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THE L^2 $\overline{\partial}$-CAUCHY PROBLEM ON WEAKLY q-PSEUDOCONVEX DOMAINS IN STEIN MANIFOLDS

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Abstract. Let X be a Stein manifold of complex dimension $n \geqslant 2$ and $\Omega \Subset X$ be a relatively compact domain with C^2 smooth boundary in X. Assume that Ω is a weakly q-pseudoconvex domain in X. The purpose of this paper is to establish sufficient conditions for the closed range of $\overline{\partial}$ on Ω. Moreover, we study the $\overline{\partial}$-problem on Ω. Specifically, we use the modified weight function method to study the weighted $\overline{\partial}$-problem with exact support in Ω. Our method relies on the L^2-estimates by Hörmander (1965) and by Kohn (1973).

Keywords: $\overline{\partial}$ operator; $\overline{\partial}$-Neumann operator; q-convex domain; Stein manifold

MSC 2010: 32F10, 32W05

1. Introduction

The solution of the $\overline{\partial}$-Neumann problem has many important applications in the theory of several complex variables and in partial differential equations, particulary in the study of the $\overline{\partial}$-problem with exact support. On domains with certain geometric conditions on the boundary, the question of existence of a solution to the $\overline{\partial}$-Neumann problem was settled through the works of Hörmander [10] and Kohn [11], [12]. In fact, Hörmander's results in [10] imply that there exists a bounded operator N on $L^2_{r,s}(\Omega)$, which inverts the complex Laplacian under the assumption that Ω is a bounded, pseudoconvex domain.

Following Hörmander [10], the $\overline{\partial}$-problem can be solved in L^2 if $\overline{\partial}$ satisfies $Z(q)$. As shown in Theorem 1.9.9 in [18], q-pseudoconvexity implies that for $L^2_{0,s+1}$-forms f in the kernel of $\overline{\partial}$, there exists an $L^2_{0,s}$-form u solving the $\overline{\partial}$-problem $\overline{\partial}u = f$. It has been proved recently, by several authors including Harrington-Raich [8], that N exists on q-forms in a q-pseudoconvex domain. Establishing the existence of the $\overline{\partial}$-Neumann operator leads to a particular solution to the $\overline{\partial}$-problem with support condition. Here, we are interested in the existence of such an L^2-solution u for given data f. More precisely, we prove the following result:

Theorem 1.1. *Let $\Omega \Subset X$ be a weakly q-pseudoconvex domain with C^2 boundary $b\Omega$ in a Stein manifold X of complex dimension $n \geqslant 2$. For any $q \leqslant s \leqslant n$ and for $f \in L^2_{r,s}(\Omega)$, $\operatorname{supp} f \subset \bar{\Omega}$, satisfying $\overline{\partial} f = 0$ in the distribution sense in X, there exists $u \in L^2_{r,s-1}(\Omega)$, $\operatorname{supp} u \subset \bar{\Omega}$ such that $\overline{\partial} u = f$ in the distribution sense in X.*

The $\overline{\partial}$-problem with exact support was considered by Derridj [6], [7] using Carleman type estimates for smooth domains with plurisubharmonic defining functions. Shaw [17] has obtained a solution to this problem in a pseudoconvex domain Ω with C^1 boundary in $\mathbb{C}^n$. If Ω is locally Stein in the complex projective space, Cao-Shaw-Wang [2] obtained a solution to this problem in Ω.

Also, in the setting of strictly q-convex (or concave) domains, the $\overline{\partial}$-problem with exact support has been studied by Sambou in his thesis, where he proves some Dolbeault isomorphism between the tangential Cauchy-Riemann cohomology groups of smooth forms and currents on hypersurfaces (see [16]). Abdelkader and Saber [1] studied this problem on strictly q-convex domains in a complex manifold. Saber [15] (respectively [14]) studied this problem on a weakly q-pseudoconvex domain with C^1-smooth boundary (respectively with Lipschitz boundary) in $\mathbb{C}^n$.

