$.

R e m a r k 2.3. By Proposition 2.1 we can obtain an equivalent definition for tense Boolean algebras. This shows that tense Boolean algebras form a variety.

Definition 2.2. A frame is a pair (X, R), where X is a nonempty set and R is a binary relation on X.

Let (X, R) be a frame. We define the operations $G\colon \mathbf{2}^X \to \mathbf{2}^X$ and $H\colon \mathbf{2}^X \to \mathbf{2}^X$ by

$$G(p)(x) = \bigwedge\{p(y);\ y \in X, x\ R\ y\} \quad \text{and} \quad H(p)(x) = \bigwedge\{p(y);\ y \in X, y\ R\ x\},$$

for all $p \in \mathbf{2}^X$ and $x \in X$.

Proposition 2.2. *For any frame (X, R), $(\mathbf{2}^X, G, H)$ is a tense Boolean algebra.*

R e m a r k 2.4. In the tense Boolean algebra $(\mathbf{2}^X, G, H)$ the tense operators F and P are given by:

$$F(p)(x) = \bigvee\{p(y);\ y \in X, x\ R\ y\} \quad \text{and} \quad P(p)(x) = \bigvee\{p(y);\ y \in X, y\ R\ x\},$$

for all $p \in \mathbf{2}^X$ and $x \in X$.

Definition 2.3. Let $(\mathcal{B}, G, H)$ and $(\mathcal{B}', G', H')$ be two tense Boolean algebras. A function $f\colon B \to B'$ is a morphism of tense Boolean algebras if f is a Boolean morphism and satisfies the conditions: $f(G(x)) = G'(f(x))$ and $f(H(x)) = H'(f(x))$, for any $x \in B$.

By this definition, it follows that a morphism of tense Boolean algebras commutes with the tense operators F and P.

Theorem 2.1 (The representation theorem for tense Boolean algebras)**.** *For any tense Boolean algebra $(\mathcal{B}, G, H)$, there exist a frame (X, R) and an injective morphism of tense Boolean algebras*

$$d\colon (\mathcal{B}, G, H) \to (\mathbf{2}^X, G, H)$$

where operators G and H are defined as in Proposition 2.2.

2.2. Tense Łukasiewicz-Moisil algebras. In this subsection we will recall some basic definitions and results on the representation of tense Łukasiewicz-Moisil algebras (see [10]).

Definition 2.4. An algebra $(\mathcal{L}, G, H)$ is a tense Łukasiewicz-Moisil algebra (or tense LM$_n$-algebra) if $\mathcal{L} = \langle L, \wedge, \vee, \sim, \varphi_1, \dots, \varphi_{n-1}, 0_L, 1_L\rangle$ is an LM$_n$-algebra and G and H are two unary operators on L such that

(tlm1) $G(1_L) = 1_L$ and $H(1_L) = 1_L$,
(tlm2) $G(x \wedge y) = G(x) \wedge G(y)$ and $H(x \wedge y) = H(x) \wedge H(y)$,
(tlm3) $x \leqslant GP(x)$ and $x \leqslant HF(x)$, where $F(x) = {\sim}G({\sim}x)$ and $P(x) = {\sim}H({\sim}x)$,
(tlm4) $G(\varphi_i(x)) = \varphi_i(G(x))$ and $H(\varphi_i(x)) = \varphi_i(H(x))$ for all $i = 1, \dots, n-1$.

Let $\mathcal{L} = \langle L, \wedge, \vee, \sim, \varphi_1, \dots, \varphi_{n-1}, 0_L, 1_L\rangle$ be an LM$_n$-algebra. In the following we will denote by id_L, O_L and I_L the functions $\mathrm{id}_L, O_L, I_L\colon L \to L$, defined by $\mathrm{id}_L(x) = x$, $O_L(x) = 0_L$ and $I_L(x) = 1_L$ for all $x \in L$.

We also denote by L_n the chain of n rational fractions $\mathrm{L}_n = \{j/(n-1); 1 \leqslant j \leqslant n-1\}$ endowed with the natural lattice structure and the unary operations $\sim$ and φ_i, defined as follows: ${\sim}(j/(n-1)) = 1 - j/(n-1)$ and $\varphi_i(j/(n-1)) = 0$ if $i + j < n$ or $\varphi_i(j/(n-1)) = 1$ in the other cases.

Remark 2.5. Let (L_n, G, H) be a tense LM$_n$-algebra. Then $G = H = \mathrm{id}_{\mathrm{L_n}}$ or $G = H = I_{\mathrm{L_n}}$.

Definition 2.5. Let (X, R) be a frame. We define the operations $G\colon \mathrm{L}_n^X \to \mathrm{L}_n^X$ and $H\colon \mathrm{L}_n^X \to \mathrm{L}_n^X$ by:

$$G(p)(x) = \bigwedge\{p(y);\ y \in X, x\ R\ y\} \quad \text{and} \quad H(p)(x) = \bigwedge\{p(y);\ y \in X, y\ R\ x\},$$

for all $p \in \mathrm{L}_n^X$ and $x \in X$.

Proposition 2.3. *For any frame* (X, R), (L_n^X, G, H) *is a tense* LM_n*-algebra.*

Definition 2.6. Let $(\mathcal{L}, G, H)$ and $(\mathcal{L}', G', H')$ be two tense LM_n-algebras. A function $f\colon L \to L'$ is a morphism of tense LM_n-algebras if f is an LM_n-algebra morphism and satisfies the conditions $f(G(x)) = G'(f(x))$ and $f(H(x)) = H'(f(x))$ for any $x \in L$.

Now we will recall a representation theorem for tense LM_n-algebras that generalizes Theorem 2.1.

Theorem 2.2 (The representation theorem for tense LM_n-algebras)**.** *For any tense* LM_n*-algebra* $(\mathcal{L}, G, H)$*, there exist a frame* (X, R) *and an injective morphism of tense* LM_n*-algebras*

$$\Phi\colon\ (\mathcal{L}, G, H) \to (\mathrm{L}_n^X, G, H)$$

where the operators G *and* H *are defined as in Proposition* 2.3.

2.3. $n \times m$**-valued Łukasiewicz-Moisil algebras.** In this subsection we will recall some basic definitions and results on $n \times m$-valued Łukasiewicz-Moisil algebras (see [15], [16], [21]–[24]).

