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ON A GENERALIZATION OF A THEOREM OF BURNSIDE

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Abstract. A theorem of Burnside asserts that a finite group G is p-nilpotent if for some prime p a Sylow p-subgroup of G lies in the center of its normalizer. In this paper, let G be a finite group and p the smallest prime divisor of $|G|$, the order of G. Let $P \in \mathrm{Syl}_p(G)$. As a generalization of Burnside's theorem, it is shown that if every non-cyclic p-subgroup of G is self-normalizing or normal in G then G is solvable. In particular, if $P \ncong \langle a, b;\ a^{p^{n-1}} = 1,$ $b^2 = 1, b^{-1}ab = a^{1+p^{n-2}}\rangle$, where $n \geqslant 3$ for $p > 2$ and $n \geqslant 4$ for $p = 2$, then G is p-nilpotent or p-closed.

Keywords: non-cyclic p-subgroup; p-nilpotent; self-normalizing subgroup; normal subgroup

MSC 2010: 20D10

1. Introduction

Recall that a finite group G is said to be p-nilpotent if the Sylow p-subgroup P of G has a normal complement in G. For criteria for p-nilpotence of finite groups, a classical result is due to Burnside:

Theorem 1.1 ([2], Theorem 10.1.8). *If for some prime p a Sylow p-subgroup P of G lies in the center of its normalizer, then G is p-nilpotent.*

Following Burnside's theorem, a well-known result for p-nilpotence of finite groups is:

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Theorem 1.2 ([2], Theorem 10.1.9)**.** *Let p be the smallest prime divisor of $|G|$, the order of G. If the Sylow p-subgroup of G is cyclic, then G is p-nilpotent.*

Let G be a finite group and H a subgroup of G. By $N_G(H)$ we denote the normalizer of H in G. It is obvious that the following inequality holds for any subgroup H of G:

$$H \leqslant N_G(H) \leqslant G.$$

If $H = N_G(H)$ then H is said to be self-normalizing in G. And if $N_G(H) = G$ then H is said to be normal in G.

As a generalization of Theorems 1.1 and 1.2, consider finite groups with every non-cyclic p-subgroup being self-normalizing or normal. Then we have the following result, the proof of which is given in Section 2.

Theorem 1.3. *Let G be a finite group and p the smallest prime divisor of $|G|$. Let $P \in \mathrm{Syl}_p(G)$. If every non-cyclic p-subgroup of G is self-normalizing or normal in G, then G is solvable. In particular, if $P \not\cong \langle a, b;\ a^{p^{n-1}} = 1,\ b^2 = 1,\ b^{-1}ab = a^{1+p^{n-2}}\rangle$, where $n \geqslant 3$ for $p > 2$ and $n \geqslant 4$ for $p = 2$, then G is p-nilpotent or p-closed (that is, P is normal in G).*

Remark 1.4. (1) The group in Theorem 1.3 may be non-supersolvable, even if we assume that every non-cyclic subgroup of G of prime-power order is self-normalizing or normal. For example, every non-cyclic subgroup of A_4 of prime-power order is normal but A_4 is non-supersolvable.

(2) In Theorem 1.3, the hypothesis that p is the smallest prime divisor of $|G|$ cannot be removed. For example, take $p = 3$, it is obvious that A_5 satisfies the hypothesis since A_5 has no non-cyclic 3-subgroups. However, A_5 is non-solvable.

(3) In Theorem 1.3, if we assume that every non-abelian p-subgroup of G is self-normalizing or normal, we cannot conclude that G is solvable. For example, it is obvious that A_5 satisfies the hypothesis since A_5 has no non-abelian 2-subgroups. However, A_5 is non-solvable.

(4) In Theorem 1.3, if we assume that every abelian non-cyclic p-subgroup of G is self-normalizing or normal, we cannot claim that G is solvable. For example, it is obvious that $\mathrm{SL}_2(5)$ satisfies the hypothesis since $\mathrm{SL}_2(5)$ has no abelian non-cyclic 2-subgroups. However, $\mathrm{SL}_2(5)$ is non-solvable.