2. Notation and definitions

Let X be a complex manifold of complex dimension n with a Hermitian metric g. Let $\Omega \Subset X$ be an open submanifold with smooth boundary $b\Omega$ and defining function ϱ. Denote by $L_1, L_2, \ldots, L_n$ a C^∞ special boundary coordinate chart in a small neighborhood U of some point $z_0 \in b\Omega$, i.e., $L_i \in T^{1,0}$ on $U \cap \bar{\Omega}$ with L_i tangential for $1 \leqslant i \leqslant n-1$ and $\langle L_i, L_j \rangle = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. Denote $\bar{L}_1, \bar{L}_2, \ldots, \bar{L}_n$ the conjugate of $L_1, L_2, \ldots, L_n$, respectively; these form an orthonormal basis of $T^{1,0}$ on U. The dual basis of $(1,0)$ forms are $\omega^1, \ldots, \omega^n = \sqrt{2}\partial\varrho$. The Levi form associated to ϱ is defined by

$$\varrho_{jk} = \langle L_j \wedge \bar{L}_k, \partial\overline{\partial}\varrho \rangle, \quad j,k = 1, 2, \ldots, n-1.$$

Let $\big(\partial^2\varrho(z)/\partial z_j \overline{\partial} z_k\big)_{j,k=1}^{n-1}$ be the matrix of the Levi form $\partial\overline{\partial}\varrho(z)$ in the complex tangential direction at z. Let $\lambda_1(z) \leqslant \ldots \leqslant \lambda_{n-1}(z)$ be the eigenvalues of $(\varrho_{jk}(z))_{j,k=1}^{n-1}$.

A complex-valued differential form u of type (r,s) on X can be expressed as $u = \sum\limits_{I,J} u_{I,J}\,\mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J$, where I and J are strictly increasing multi-indices with lengths r and s, respectively. Let $C^\infty_{r,s}(X)$ be the space of complex-valued differential forms of class C^∞ and of type (r,s) on X. For $u,v \in C^\infty_{r,s}(X)$, we define a local inner product (u,v) induced by the Hermitian metric by $(u,v)=\sum\limits_{I,J} u_{I,J}\overline{v}_{I,J}$.

The Hodge star operator $\star$ is a linear map $\star\colon\ C^\infty_{r,s}(X)\to C^\infty_{n-s,n-r}(X)$ which satisfies $\overline{\star u}=\star\bar{u}$ (that is, $\star$ is a real operator) and $\star\star u=(-1)^{r+s}u$; for the proof cf. [13], Theorem 2.1. Let $C^\infty_0(\Omega)$ be the space of C^∞-functions with compact support in Ω. Let $C^\infty_{r,s}(\bar\Omega)=\{u|_{\bar\Omega};\ u\in C^\infty_{r,s}(X)\}$ be the subspace of $C^\infty_{r,s}(\Omega)$ whose elements can be extended smoothly up to the boundary $b\Omega$. Let $L^2_{r,s}(\Omega)$ be the space of (r,s)-forms on Ω with square-integrable coefficients. If φ is a smooth function in Ω, the weighted L^2-inner product and norms are defined by

$$\langle u,v\rangle_\varphi=\int_\Omega (u,v)\mathrm{e}^{-\varphi}\,\mathrm{d}V \quad\text{and}\quad \|u\|^2_\varphi=\langle u,u\rangle_\varphi,$$

where $\mathrm{d}V$ is the volume element. We write

$$\mathrm{d}\varphi=\sum_{j=1}^n L_j(\varphi)\omega_j+\sum_{j=1}^n \bar{L}_j(\varphi)\overline{\omega}_j.$$

Then one defines

$$\partial\varphi=\sum_{j=1}^n L_j(\varphi)\omega_j \quad\text{and}\quad \overline{\partial}\varphi=\sum_{j=1}^n \bar{L}_j(\varphi)\overline{\omega}_j.$$

We denote by φ_{jk} the coefficients in $\partial\overline{\partial}\varphi=\sum\limits_{jk}\varphi_{jk}\omega_j\wedge\overline{\omega}_k$, that is,

$$\varphi_{jk}=\langle L_j\wedge\bar{L}_k,\partial\overline{\partial}\varphi\rangle,\quad j,k=1,2,\ldots,n.$$