Definition 2.7. An $n \times m$-valued Łukasiewicz-Moisil algebra (or $\mathrm{LM}_{n\times m}$-algebra), in which n and m are integers, $n \geqslant 2$, $m \geqslant 2$, is an algebra $\langle L, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, 0_L, 1_L\rangle$ where $(n\times m)$ is the cartesian product $\{1, \ldots, n-1\} \times \{1, \ldots, m-1\}$, the reduct $\langle L, \wedge, \vee, \sim, 0_L, 1_L\rangle$ is a De Morgan algebra and $\{\sigma_{ij}\}_{(i,j)\in(n\times m)}$ is a family of unary operations on L which fulfils the conditions

(C1) $\sigma_{ij}(x \vee y) = \sigma_{ij}x \vee \sigma_{ij}y$,
(C2) $\sigma_{ij}x \leqslant \sigma_{(i+1)j}x$,
(C3) $\sigma_{ij}x \leqslant \sigma_{i(j+1)}x$,
(C4) $\sigma_{ij}\sigma_{rs}x = \sigma_{rs}x$,
(C5) $\sigma_{ij}x = \sigma_{ij}y$ for all $(i, j) \in (n \times m)$ implies $x = y$,
(C6) $\sigma_{ij}x \vee {\sim}\sigma_{ij}x = 1_L$,
(C7) $\sigma_{ij}({\sim}x) = {\sim}\sigma_{(n-i)(m-j)}x$.

Let $\mathcal{L} = \langle L, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, 0_L, 1_L\rangle$ be an $\mathrm{LM}_{n\times m}$-algebra. In the following we will denote by id_L, O_L and I_L the functions $\mathrm{id}_L, O_L, I_L\colon L \to L$, defined by $\mathrm{id}_L(x) = x$, $O_L(x) = 0_L$ and $I_L(x) = 1_L$ for all $x \in L$.

The results announced here for $\mathrm{LM}_{n\times m}$-algebras will be used throughout the paper.

(LM1) A set $\sigma_{ij}(L) = C(L)$ for all $(i, j) \in (n \times m)$, where $C(L)$ is the set of all complemented elements of L ([24], Proposition 2.5).

(LM2) Every $\mathrm{LM}_{n\times 2}$-algebra is isomorphic to an n-valued Łukasiewicz-Moisil algebra. It is worth noting that $\mathrm{LM}_{n\times m}$-algebras constitute a nontrivial generalization of the latter (see [22], Remark 2.1).

(LM3) Let $\langle L, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, 0_L, 1_L\rangle$ be an $\mathrm{LM}_{n\times m}$-algebra and $(i,j) \in (n\times m)$. We define the binary operation, called weak implication, $\hookrightarrow_{i,j}$ on L, as follows: $a \hookrightarrow_{i,j} b = \sim\sigma_{ij}a \vee \sigma_{ij}b$ for all $a, b \in L$. The implication $\hookrightarrow_{i,j}$ has the following properties:

- (WI1) $a \hookrightarrow_{i,j} (b \hookrightarrow_{i,j} a) = 1_L$,
- (WI2) $a \hookrightarrow_{i,j} (b \hookrightarrow_{i,j} (a \wedge b)) = 1_L$,
- (WI3) $a \hookrightarrow_{i,j} (b \hookrightarrow_{i,j} c) = (a \hookrightarrow_{i,j} b) \hookrightarrow_{i,j} (a \hookrightarrow_{i,j} c)$,
- (WI4) $(a \wedge b) \hookrightarrow_{i,j} a = 1_L$ and $(a \wedge b) \hookrightarrow_{i,j} b = 1_L$,
- (WI5) $a \hookrightarrow_{i,j} (a \vee b) = 1_L$ and $b \hookrightarrow_{i,j} (a \vee b) = 1_L$,
- (WI6) $a \leqslant b$ implies $a \hookrightarrow_{i,j} b = 1_L$,
- (WI7) if $a \hookrightarrow_{i,j} b = 1_L$ for all $(i,j) \in (n\times m)$ then $a \leqslant b$,
- (WI8) $a \hookrightarrow_{i,j} 1_L = 1_L$,
- (WI9) $a \hookrightarrow_{i,j} (b \wedge c) = (a \hookrightarrow_{i,j} b) \wedge (a \hookrightarrow_{i,j} c)$,
- (WI10) $\sigma_{rs}(a) \hookrightarrow_{i,j} \sigma_{rs}(b) = a \hookrightarrow_{r,s} b$,
- (WI11) $\sigma_{rs}(a) \hookrightarrow_{i,j} \sigma_{rs}(a) = 1_L$,
- (WI12) $a \hookrightarrow_{i,j} (b \hookrightarrow_{i,j} c) = (a \wedge b) \hookrightarrow_{i,j} c$,
- (WI13) if $a \leqslant b \hookrightarrow_{i,j} c$ for all $(i,j) \in (n\times m)$ then $a \wedge b \leqslant c$ (see [23]).

(LM4) The class of $\mathrm{LM}_{n\times m}$-algebras is a variety and two equational bases for it can be found in [24], Theorem 2.7, and [22], Theorem 4.6.

(LM5) Let $\langle L, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, 0_L, 1_L\rangle$ be an $\mathrm{LM}_{n\times m}$-algebra. Let X be a nonempty set and let L^X be the set of all functions from X into L. Then L^X is an $\mathrm{LM}_{n\times m}$-algebra where the operations are defined componentwise (see [23]).

(LM6) Let $\langle L, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, 0_L, 1_L\rangle$ be an $\mathrm{LM}_{n\times m}$-algebra. We say that L is complete if the lattice $\langle L, \wedge, \vee, 0_L, 1_L\rangle$ is complete. Also, we say that L is completely chrysippian if, for every $\{x_s\}_{s\in S} \subseteq L$ such that $\bigwedge\limits_{s\in S} x_s$ and $\bigvee\limits_{s\in S} x_s$ exist, the following conditions hold: $\sigma_{ij}\Big(\bigwedge\limits_{s\in S} x_s\Big) = \bigwedge\limits_{s\in S} \sigma_{ij}(x_s)$ for all $(i,j) \in (n\times m)$ and $\sigma_{ij}\Big(\bigvee\limits_{s\in S} x_s\Big) = \bigvee\limits_{s\in S} \sigma_{ij}(x_s)$ for all $(i,j) \in (n\times m)$ (see [23]).