2. Proof of Theorem 1.3

P r o o f. (1) We first prove that G is solvable. Let G be a counterexample of minimal order. It follows that G is a minimal non-solvable group. Then $G/\Phi(G)$ is a minimal non-abelian simple group, where $\Phi(G)$ is the Frattini subgroup of G. Let $P \in \mathrm{Syl}_p(G)$.

(i) *Claim*: P is non-cyclic. Otherwise, assume that P is cyclic. Since p is the smallest prime divisor of $|G|$, G is p-nilpotent by [2], Theorem 10.1.9. Then P has a normal complement N in G. It follows that $N\Phi(G)/\Phi(G)$ is a nontrivial normal subgroup of $G/\Phi(G)$, a contradiction. So P is non-cyclic.

(ii) *Claim*: Every maximal subgroup of P is cyclic. Otherwise, assume that P_1 is a non-cyclic maximal subgroup of P. It is obvious that P_1 is not self-normalizing in G since $P \leqslant N_G(P_1)$. By the hypothesis, $P_1 \trianglelefteq G$. Since $G/\Phi(G)$ is a non-abelian simple group, $P_1\Phi(G)/\Phi(G)$ is a trivial normal subgroup of $G/\Phi(G)$. It follows that $P_1 \leqslant \Phi(G)$. It is obvious that $P \nleqslant \Phi(G)$. Then the Sylow p-subgroup of $G/\Phi(G)$ has order p. It follows that $G/\Phi(G)$ is p-nilpotent by [2], Theorem 10.1.9, a contradiction. So every maximal subgroup of P is cyclic.

(iii) *Claim*: Every proper subgroup of G is p-nilpotent. Otherwise, G has a proper subgroup M such that M is a minimal non-p-nilpotent group. By [2], Theorems 9.1.9 and 10.3.3, $M = P_2 \rtimes Q$, where $P_2 \in \mathrm{Syl}_p(M)$ and $Q \in \mathrm{Syl}_q(M)$, $p \neq q$. It is obvious that P_2 is non-cyclic. By (i) and (ii), we can assume $P = P_2$. Then $P < M \leqslant N_G(P)$. By the hypothesis, $P \trianglelefteq G$. It follows that $P\Phi(G)/\Phi(G)$ is a nontrivial normal subgroup of $G/\Phi(G)$, a contradiction. So every proper subgroup of G is p-nilpotent.

(iv) Final conclusion. It follows that G is a minimal non-p-nilpotent group. By [2], Theorem 10.3.3, any minimal non-p-nilpotent group is a minimal non-nilpotent group. Then any minimal non-p-nilpotent group is solvable by [2], Theorem 9.1.9, a contradiction. So G is solvable.

(2) In the sequel, suppose $P \ncong \langle a, b;\ a^{p^{n-1}} = 1, b^2 = 1, b^{-1}ab = a^{1+p^{n-2}}\rangle$, where $n \geqslant 3$ for $p > 2$ and $n \geqslant 4$ for $p = 2$. Assume that G is neither p-nilpotent nor p-closed. It follows that there exists a subgroup M of G such that M is a minimal non-p-nilpotent group. By [2], Theorems 9.1.9 and 10.3.3, $M = P_3 \rtimes Q$, where $P_3 \in \mathrm{Syl}_p(M)$ and $Q \in \mathrm{Syl}_q(M)$, $p \neq q$. Since M is non-p-nilpotent, P_3 is non-cyclic by [2], Theorem 10.1.9. Let $P \in \mathrm{Syl}_p(G)$ be such that $P_3 \leqslant P$.

(i) Suppose $P_3 = P$. Then $P < M \leqslant N_G(P)$. By the hypothesis, we have $P \trianglelefteq G$, that is G is p-closed, a contradiction.