The Cauchy-Riemann operator $\overline{\partial}\colon\ C^\infty_{r,s-1}(\Omega)\to C^\infty_{r,s}(\Omega)$ satisfies

$$\overline{\partial}u=\sum_{I,J}\sum_{k=1}^n \bar{L}_k u_{I\overline{J}}\overline{\omega}^k\wedge\omega^I\wedge\overline{\omega}^J+\ldots, \tag{2.1}$$

where the dots refer to terms of order zero in u. Let $\mathcal{D}^{r,s}(U)$ be the space of (r,s)-forms u on U such that

$$u_{I,J}=0\quad\text{on } b\Omega \text{ when } n\in J. \tag{2.2}$$

Then, for forms $u\in\mathcal{D}^{r,s}(U)$, we have

$$\overline{\partial}^\star_\varphi u=(-1)^{r-1}\sum_{I,K}\sum_{j=1}^n \delta^\varphi_j u_{IjK}\omega^I\wedge\overline{\omega}^K+\ldots, \tag{2.3}$$

where $\delta_j^\varphi = \mathrm{e}^\varphi L_j(\mathrm{e}^{-\varphi})$ and the dots refer to terms of order zero in u. Let $\overline{\partial}\colon \operatorname{dom}\overline{\partial} \subset L^2_{r,s}(\Omega) \to L^2_{r,s+1}(\Omega)$ be the maximal closure of the Cauchy-Riemann operator and $\overline{\partial}^*_\varphi$ be its Hilbert space adjoint of $\overline{\partial}$. For $1 \leqslant s \leqslant n$, we denote by $\Box_\varphi = \overline{\partial}\overline{\partial}^*_\varphi + \overline{\partial}^*_\varphi\overline{\partial}\colon \operatorname{dom}\Box_\varphi \to L^2_{r,s}(\Omega)$ the Laplace-Beltrami operator, where $\operatorname{dom}\Box_\varphi = \{u \in \operatorname{dom}\overline{\partial} \cap \operatorname{dom}\overline{\partial}^*_\varphi;\ \overline{\partial}u \in \operatorname{dom}\overline{\partial}^*_\varphi$ and $\overline{\partial}^*_\varphi u \in \operatorname{dom}\overline{\partial}\}$. Thus

$$\mathcal{H}_\varphi(\Omega) = \{u \in \operatorname{dom}(\Box_\varphi);\ \overline{\partial}u = \overline{\partial}^*_\varphi u = 0\}.$$

Then $\mathcal{H}_\varphi(\Omega)$ is a closed subspace of $\operatorname{dom}(\Box_\varphi)$ since $\Box_\varphi$ is a closed operator. One defines the $\overline{\partial}$-Neumann operator $N_\varphi\colon L^2_{r,s}(\Omega) \to L^2_{r,s}(\Omega)$ as the inverse of the restriction of $\Box_\varphi$ to $(\mathcal{H}_\varphi(\Omega))^\perp$.

Definition 2.1. We say that $u \in L^2_{r,s}(\Omega)$ is supported in $\bar{\Omega}$ ($\operatorname{supp} u \subset \bar{\Omega}$) or u vanishes to infinite order at the boundary of Ω if u vanishes on $b\Omega$.

Definition 2.2 (Ho [9])**.** We say that Ω is weakly q-pseudoconvex domain ($q \geqslant 1$) if at every point $x_0 \in b\Omega$ we have

$$\sum_{|K|}{}' \sum_{j,k} \frac{\partial^2 \varrho}{\partial z_j \partial \overline{z}_k} u_{jK} \bar{u}_{kK} \geqslant 0 \quad \text{for every } (0,q)\text{-form } u = \sum_{|J|=q} u_J \, \mathrm{d}\overline{z}^J$$

such that $\sum_{j=1}^n (\partial\varrho/\partial z_j) u_{jK} = 0$ for all $|K| = q-1$.

Definition 2.3. A complex manifold X is said to be a Stein manifold if there exists an exhaustion function $\mu \in C^2(X, \mathbb{R})$ such that $\mathrm{i}\partial\overline{\partial}\mu > 0$ on X.

Remark 2.4. If we take $\varphi_t = t\mu$, $t \geqslant 0$ and use the notation $\|\cdot\|_t = \|\cdot\|_{\varphi_t}$, $\langle , \rangle_t = \langle , \rangle_{\varphi_t}$ and $\overline{\partial}^\star_t = \overline{\partial}^\star_{\varphi_t}$, $\Box_{\varphi_t} = \Box^t$, $N_{\varphi_t} = N^t$ and $\mathcal{H}_{\varphi_t}(\Omega) = \mathcal{H}_t(\Omega)$, it is known that $\operatorname{dom}\overline{\partial}^\star_t = \operatorname{dom}\overline{\partial}^\star$ (e.g., [3], Chapter 4). In that case $\langle f, g\rangle_t$ denotes $\langle f, g\rangle_{\varphi_t}$, that is, we use subscripts t instead of φ_t. The inner product $\langle f, g\rangle_t$ and the norm $\|f\|_t^2$, in $L^2_{p,q}(\Omega)$, are denoted by

$$\langle f, g\rangle_t = \int_\Omega f \wedge \star_t \overline{g} \quad \text{and} \quad \|f\|_t^2 = \langle f, f\rangle_t, \quad \text{where } \star_t = \mathrm{e}^{-\varphi_t}\star = \star\mathrm{e}^{-\varphi_t}.$$