(LM7) Let $C(L)\!\uparrow^{(n\times m)} = \{f\colon (n\times m) \longrightarrow C(L)$ such that for arbitrary i, j if $r \leqslant s$, then $f(r,j) \leqslant f(s,j)$ and $f(i,r) \leqslant f(i,s)\}$. Then

$$\langle C(L)\!\uparrow^{(n\times m)}, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, O, I\rangle$$

is an $\mathrm{LM}_{n\times m}$-algebra where for all $f \in C(L)\!\uparrow^{(n\times m)}$ and $(i,j) \in (n\times m)$ the operations $\sim$ and σ_{ij} are defined as follows: $(\sim f)(i,j) = \neg f(n-i,\, m-j)$, where $\neg x$ denotes the Boolean complement of x, $(\sigma_{ij}f)(r,s) = f(i,j)$ for all $(r,s) \in (n \times m)$, and the remaining operations are defined componentwise ([22], Proposition 3.2). It is worth noting that this result can be generalized by replacing $C(L)$ by any Boolean algebra B. Furthermore, if B is a complete Boolean algebra, it is simple to check that $B{\uparrow}^{(n\times m)}$ is also a complete $\mathrm{LM}_{n\times m}$-algebra.

(LM8) Let $\mathcal{L}$ and $\mathcal{L}'$ be two $\mathrm{LM}_{n\times m}$-algebras. A morphism of $\mathrm{LM}_{n\times m}$-algebras is a function $f\colon L \to L'$ such that following conditions hold for all $x, y \in L$:
- (i) $f(0_L) = 0_{L'}$ and $f(1_L) = 1_{L'}$;
- (ii) $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$;
- (iii) $f(\sigma_{ij}(x)) = \sigma'_{ij}(f(x))$ for every $(i,j) \in (n \times m)$;
- (iv) $f({\sim}x) = {\sim}' f(x)$.

Let us observe that condition (iv) is a direct consequence of (C5), (C7) and the conditions (i) to (iii).

(LM9) Every $\mathrm{LM}_{n\times m}$-algebra L can be embedded into $C(L){\uparrow}^{(n\times m)}$ ([22], Theorem 3.1). Besides, L is isomorphic to $C(L){\uparrow}^{(n\times m)}$ if and only if L is centred ([22], Corollary 3.1) where L is centred if for each $(i,j) \in (n \times m)$ there exists $c_{ij} \in L$ such that

$$\sigma_{rs}c_{ij} = \begin{cases} 0 & \text{if } i > r \text{ or } j > s, \\ 1 & \text{if } i \leqslant r \text{ and } j \leqslant s. \end{cases}$$

(LM10) Identifying the set $(n\times 2)$ with $\mathbf{n} = \{1, \ldots, n-1\}$ we have that $\tau_{\mathrm{L}_n}\colon \mathrm{L}_n \to \mathbf{2}{\uparrow}^{\mathbf{n}}$ is an isomorphism which in this case is defined by $\tau_{\mathrm{L}_n}(j/(n-1)) = f_j$ where $f_j(i) = 0$ if $i + j < n$ and $f_j(i) = 1$ in the other case (see [23]).

3. Tense $n \times m$-valued Łukasiewicz-Moisil algebras

In this section we introduce tense $\mathrm{LM}_{n\times m}$-algebras. The notion of the tense $\mathrm{LM}_{n\times m}$-algebra is obtained by endowing an $\mathrm{LM}_{n\times m}$-algebra with two unary operations G and H, similar to the tense operators on an n-valued Łukasiewicz-Moisil algebra. Here are the basic definitions and properties.

Definition 3.1. An algebra $(\mathcal{L}, G, H)$ is a tense $n\times m$-valued Łukasiewicz-Moisil algebra (or tense $\mathrm{LM}_{n\times m}$-algebra) if

$$\mathcal{L} = \langle L, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, 0_L, 1_L\rangle$$

is an $\mathrm{LM}_{n\times m}$-algebra and G and H are two unary operators on L such that:

(T1) $G(1_L) = 1_L$ and $H(1_L) = 1_L$,
(T2) $G(x \wedge y) = G(x) \wedge G(y)$ and $H(x \wedge y) = H(x) \wedge H(y)$,
(T3) $x \leqslant GP(x)$ and $x \leqslant HF(x)$, where $F(x) = \sim G(\sim x)$ and $P(x) = \sim H(\sim x)$,
(T4) $G(\sigma_{ij}(x)) = \sigma_{ij}(G(x))$ and $H(\sigma_{ij}(x)) = \sigma_{ij}(H(x))$ for all $(i,j) \in (n \times m)$.

In the following we will indicate the class of tense $\mathrm{LM}_{n\times m}$-algebras with $\mathbf{tLM}_{n\times m}$ and we will denote its elements simply by L or (L, G, H) in case we need to specify the tense operators.

Remark 3.1. (i) From Definition 3.1 and (LM4) we infer that $\mathbf{tLM}_{n\times m}$ is a variety and two equational bases for it can be obtained.

(ii) If (L, G, H) is a tense $\mathrm{LM}_{n\times m}$-algebra, then from (LM1) and (T4) we have that $(C(L), C(G), C(H))$ is a Boolean algebra, where the unary operations $C(G)\colon C(L) \to C(L)$ and $C(H)\colon C(L) \to C(L)$, are defined by $C(G) = G|_{C(L)}$ and $C(H) = H|_{C(L)}$.

(iii) Taking into account (LM2), we infer that every tense $\mathrm{LM}_{n\times 2}$-algebra is isomorphic to a tense n-valued Łukasiewicz-Moisil algebra.