(ii) Suppose $P_3 < P$. Then $P_3 < N_P(P_3) \leqslant N_G(P_3)$. By the hypothesis, one has $P_3 \trianglelefteq G$. Similarly, we have that every non-cyclic maximal subgroup of P is normal in G. Let P have at least two non-cyclic maximal subgroups. Suppose that they are P_4 and P_5. Then $P = P_4P_5 \trianglelefteq G$, a contradiction. Thus, P has a unique non-cyclic maximal subgroup. It follows that P must have at least one cyclic maximal subgroup. Then by [1], Chapter I, Theorem 14.9, we can easily get that $P \cong \langle a, b;\ a^{p^{n-1}} = 1, b^2 = 1, b^{-1}ab = a^{1+p^{n-2}} \rangle$, where $n \geqslant 3$ for $p > 2$ and $n \geqslant 4$ for $p = 2$, a contradiction.

So G is p-nilpotent or p-closed. □

3. Some remarks

In this section, we give some remarks on two simple propositions.

Proposition 3.1. *Let G be a finite group and p the smallest prime divisor of $|G|$. If every non-cyclic p-subgroup of G is self-normalizing in G, then G is p-nilpotent.*

Proof. Let G be a counterexample of minimal order. Then G is a minimal non-p-nilpotent group. By [2], Theorems 9.1.9 and 10.3.3, one has $G = P \rtimes Q$, where $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$, $p \neq q$. Since G is non-p-nilpotent, P is non-cyclic by [2], Theorem 10.1.9. Then by the hypothesis, $P = N_G(P)$. However, this is a contradiction since $N_G(P) = G > P$. So G is p-nilpotent. □

Remark 3.2. (1) In Proposition 3.1, the hypothesis that p is the smallest prime divisor of $|G|$ cannot be removed. For example, taking $p = 3$, it is obvious that A_5 satisfies the hypothesis since every 3-subgroup of A_5 is cyclic. However, A_5 is non-3-nilpotent.

(2) In Proposition 3.1, if we assume that every non-abelian p-subgroup of G is self-normalizing in G, we cannot claim that G is p-nilpotent. For example, every non-abelian 2-subgroup of the symmetric group S_4 is self-normalizing but S_4 is non-2-nilpotent.

(3) In Proposition 3.1, if we assume that every abelian non-cyclic p-subgroup of G is self-normalizing in G, we cannot claim that G is p-nilpotent. For example, it is obvious that $\mathrm{SL}_2(3)$ satisfies the hypothesis since $\mathrm{SL}_2(3)$ has no abelian non-cyclic 2-subgroups. However, $\mathrm{SL}_2(3)$ is non-2-nilpotent.

Proposition 3.3. *Let G be a finite group and p the smallest prime divisor of $|G|$. If every non-cyclic p-subgroup of G is normal in G, then G is p-nilpotent or p-closed.*

Proof. Let $P \in \mathrm{Syl}_p(G)$. If P is cyclic, then G is p-nilpotent by [2], Theorem 10.1.9. If P is non-cyclic, then $P \trianglelefteq G$ by the hypothesis. That is, G is p-closed. □

Remark 3.4. (1) In Proposition 3.3, the hypothesis that p is the smallest prime divisor of $|G|$ cannot be removed. For example, taking $p = 3$, it is obvious that A_5 satisfies the hypothesis since A_5 has no non-cyclic 3-subgroups. However, A_5 is neither 3-nilpotent nor 3-closed.

(2) In Proposition 3.3, if we assume that every non-abelian p-subgroup of G is normal in G, we cannot assert that G is p-nilpotent or p-closed. For example, it is obvious that A_5 satisfies the hypothesis since A_5 has no non-abelian 2-subgroups. However, A_5 is neither 2-nilpotent nor 2-closed.

(3) In Proposition 3.3, if we assume that every abelian non-cyclic p-subgroup of G is normal in G, we cannot assert that G is p-nilpotent or p-closed. For example, it is obvious that $\mathrm{SL}_2(5)$ satisfies the hypothesis since $\mathrm{SL}_2(5)$ has no abelian non-cyclic 2-subgroups. However, $\mathrm{SL}_2(5)$ is neither 2-nilpotent nor 2-closed.

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