Lemma 2.5. *Let $\Omega \Subset X$ be a smooth domain in a Stein manifold X and ϱ be its defining function. The following two conditions are equivalent:*

(i) *Ω is weakly q-pseudoconvex.*

(ii) *For any $z \in b\Omega$ the sum of any q eigenvalues $\varrho_{i_1}, \ldots, \varrho_{i_q}$, with distinct subscripts, of the Levi-form at z satisfies $\sum_{j=1}^q \varrho_{i_j} \geqslant 0$.*

3. Closed range for $\overline{\partial}$

The purpose of this section is to establish sufficient conditions for the closed range of $\overline{\partial}$ on not necessarily pseudoconvex domains (and their boundaries) in Stein manifolds.

Theorem 3.1 (cf. Zampieri [18])**.** *Let $\Omega \Subset X$ be the same as in Theorem 1. If $\varphi_t = t\mu$, $t > 0$, for any (r,s)-form $u \in \operatorname{dom}\overline{\partial} \cap \operatorname{dom}\overline{\partial}_t^\star$, $q \leqslant s \leqslant n$, we have*

$$\|\overline{\partial}u\|_t^2 + \|\overline{\partial}_t^* u\|_t^2 \geqslant C_0 t \|u\|_t^2. \tag{3.1}$$

From (3.1), we get $\sqrt{t}\|u\|_t \lesssim \|\Box^t u\|_t$; thus $\Box^t$ has closed range and there is well defined a continues inverse operator N_t. Moreover, $\overline{\partial}N_t$ and $\overline{\partial}^\star N_t$ are also continuous. Finally, for $\overline{\partial}f = 0$ in degree $\geqslant q+1$, the form $u := \overline{\partial}^\star N_t f$ is the $L^2(\Omega, \varphi_t)$-canonical solution of the equation $\overline{\partial}u = f$, that is, the one orthogonal to $\ker\overline{\partial}$. More precisely, we have the following theorem:

Theorem 3.2 (cf. Chen-Shaw [3], Demailly [4], [5])**.** *Let $\Omega \Subset X$ be the same as in Theorem 1.1. For t sufficiently large, and for any $q \leqslant s \leqslant n$, we have the following:*

(1) *$\mathcal{H}_t(\Omega)$ is finite dimensional,*
(2) *the Laplace-Beltrami operator $\Box^t$ has closed range in $L^2_{r,s}(\Omega)$,*
(3) *the $\overline{\partial}$-Neumann operator $N^t\colon L^2_{r,s}(\Omega) \to L^2_{r,s}(\Omega)$ exists and is bounded,*
(4) *$\operatorname{Ran} N^t \subset \operatorname{dom}\Box^t$, $N^t\Box^t = I$ on $\operatorname{dom}\Box^t$,*
(5) *for $f \in L^2_{r,s}(\Omega)$, we have $f = \overline{\partial}\overline{\partial}_t^\star N^t f \oplus \overline{\partial}_t^\star\overline{\partial}N^t f$,*
(6) *$\overline{\partial}N^t = N^t\overline{\partial}$, $q \leqslant s \leqslant n-1$ and $\overline{\partial}_t^\star N^t = N^t\overline{\partial}_t^\star$, $q+1 \leqslant s \leqslant n$,*
(7) *the operator $\overline{\partial}$ has closed range in $L^2_{r,s}(\Omega)$ and $L^2_{r,s+1}(\Omega)$,*
(8) *the operator $\overline{\partial}_t^\star$ has closed range in $L^2_{r,s}(\Omega)$ and $L^2_{r,s-1}(\Omega)$,*
(9) *the canonical solution operators to $\overline{\partial}$ given by $\overline{\partial}_t^\star N^t\colon L^2_{r,s}(\Omega) \to L^2_{r,s-1}(\Omega)$ and $N^t\overline{\partial}_t^\star\colon L^2_{r,s+1}(\Omega) \to L^2_{r,s}(\Omega)$ are continuous,*
(10) *the canonical solution operators to $\overline{\partial}_t^\star$ given by $\overline{\partial}N^t\colon L^2_{r,s}(\Omega) \to L^2_{r,s+1}(\Omega)$ and $N^t\overline{\partial}\colon L^2_{r,s-1}(\Omega) \to L^2_{r,s}(\Omega)$ are continuous,*
(11) *for any $f \in L^2_{r,s}(\Omega)$, where $q \leqslant s \leqslant n$, such that $\overline{\partial}f = 0$ in Ω, there exists $u \in L^2_{r,s-1}(\Omega)$ satisfying $\overline{\partial}u = f$ with $\|u\|_t \lesssim \|f\|_t$.*