According to this remark one gets the following result:

Lemma 3.1. *The following conditions hold in any tense* $\mathrm{LM}_{n\times m}$*-algebra* (L, G, H)*:*

(T5) $x \leqslant y$ *implies* $G(x) \leqslant G(y)$ *and* $H(x) \leqslant H(y)$,
(T6) $x \leqslant y$ *implies* $F(x) \leqslant F(y)$ *and* $P(x) \leqslant P(y)$,
(T7) $F(0_L) = 0_L$ *and* $P(0_L) = 0_L$,
(T8) $F(x \vee y) = F(x) \vee F(y)$ *and* $P(x \vee y) = P(x) \vee P(y)$,
(T9) $FH(x) \leqslant x$ *and* $PG(x) \leqslant x$,
(T10) $GP(x) \wedge F(y) \leqslant F(P(x) \wedge y)$ *and* $HF(x) \wedge P(y) \leqslant P(F(x) \wedge y)$,
(T11) $G(x) \wedge F(y) \leqslant F(x \wedge y)$ *and* $H(x) \wedge P(y) \leqslant P(x \wedge y)$,
(T12) $G(x) \wedge F(y) \leqslant G(x \wedge y)$ *and* $H(x) \wedge P(y) \leqslant H(x \wedge y)$,
(T13) $G(x \vee y) \leqslant G(x) \vee F(y)$ *and* $H(x \vee y) \leqslant H(x) \vee P(y)$.

Proposition 3.1. *Let* $\langle L, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, 0, 1\rangle$ *be an* $\mathrm{LM}_{n\times m}$*-algebra and* G, H *two unary operations on* L *that satisfy conditions* (T1) *and* (T4)*. Then condition* (T2) *is equivalent to:* (T2)$'$ $G(a \hookrightarrow_{i,j} b) \leqslant G(a) \hookrightarrow_{i,j} G(b)$ *and* $H(a \hookrightarrow_{i,j} b) \leqslant H(a) \hookrightarrow_{i,j} H(b)$ *for all* $(i,j) \in (n \times m)$.

Proof. We will only prove the equivalence between (T2) and (T2)$'$ in the case of G.

(T2) $\Rightarrow$ (T2)$'$. Let $(i,j) \in (n \times m)$. We obtain that $G(a \hookrightarrow_{i,j} b) \in C(L)$ and $G(a) \hookrightarrow_{i,j} G(b) \in C(L)$, so $G(a) \hookrightarrow_{i,j} G(b)$ has a complement $\neg(G(a) \hookrightarrow_{i,j} G(b)) =$

$\sim\sigma_{rs}(G(a) \hookrightarrow_{i,j} G(b))$ for all $(r,s) \in (n \times m)$. Then, we have: $G(a \hookrightarrow_{i,j} b) \wedge \sim\sigma_{rs}(G(a) \hookrightarrow_{i,j} G(b)) = G(\sim\sigma_{ij}(a) \vee \sigma_{ij}(b)) \wedge \sim\sigma_{rs}(\sim\sigma_{ij}(G(a)) \vee \sigma_{ij}(G(b))) = G(\sim\sigma_{ij}(a) \vee \sigma_{ij}(b)) \wedge \sigma_{ij}(G(a)) \wedge \sim\sigma_{ij}G(b) = G(\sim\sigma_{ij}(a) \vee \sigma_{ij}(b)) \wedge G\sigma_{ij}(a) \wedge \sim\sigma_{ij}G(b) = G((\sim\sigma_{ij}(a) \vee \sigma_{ij}(b)) \wedge \sigma_{ij}(a)) \wedge \sim\sigma_{ij}G(b) = G(\sigma_{ij}(a \wedge b)) \wedge \sim\sigma_{ij}G(b) = \sigma_{ij}G(a \wedge b) \wedge \sim\sigma_{ij}G(b) \leqslant \sigma_{ij}G(b) \wedge \sim\sigma_{ij}G(b) = 0_L$, so $G(a \hookrightarrow_{i,j} b) \wedge \sim\sigma_{rs}(G(a) \hookrightarrow_{i,j} G(b)) = 0_L$. It follows that $G(a \hookrightarrow_{i,j} b) \leqslant G(a) \hookrightarrow_{i,j} G(b)$.

(T2)$'$ $\Rightarrow$ (T2). Let $a, b \in L$ be such that $a \leqslant b$. By (WI6) we obtain that $a \hookrightarrow_{i,j} b = 1_L$ for all $(i,j) \in (n \times m)$, so $1_L = G(1_L) = G(a \hookrightarrow_{i,j} b) \leqslant G(a) \hookrightarrow_{i,j} G(b)$ for all $(i,j) \in (n \times m)$. By using (WI7) we have that $G(a) \leqslant G(b)$, so G is increasing. It follows that $G(a \wedge b) \leqslant G(a) \wedge G(b)$. From (WI2) and (WI7) we obtain $a \leqslant b \hookrightarrow_{i,j} (a \wedge b)$ for all $(i,j) \in (n \times m)$, then $G(a) \leqslant G(b \hookrightarrow_{i,j} (a \wedge b)) \leqslant G(b) \hookrightarrow_{i,j} G(a \wedge b)$ for all $(i,j) \in (n \times m)$. By (WI13) it follows that $G(a) \wedge G(b) \leqslant G(a \wedge b)$. Therefore, $G(a \wedge b) = G(a) \wedge G(b)$. □

Thus, if in Definition 3.1 we replace axiom (T2) by (T2$'$), we obtain an equivalent definition for tense $\mathrm{LM}_{n\times m}$-algebras.

Definition 3.2. Let (X, R) be a frame and $\langle L, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, 0_L, 1_L\rangle$ a complete and completely chrysippian $\mathrm{LM}_{n\times m}$-algebra. We will define on L^X the following operations:

$$G(p)(x) = \bigwedge\{p(y);\ y \in X, x\ R\ y\} \quad \text{and} \quad H(p)(x) = \bigwedge\{p(y);\ y \in X, y\ R\ x\},$$

for all $p \in L^X$ and $x \in X$.

Proposition 3.2. *For any frame (X, R), (L^X, G, H) is an $\mathrm{LM}_{n\times m}$-algebra.*

P r o o f. Since L is an $\mathrm{LM}_{n\times m}$-algebra hence by (LM5) we have that L^X is an $\mathrm{LM}_{n\times m}$-algebra. Now, we will prove that G and H satisfy conditions (T1)–(T4) in Definition 3.1. Note that properties (T1)–(T3) are already proved in the n-valued Łukasiewicz-Moisil case. We will prove only (T4). Let $f \in L^X$, $x \in X$ and $(i,j) \in (n\times m)$. Using the fact that L is completely chrysippian we have that: $G(\sigma_{ij}(f))(x) = \bigwedge\{\sigma_{ij}(f)(y);\ y \in X, x\ R\ y\} = \sigma_{ij}(\bigwedge\{f(y);\ y \in X,\ x\ R\ y\}) = \sigma_{ij}(G(f)(x)) = \sigma_{ij}(G(f))(x)$. □

R e m a r k 3.2. In the tense $\mathrm{LM}_{n\times m}$-algebra (L^X, G, H) the tense operators P and F are defined in the following way: $P(p)(x) = \bigvee\{p(y);\ y \in X, y\ R\ x\}$ and $F(p)(x) = \bigvee\{p(y);\ y \in X, x\ R\ y\}$.

Definition 3.3. Let $(\mathcal{L}, G, H)$ and $(\mathcal{L}', G', H')$ be two tense $\mathrm{LM}_{n\times m}$-algebras. A function $f\colon L \to L'$ is a morphism of tense $\mathrm{LM}_{n\times m}$-algebras if f is an $\mathrm{LM}_{n\times m}$-algebra morphism and satisfies the conditions $f(G(x)) = G'(f(x))$ and $f(H(x)) = H'(f(x))$ for any $x \in L$.

Definition 3.4. Let (X,R) and (Y,Q) be two frames. A function $u\colon (X,R)\to (Y,Q)$ is a frame morphism if the following condition is satisfied: $a\ R\ b$ implies $u(a)\ Q\ u(b)$ for all $a,b\in X$.

Let $\langle L,\wedge,\vee,\sim,\{\sigma_{ij}\}_{(i,j)\in(n\times m)},0_L,1_L\rangle$ be an $\mathrm{LM}_{n\times m}$-algebra and $u\colon (X,R)\to (Y,Q)$ a frame morphism. We consider the function $u^*\colon L^Y\to L^X$, defined by: $u^*(p)=p\circ u$ for all $p\in L^Y$.

Proposition 3.3. *Let $\mathcal{L}$ be a complete and completely chrysippian $\mathrm{LM}_{n\times m}$-algebra, (X,R), (Y,Q) two frames and $u\colon (X,R)\to(Y,Q)$ a frame morphism which satisfies the following conditions:*

(a) *A morphism $u\colon X\to Y$ is surjective.*
(b) *If $u(a)\ Q\ u(b)$ then $a\ R\ b$ for all $a,b\in X$.*

Then u^* is a morphism of tense $\mathrm{LM}_{n\times m}$-algebras.

Proof. We will only prove that $u^*\circ G=G\circ u^*$. Let $p\in L^Y$ and $x\in X$. We have $u^*(G(p))(x)=(G(p)\circ u)(x)=G(p)(u(x))=\bigwedge\{p(b);\ b\in Y, u(x)\ Q\ b\}$ and $G(u^*(p))(x)=G(p\circ u)(x)=\bigwedge\{(p(u(a)));\ a\in X, x\ R\ a\}$.

(1) Let $a\in X$ with $x\ R\ a$. It follows that $u(a)\in Y$ and $u(x)\ Q\ u(a)$, so $\{p(u(a));\ a\in X, x\ R\ a\}\subseteq\{p(b);\ b\in Y, u(x)\ Q\ b\}$, hence $\bigwedge\{p(b);\ b\in Y, u(x)\ Q\ b\}\leqslant\bigwedge\{p(u(a));\ a\in X, x\ R\ a\}$.
(2) Let $b\in Y$ with $u(x)\ Q\ b$. By conditions (a) and (b) it follows that there exists $a\in X$ such that $b=u(a)$ and $x\ R\ a$. We get that $\{p(b);\ b\in Y, u(x)\ Q\ b\}\subseteq\{p(u(a));\ a\in X, x\ R\ a\}$, so $\bigwedge\{p(u(a));\ a\in X, x\ R\ a\}\leqslant\bigwedge\{p(b);\ b\in Y, u(x)\ Q\ b\}$.

By (1) and (2) it results that $u^*(G(p))(x)=G(u^*(p))(x)$, so $u^*\circ G=G\circ u^*$. $\square$

4. Representation theorem for tense $\mathrm{LM}_{n\times m}$-algebras

In this section we give a representation theorem for tense $\mathrm{LM}_{n\times m}$-algebras. To prove this theorem we use the representation theorem for tense Boolean algebras.

Let $(\mathcal{B},G,H)$ be a tense Boolean algebra. We consider the set of all increasing functions in each component from $(n\times m)$ to B, that is, $D(B)=B{\uparrow}^{(n\times m)}=\{f\colon (n\times m)\longrightarrow B$ such that for arbitrary i, j if $r\leqslant s$, then $f(r,j)\leqslant f(s,j)$ and $f(i,r)\leqslant f(i,s)\}$.

We define on $D(B)$ unary operations $D(G)$ and $D(H)$ by:

$$D(G)(f)=G\circ f\quad\text{and}\quad D(H)(f)=H\circ f\quad\text{for all } f\in D(L).$$

The following result is necessary for the proof of Theorem 4.1.

Lemma 4.1. *If* $(\mathcal{B}, G, H)$ *is a Boolean algebra then* $(D(B), D(G), D(H))$ *is a tense* $\mathrm{LM}_{n\times m}$*-algebra.*

P r o o f. By (LM7), $D(B)$ is an $\mathrm{LM}_{n\times m}$-algebra. We will prove that $D(G)$ and $D(H)$ verify (T1)–(T4) of Definition 3.1.

(T1): Let $f \in D(B)$ and $(i,j) \in (n \times m)$. Then $D(G)(1_{D(B)})(i,j) = (G \circ 1_{D(B)})(i,j) = G(1_{D(B)})(i,j) = G(1_B) = 1_B$, hence $D(G)(1_{D(B)}) = 1_{D(B)}$.

(T2): Let $f, g \in D(B)$ and $(i,j) \in (n \times m)$. We have $D(G)(f \wedge g)(i,j) = (G \circ (f \wedge g))(i,j) = G((f \wedge g)(i,j)) = G(f(i,j) \wedge g(i,j)) = Gf(i,j) \wedge Gg(i,j) = (G \circ f)(i,j) \wedge (G \circ g)(i,j) = D(G)(f)(i,j) \wedge D(G)(g)(i,j) = (D(G)(f) \wedge D(G)(g))(i,j)$, so $D(G)(f \wedge g) = D(G)(f) \wedge D(G)(g)$.