4. Proof of Theorem 1.1

Following Theorem 3.2, N^t exists for forms in $L^2_{n-r,n-s}(\Omega)$. Thus, we can define $u \in L^2_{r,s-1}(\Omega)$ by

$$u = -\star_{(t)} \overline{\overline{\partial} N^t_{n-r,n-s} \star_{(-t)} \overline{f}}. \tag{4.1}$$

Thus $\operatorname{supp} u \subset \bar{\Omega}$. Thus, u vanishes on $b\Omega$. Now, we extend u to X by defining $u = 0$ in $X \setminus \Omega$. We want to prove that the extended form u satisfies the equation $\overline{\partial} u = f$ in the distribution sense in X.

For $\eta \in L^2_{n-r,n-s-1}(\Omega) \cap \operatorname{dom} \overline{\partial}$, we have

$$\begin{aligned}\langle \overline{\partial}\eta, \star_{-t} f\rangle_{(t)\Omega} &= \int_\Omega \overline{\partial}\eta \wedge \star_t(\star_{-t} f) = (-1)^{r+s} \int_\Omega \overline{\partial}\eta \wedge f = (-1)^{(r+s)(r+s-1)} \int_\Omega f \wedge \overline{\partial}\eta \\ &= \int_\Omega f \wedge \overline{\partial}\eta = (-1)^{r+s} \langle f, \star_{-t}\overline{\partial}\eta\rangle_{(t)\Omega} = (-1)^{r+s} \langle f, \star_{-t}\overline{\partial}\eta\rangle_{(t)X},\end{aligned}$$

because $\operatorname{supp} f \subset \bar{\Omega}$. Since $\overline{\partial}^\star_t = \mathrm{e}^{\varphi_t} \vartheta \mathrm{e}^{-\varphi_t} = -\star_{-t}\overline{\partial}\star_t$ and $\vartheta|_\Omega = \overline{\partial}^\star|_\Omega$, when ϑ acts in the distribution sense (see [10]), we obtain

$$\langle \overline{\partial}\eta, \star_{-t} f\rangle_{(t)\Omega} = \langle f, \vartheta \star_{-t} \eta\rangle_{(t)X} = \langle \overline{\partial} f, \star_{-t}\eta\rangle_{(t)X} = 0.$$

It follows that $\overline{\partial}^\star_t(\star_{-t} f) = 0$ on Ω. Using Theorem 3.2, we have

$$\overline{\partial}^\star_t N^t(\star_{(-t)} f) = N^t \overline{\partial}^\star_t(\star_{(-t)} f) = 0. \tag{4.2}$$

Thus, from (4.1), and (4.2), we obtain

$$\begin{aligned}\overline{\partial} u &= -\overline{\partial \star_t \overline{\partial} N^t_{n-r,n-s} \star_{-t} \overline{f}} = (-1)^{r+s+1} \overline{\star_t \star_{-t} \partial \star_t \overline{\partial} N^t_{n-r,n-s} \star_{-t} \overline{f}} \\ &= (-1)^{r+s} \overline{\star_t \overline{\partial}^\star_t \overline{\partial} N^t_{n-r,n-s} \star_{-t} \overline{f}} = (-1)^{r+s} \overline{\star_t (\overline{\partial}^\star_t \overline{\partial} + \overline{\partial}\overline{\partial}^\star_t) N^t_{n-r,n-s} \star_{-t} \overline{f}} \\ &= (-1)^{r+s} \overline{\star_t \star_{-t} \overline{f}} = f\end{aligned}$$

in the distribution sense in Ω. Since $u = 0$ in $X \setminus \Omega$, then for $v \in L^2_{r,s}(X) \cap \operatorname{dom} \overline{\partial}^\star_t$, we obtain