(T3): Let $f \in D(B)$ and $(i,j) \in (n \times m)$. Then $D(G) \sim D(H)(\sim f)(i,j) = D(G) \sim D(H)(\neg f)(n-i, m-j) = D(G) \sim (H \circ \neg f)(n-i, m-j) = D(G)\neg(H \circ \neg f)(i,j) = (G \circ \neg H \circ \neg f)(i,j)$. Since (L, G, H) is a tense Boolean algebra we have that $f(i,j) \leqslant G\neg H \neg f(i,j)$. Therefore, $f \leqslant D(G) \sim D(H) \sim f$.

(T4): Let $f \in D(B)$ and $(i,j), (r,s) \in (n \times m)$. Then $D(G)(\sigma_{rs}(f)(i,j)) = (G \circ (\sigma_{rs} f))(i,j) = G((\sigma_{rs} f)(i,j)) = Gf(r,s) = (G \circ f)(r,s) = D(G)(f)(r,s) = \sigma_{rs}(D(G)(f))(i,j)$, so $D(G)(\sigma_{rs}) = \sigma_{rs}(D(G))$.

□

Definition 4.1. Let $(\mathcal{B}, G, H)$, $(\mathcal{B}', G, H)$ be two tense Boolean algebras, $f\colon B \to B'$ a tense Boolean morphism and $D(B)$ and $D(B')$ the corresponding tense $\mathrm{LM}_{n\times m}$-algebras. We will extend the function f to a function $D(f)\colon D(B) \to D(B')$ in the following way: $D(f)(u) = f \circ u$ for every $u \in D(B)$.

Lemma 4.2. *The function* $D(f)\colon D(B) \to D(B')$ *is a morphism of tense* $\mathrm{LM}_{n\times m}$*-algebras.*

P r o o f. Since f is a Boolean morphism it is easy to prove that $D(f)$ is a bounded lattice homomorphism. Let $u \in D(B)$ and $(i,j), (r,s) \in (n \times m)$. Then we have that

$$D(f)(\sigma_{rs} u)(i,j) = f((\sigma_{rs} u)(i,j)) = f(u(r,s))$$

and $\sigma_{rs}(D(f)(u))(i,j) = D(f)(u)(r,s) = f(u(r,s))$. It follows that $D(f) \circ \sigma_{rs} = \sigma_{rs} \circ D(f)$. On the other hand, $D(f)(D(G)u)(r,s) = (f \circ (D(G)u))(r,s) = f(D(G)u)(r,s)$.
□

Lemma 4.3. *If $f\colon B \to B'$ is an injective morphism of tense Boolean algebras then $D(f)\colon D(B) \to D(B')$ is an injective morphism of tense* $\mathrm{LM}_{n\times m}$*-algebras.*

Proof. By Lemma 4.2, it remains to prove that $D(f)$ is injective. Let $u, v \in D(B)$ be such that $D(f)(u) = D(f)(v)$, then $f(u(i,j)) = f(v(i,j))$ for all $(i,j) \in (n \times m)$. Since f is injective we obtain that $u(i,j) = v(i,j)$ for all $(i,j) \in (n \times m)$. Therefore, $u = v$. $\square$

Definition 4.2. Let (L, G, H) be a tense $\mathrm{LM}_{n\times m}$-algebra. We consider the function $\tau_L\colon L \to D(C(L))$, defined by $\tau_L(x)(i,j) = \sigma_{ij}(x)$ for all $x \in L$, $(i,j) \in (n \times m)$.

Lemma 4.4. *A mapping τ_L is an injective morphism in* $\mathbf{tLM}_{n\times m}$.

Proof. Taking into account [22], Theorem 3.1, the mapping $\tau_L\colon L \to D(C(L))$ is a one-to-one $\mathrm{LM}_{n\times m}$-morphism. Besides, from (T4) it is simple to check that $\tau_L(G(x)) = G(\tau_L(x))$ and $\tau_L(H(x)) = H(\tau_L(x))$ for all $x \in L$. $\square$

Definition 4.3. Let $(\mathcal{B}, G, H)$ be a tense Boolean algebra. We consider the function $\phi_B\colon B \to C(D(B))$, defined by $\phi_B(x) = f_x$ where $f_x\colon (n \times m) \to B$, $f_x(i,j) = x$ for all $(i,j) \in (n \times m)$.

Lemma 4.5. *ϕ_B is an isomorphism of tense Boolean algebras.*

Proof. Let $x \in B$ and $f_x\colon (n\times m) \to B$ with $f_x(i,j) = x$ for all $(i,j) \in (n\times m)$. It follows that f_x is increasing in each component and $\sigma_{rs}(f_x) = f_x$ for all $(r,s) \in (n \times m)$, so $f_x \in C(D(B))$. We obtain that ϕ_B is well defined. It is easy to prove that ϕ_B is a Boolean morphism. Let us check that ϕ_B commutes with G and H. Let $x \in B$ and $(i,j) \in (n \times m)$. We have:

(a) $\phi_B(G(x))(i,j) = f_{G(x)}(i,j) = G(x)$.

(b) $C(D(G))(\phi_B(x))(i,j) = D(G)|_{C(D(B))}(\phi_B(x))(i,j) = (G \circ \phi_B(x))(i,j) = G(\phi_B(x)(i,j)) = G(f_x(i,j)) = G(x)$.

By (a) and (b) we obtain that $\phi_B \circ G = C(D(G)) \circ \phi_B$. The homomorphism $\phi_B\colon B \to C(D(B))$ is injective because $\phi_B(x) = \phi_B(y)$ implies $f_x = f_y$, hence $f_x(i,j) = f_y(i,j)$ for all $(i,j) \in (n \times m)$, so $x = y$. To prove surjectivity we take $g \in C(D(B))$. Then $\sigma_{ij}(g) = g$ for all $(i,j) \in (n\times m)$, which means that $\sigma_{ij}(g)(r,s) = g(r,s)$ for all $(i,j), (r,s) \in (n \times m)$. But $\sigma_{ij}(g)(r,s) = g(i,j)$, hence $g(i,j) = g(r,s)$ for all $(i,j), (r,s) \in (n \times m)$, hence g is constant. Therefore $\phi_B(g) = g$. It follows that ϕ_B is an isomorphism. $\square$

Lemma 4.6. *Let (L, G, H) be a tense* $\mathrm{LM}_{n\times m}$*-algebra. The following implications hold:*

(i) *If $C(G) = \mathrm{id}_{C(L)}$ then $G = \mathrm{id}_L$.*
(ii) *If $C(H) = \mathrm{id}_{C(L)}$ then $H = \mathrm{id}_L$.*

P r o o f. (i) Let $x \in L$. We have that $\sigma_{ij}(x) \in C(L)$ for all $(i, j) \in (n \times m)$. By the hypothesis it follows that $G\sigma_{ij}(x) = \sigma_{ij}(x)$. Using the fact that G commutes with σ_{ij}, we obtain $\sigma_{ij}G(x) = \sigma_{ij}(x)$ for all $(i, j) \in (n \times m)$. By (C5) it follows that $G(x) = x$. Therefore, $G = \mathrm{id}_L$. □

Lemma 4.7. *Let (L, G, H) be a tense* $\mathrm{LM}_{n\times m}$*-algebra. The following implications hold:*

(i) *If $C(G) = I_{C(L)}$ then $G = I_L$.*
(ii) *If $C(H) = I_{C(L)}$ then $H = I_L$.*

P r o o f. (i) Let $x \in L$. We have that $\sigma_{ij}(x) \in C(L)$ for all $(i, j) \in (n \times m)$. By the hypothesis it follows that $C(G)(\sigma_{ij}(x)) = 1_{C(L)}$. Since $C(G) = G|_{C(L)}$ we obtain that $C(G)(\sigma_{ij}x) = G(\sigma_{ij}x) = \sigma_{ij}G(x) = 1_{C(L)} = \sigma_{ij}(1_L)$ for all $(i, j) \in (n \times m)$. By (C5) it results that $G(x) = 1_L$. Therefore $G = I_L$. □

Proposition 4.1. *Let $(D(\mathbf{2}), G, H)$ be a tense* $\mathrm{LM}_{n\times m}$*-algebra. Then $G = H = \mathrm{id}_{D(\mathbf{2})}$ or $G = H = I_{D(\mathbf{2})}$.*

P r o o f. Let $(C(D(\mathbf{2})), C(G), C(H))$ be the tense Boolean algebra obtained from the tense $\mathrm{LM}_{n\times m}$-algebra $(D(\mathbf{2}), G, H)$ and $\phi_{\mathbf{2}}\colon \mathbf{2} \to C(D(\mathbf{2}))$ as defined in Definition 4.3. Let us consider the functions $G^*, H^*\colon \mathbf{2} \to \mathbf{2}$, defined by: $G^* = \phi_{\mathbf{2}}^{-1} \circ C(G) \circ \phi_{\mathbf{2}}$ and $H^* = \phi_{\mathbf{2}}^{-1} \circ C(H) \circ \phi_{\mathbf{2}}$. First, we will prove that $(\mathbf{2}, G^*, H^*)$ is a tense Boolean algebra. We have to verify the axioms (tb1)–(tb3) of Definition 2.1.

(tb1) We must prove that $G^*(1_{\mathbf{2}}) = 1_{\mathbf{2}}$. We have $G^*(1_{\mathbf{2}}) = (\phi_{\mathbf{2}}^{-1} \circ C(G) \circ \phi_{\mathbf{2}})(1_{\mathbf{2}}) = (\phi_{\mathbf{2}}^{-1} \circ C(G))(\phi_{\mathbf{2}}(1_{\mathbf{2}})) = \phi_{\mathbf{2}}^{-1}(C(G)(1_{C(D(\mathbf{2}))})) = \phi_{\mathbf{2}}^{-1}(1_{C(D(\mathbf{2}))}) = \phi_{\mathbf{2}}^{-1}(\phi_{\mathbf{2}}(1_{\mathbf{2}})) = 1_{\mathbf{2}}$, so $G^*(1_{\mathbf{2}}) = 1_{\mathbf{2}}$.

(tb2) By applying the fact that $C(G)$, $\phi_{\mathbf{2}}$ and $\phi_{\mathbf{2}}^{-1}$ commute with $\wedge$.

(tb3) Let $x, y \in \mathbf{2}$ be such that $G^*(x) \vee y = 1_{\mathbf{2}}$. Thus $(\phi_{\mathbf{2}}^{-1} \circ C(G) \circ \phi_{\mathbf{2}})(x) \vee y = 1_{\mathbf{2}}$. It follows that $\phi_{\mathbf{2}}^{-1}(C(G)(\phi_{\mathbf{2}}(x))) \vee y = 1_{\mathbf{2}}$, hence $C(G)(\phi_{\mathbf{2}}(x)) \vee \phi_{\mathbf{2}}(y) = 1_{C(D(\mathbf{2}))}$. Since $C(G)$ and $C(H)$ verify (tb3) we obtain that $\phi_{\mathbf{2}}(x) \vee C(H)(\phi_{\mathbf{2}}(y)) = 1_{C(D(\mathbf{2}))}$. By applying $\phi_{\mathbf{2}}^{-1}$ it results that $x \vee (\phi_{\mathbf{2}}^{-1} \circ C(H) \circ \phi_{\mathbf{2}})(y) = 1_{\mathbf{2}}$, hence $x \vee H^*(y) = 1_{\mathbf{2}}$. The converse implication can be proved similarly.

Thus $(\mathbf{2}, G^*, H^*)$ is a tense Boolean algebra. According to Remark 2.2, we will study two cases: $G' = H' = \mathrm{id}_{\mathbf{2}}$ and $G^* = H^* = I_{\mathbf{2}}$.

(i) Suppose that $G^* = H^* = \mathrm{id}_{\mathbf{2}}$. Then we have that $C(G) = \phi_{\mathbf{2}} \circ G^* \circ \phi_{\mathbf{2}}^{-1}$ and $C(H) = \phi_{\mathbf{2}} \circ H^* \circ \phi_{\mathbf{2}}^{-1}$ so $C(G) = C(H) = \mathrm{id}_{C(D(\mathbf{2}))}$. By Lemma 4.6 it follows that $G = H = \mathrm{id}_{D(\mathbf{2})}$.