$$\begin{aligned}\langle u, \overline{\partial}^\star_t v\rangle_{(t)X} &= \langle u, \overline{\partial}^\star_t v\rangle_{(t)\Omega} = \langle \star\overline{\partial}^\star_t v, \star_{-t} u\rangle_{(t)\Omega} = (-1)^{r+s} \langle \overline{\partial} \star_t v, \star_{-t} u\rangle_{(t)\Omega} \\ &= (-1)^{r+s} \langle \star v, \overline{\partial}^\star \star_{-t} u\rangle_{(t)\Omega} = \langle \star v, \star_{-t}\overline{\partial} u\rangle_{(t)\Omega} = \langle f, v\rangle_{(t)\Omega} = \langle f, v\rangle_{(t)X},\end{aligned}$$

where the third equality holds since $\star_{-t} u \in \operatorname{dom} \overline{\partial}^\star_t$. Thus $\overline{\partial} u = f$ in the distribution sense in X. □

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References

[1] *O. Abdelkader, S. Saber*: Solution to $\overline{\partial}$-equations with exact support on pseudo-convex manifolds. Int. J. Geom. Methods Mod. Phys. *4* (2007), 339–348.
[2] *J. Cao, M.-C. Shaw, L. Wang*: Estimates for the $\overline{\partial}$-Neumann problem and nonexistence of C^2 Levi-flat hypersurfaces in $\mathbb{C}P^n$. Math. Z. *248* (2004), 183–221; errata dtto *248* (2004), 223–225.
[3] *S.-C. Chen, M.-C. Shaw*: Partial Differential Equations in Several Complex Variables. AMS/IP Studies in Advanced Mathematics 19, American Mathematical Society, Providence; International Press, Somerville, 2001.
[4] *J.-P. Demailly*: Complex analytic and differential geometry. Preprint (2009) available at `http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf`.
[5] *J.-P. Demailly*: Estimations L^2 pour l'opérateur $\overline{\partial}$ d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne complète. Ann. Sci. Éc. Norm. Supér. (4) *15* (1982), 457–511. (In French.)
[6] *M. Derridj*: Inégalités de Carleman et extension locale des fonctions holomorphes. Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. *9* (1982), 645–669. (In French.)
[7] *M. Derridj*: Regularité pour $\overline{\partial}$ dans quelques domaines faiblement pseudo-convexes. J. Differ. Geom. *13* (1978), 559–576. (In French.)
[8] *P. S. Harrington, A. Raich*: Closed range for $\overline{\partial}$ and $\overline{\partial}_b$ on bounded hypersurfaces in Stein manifolds. arXiv:1106.0629.
[9] *L.-H. Ho*: $\overline{\partial}$-problem on weakly q-convex domains. Math. Ann. *290* (1991), 3–18.
[10] *L. Hörmander*: L^2 estimates and existence theorems for the $\overline{\partial}$ operator. Acta Math. *113* (1965), 89–152.
[11] *J. J. Kohn*: Harmonic integrals on strongly pseudo-convex manifolds. II. Ann. Math. (2) *79* (1964), 450–472.
[12] *J. J. Kohn*: Harmonic integrals on strongly pseudo-convex manifolds. I. Ann. Math. (2) *78* (1963), 112–148.
[13] *J. Morrow, K. Kodaira*: Complex Manifolds. Athena Series. Selected Topics in Mathematics. Holt, Rinehart and Winston, New York, 1971.
[14] *S. Saber*: The $\overline{\partial}$-problem on q-pseudoconvex domains with applications. Math. Slovaca *63* (2013), 521–530.
[15] *S. Saber*: Solution to $\overline{\partial}$ problem with exact support and regularity for the $\overline{\partial}$-Neumann operator on weakly q-convex domains. Int. J. Geom. Methods Mod. Phys. *7* (2010), 135–142.
[16] *S. Sambou*: Résolution du $\overline{\partial}$ pour les courants prolongeables définis dans un anneau. Ann. Fac. Sci. Toulouse, Math. (6) *11* (2002), 105–129. (In French.)
[17] *M.-C. Shaw*: Local existence theorems with estimates for $\overline{\partial}_b$ on weakly pseudo-convex CR manifolds. Math. Ann. *294* (1992), 677–700.
[18] *G. Zampieri*: Complex Analysis and CR Geometry. University Lecture Series 43, American Mathematical Society, Providence, 2008.

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