(ii) Suppose that $G^* = H^* = I_{D(\mathbf{2})}$. Let $g \in C(D(\mathbf{2}))$. Then $C(G)(g) = (\phi_{\mathbf{2}} \circ G^* \circ \phi_{\mathbf{2}}^{-1})(g) = \phi_{\mathbf{2}}(G^*(\phi_{\mathbf{2}}^{-1}(g))) = \phi_{\mathbf{2}}(1_{\mathbf{2}}) = 1_{C(D(\mathbf{2}))}$. Hence $C(G) = I_{C(D(\mathbf{2}))}$.

Similarly we can obtain that $C(H) = I_{C(D(\mathbf{2}))}$. By applying Lemma 4.7 it results that $G = H = I_{D(\mathbf{2})}$. □

Definition 4.4. Let (X, R) be a frame and $(\mathbf{2}^X, G, H)$ the tense Boolean algebra of Proposition 2.2. We consider the function

$$\beta\colon (D(\mathbf{2}^X), D(G), D(H)) \to (D(\mathbf{2})^X, G', H')$$

defined by $\beta(f)(x)(i,j) = f(i,j)(x)$ for all $f \in D(\mathbf{2}^X)$, $x \in X$, $(i,j) \in (n \times m)$, where G' and H' are defined by $G'(p)(x) = \bigwedge\{p(y);\ y \in X, x\ R\ y\}$ and $H'(p)(x) = \bigwedge\{p(y);\ y \in X, y\ R\ x\}$.

Lemma 4.8. *β is an isomorphism of tense* $\mathrm{LM}_{n\times m}$*-algebras.*

Proof. It is easy to see that β is an injective morphism of $\mathrm{LM}_{n\times m}$-algebras. It remains to prove that β commutes with the tense operators.

Let $f \in D(\mathbf{2}^X)$, $x \in X$ and $(i,j) \in (n \times m)$. We have:

(a) $\beta(D(G)(f))(x)(i,j) = D(G)(f)(i,j)(x) = G(f(i,j))(x) = \bigwedge\{f(i,j)(y);\ y \in X, x\ R\ y\}$.

(b) $G'(\beta(f))(x)(i,j) = \bigwedge\{\beta(f)(y)(i,j);\ y \in X, x\ R\ y\} = \bigwedge\{f(i,j)(y);\ y \in X, x\ R\ y\}$.

By (a) and (b), we obtain that $\beta(D(G)(f))(x)(i,j) = G'(\beta(f)(x))(i,j)$, so $\beta \circ D(G) = G' \circ \beta$. We define the function $\gamma\colon D(\mathbf{2})^X \to D(\mathbf{2}^X)$ by $\gamma(g)(i,j)(x) = g(x)(i,j)$ for all $g \in D(\mathbf{2})^X$, $x \in X$, $(i,j) \in (n \times m)$. Let $r \leqslant s$. For all $x \in X$ we have that $g(x) \in D(\mathbf{2})$, so $g(x)(r,j) \leqslant g(x)(s,j)$ and $g(x)(i,r) \leqslant g(x)(i,s)$. It follows that $\gamma(g)(r,j)(x) \leqslant \gamma(g)(s,j)(x)$ and $\gamma(g)(i,r)(x) \leqslant \gamma(g)(s,j)(x)$ for all $x \in X$, so $\gamma(g)(r,j) \leqslant \gamma(g)(s,j)$ and $\gamma(g)(i,r) \leqslant \gamma(g)(s,j)$. Hence, γ is well defined. We will prove that β and γ are inverse to each other. Let $g \in D(\mathbf{2})^X$, $x \in X$ and $(i,j) \in (n \times m)$. We have $(\beta \circ \gamma)(g)(x)(i,j) = \beta(\gamma(g))(x)(i,j) = \gamma(g)(i,j)(x) = g(x)(i,j)$, hence $(\beta \circ \gamma)(g) = g$. Let $f \in D(\mathbf{2}^X)$, $(i,j) \in (n \times m)$ and $x \in X$. Then $(\gamma \circ \beta)(f)(i,j)(x) = \gamma(\beta(f))(i,j)(x) = \beta(f)(x)(i,j) = f(i,j)(x)$, so $(\gamma \circ \beta)(f) = f$. □

Theorem 4.1 (The representation theorem for tense $\mathrm{LM}_{n\times m}$-algebras)**.** *For every tense* $\mathrm{LM}_{n\times m}$*-algebra* (L,G,H) *there exist a frame* (X,R) *and an injective morphism of tense* $\mathrm{LM}_{n\times m}$*-algebras* $\alpha\colon\ (L,G,H)\to(D(\mathbf{2})^X,G',H')$.

P r o o f. Let (L,G,H) be a tense $\mathrm{LM}_{n\times m}$-algebra. By Remark 3.1 we have that $(C(L),C(G),C(H))$ is a tense Boolean algebra. Applying the representation theorem for tense Boolean algebras, it follows that there exist a frame (X,R) and an injective morphism of tense Boolean algebras $d\colon\ (C(L),C(G),C(H))\to(\mathbf{2}^X,G,H)$. Let $D(d)\colon\ D(C(L))\to D(\mathbf{2}^X)$ be the corresponding morphism of d by the morphism D. Then by Lemma 4.3 we have that $D(d)$ is an injective morphism. On the other hand, using Lemma 4.4, we have an injective morphism of tense $\mathrm{LM}_{n\times m}$-algebras $\tau_L\colon\ L\to D(C(L))$. Besides, by Lemma 4.8, $\beta\colon\ D(\mathbf{2}^X)\to D(\mathbf{2})^X$ is an isomorphism of tense $\mathrm{LM}_{n\times m}$-algebras. Now, f in the diagram

$$L\longrightarrow^{\tau_L} D(C(L))\longrightarrow^{D(d)} D(\mathbf{2}^X)\longrightarrow^{\beta} D(\mathbf{2})^X$$

we consider the composition $\beta\circ D(d)\circ\tau_L$ we obtain the required injective morphism. □

Corollary 4.1. *For every tense* LM_n*-algebra* (L,G,H) *there exist a frame* (X,R) *and an injective morphism of tense* LM_n*-algebras* $\Phi\colon\ (L,G,H)\to(\mathrm{L}_n^X,G',H')$.

P r o o f. It is an immediate consequence of Remark 3.1 (ii), Theorem 4.1 and (LM10). □

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