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A NOTE ON SOLVABLE VERTEX STABILIZERS OF s-TRANSITIVE GRAPHS OF PRIME VALENCY

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Abstract. A graph X, with a group G of automorphisms of X, is said to be (G,s)-transitive, for some $s \geqslant 1$, if G is transitive on s-arcs but not on $(s+1)$-arcs. Let X be a connected (G,s)-transitive graph of prime valency $p \geqslant 5$, and G_v the vertex stabilizer of a vertex $v \in V(X)$. Suppose that G_v is solvable. Weiss (1974) proved that $|G_v| \mid p(p-1)^2$. In this paper, we prove that $G_v \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_m) \times \mathbb{Z}_n$ for some positive integers m and n such that $n \mid m$ and $m \mid p-1$.

Keywords: symmetric graph; s-transitive graph; (G,s)-transitive graph

MSC 2010: 05C25, 20B25

1. Introduction

Throughout this paper, we consider undirected finite graphs without loops or multiple edges. For a graph X, we use $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ to denote its vertex set, edge set, and its full automorphism group, respectively.

An *s-arc* in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \leqslant i \leqslant s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leqslant i \leqslant s-1$. A 1-arc is called an *arc* for short and a 0-arc is a vertex. For a subgroup $G \leqslant \mathrm{Aut}(X)$, X is said to be *(G,s)-arc-transitive* and *(G,s)-regular* if G is transitive and regular on the set of s-arcs in X, respectively. A (G,s)-arc-transitive graph is said to be *(G,s)-transitive* if the graph is not $(G,s+1)$-arc-transitive. A graph X is called *s-arc-transitive*, *s-regular* and *s-transitive* if it is $(\mathrm{Aut}(X), s)$-arc-

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transitive, $(\mathrm{Aut}(X), s)$-regular and $(\mathrm{Aut}(X), s)$-transitive, respectively. In particular, X is said to be *vertex-transitive* and *symmetric* if it is $(\mathrm{Aut}(X), 0)$-arc-transitive and $(\mathrm{Aut}(X), 1)$-arc-transitive, respectively.

Let p be a prime and n a positive integer. We denote by $\mathbb{Z}_n$ the cyclic group of order n, by $\mathbb{Z}_p^n$ the elementary abelian group of order p^n, by D_{2n} the dihedral group of order $2n$, by F_n the Frobenius group of order n, and by A_n and S_n the alternating group and the symmetric group of degree n, respectively. For two groups M and N, $N \rtimes M$ stands for a semidirect product of N by M.

Let X be a connected (G, s)-transitive graph with some positive integer s and let G_v be the stabilizer of $v \in V(X)$ in G. If X has valency 3, then by Djoković and Miller [4], G_v is isomorphic to $\mathbb{Z}_3$, S_3, $\mathrm{S}_3 \times \mathbb{Z}_2$, S_4 and $\mathrm{S}_4 \times \mathbb{Z}_2$ for $s = 1$, 2, 3, 4 and 5, respectively. If X has valency 4, then by [3], G_v is isomorphic to a 2-group for $s = 1$; by [8], Theorem 4, G_v is isomorphic to A_4 or S_4 for $s = 2$ and to $\mathbb{Z}_3 \times \mathrm{A}_4$, $\mathbb{Z}_3 \rtimes \mathrm{S}_4$ or $\mathrm{S}_3 \times \mathrm{S}_4$ for $s = 3$; by [9], Theorem 1.1, G_v is isomorphic to $\mathbb{Z}_3^2 \rtimes \mathrm{GL}(2,3)$ for $s = 4$, and to $[3^5] \rtimes \mathrm{GL}(2,3)$ for $s = 7$. If X has valency 5, then by Guo and Feng [6], Theorem 1.1, G_v is isomorphic to $\mathbb{Z}_5$, D_{10} or D_{20} for $s = 1$, F_{20}, $\mathbb{Z}_2 \times F_{20}$, A_5 or S_5 for $s = 2$, $\mathbb{Z}_4 \times F_{20}$, $\mathrm{A}_4 \times \mathrm{A}_5$, $\mathrm{S}_4 \times \mathrm{S}_5$ or $(\mathrm{A}_4 \times \mathrm{A}_5) \rtimes \mathbb{Z}_2$ with $\mathrm{A}_4 \rtimes \mathbb{Z}_2 = \mathrm{S}_4$ and $\mathrm{A}_5 \rtimes \mathbb{Z}_2 = \mathrm{S}_5$ for $s = 3$, $\mathrm{ASL}(2,4)$, $\mathrm{AGL}(2,4)$, $\mathrm{A\Sigma L}(2,4)$ or $\mathrm{A\Gamma L}(2,4)$ for $s = 4$, or $\mathbb{Z}_2^6 \rtimes \Gamma\mathrm{L}(2,4)$ for $s = 5$. Furthermore, the structure of $\mathbb{Z}_2^6 \rtimes \Gamma\mathrm{L}(2,4)$ is completely determined by Weiss [9], Theorem 1.1. For other valencies, there are many partial results, and see [10], [12] for example. Let X be a connected (G, s)-transitive graph with prime valency $p \geqslant 5$. Suppose that G_v is solvable. By Weiss [13], Theorem, $|G_v| \mid p(p-1)^2$. In this paper, we prove that $G_v \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_m) \times \mathbb{Z}_n$ for some positive integers m and n such that $\mathbb{Z}_p \rtimes \mathbb{Z}_m$ is a subgroup of $F_{p(p-1)}$ and $n \mid m$.

The structure of the vertex stabilizer G_v plays an important role in the study of (G, s)-transitive graphs. For example, Conder and Dobcsányi [1] exhausted all cubic symmetric graphs on up to 768 vertices, and cubic symmetric graphs of order np or np^2 with n a given number were classified in [5], where p is a prime.

2. Main result

In this section, we determine the structure of the solvable vertex stabilizer of connected (G, s)-transitive graph with prime valency $p \geqslant 5$.

Theorem 2.1. *Let s be a positive integer and let X be a connected (G, s)-transitive graph of prime valency $p \geqslant 5$ for some $G \leqslant \mathrm{Aut}(X)$. Suppose that G_v is solvable. Then $s \leqslant 3$ and G_v is isomorphic to $(\mathbb{Z}_p \rtimes \mathbb{Z}_m) \times \mathbb{Z}_n$, where $\mathbb{Z}_p \rtimes \mathbb{Z}_m$ is a subgroup of the Frobenius group $\mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$ and $n \mid m$. Moreover, if $m < p-1$ then $s = 1$; if $m = p-1$ and $n < p-1$ then $s = 2$; if $m = n = p-1$ then $s = 3$.*

P r o o f. Let $\{u, v\}$ be an edge of X and $N(v)$ the neighborhood of v. Denote by $G_v^{N(v)}$ the constituent of G_v acting on $N(v)$, and by G_v^* the kernel of G_v acting on $N(v)$. Then $G_v^{N(v)} = G_v/G_v^*$. Write $G_{uv}^* = G_u^* \cap G_v^*$. Since G_v is solvable and $p \geqslant 5$, we have that $|G_v| \mid p(p-1)^2$ by [13], Theorem, and $G_{uv}^* = 1$ when u and v are adjacent by [11], Theorem.

Let P be a Sylow p-subgroup of G_v. Since G_v is transitive on $N(v)$, $p \mid |G_v/G_v^*|$, and since $|G_v| \mid p(p-1)^2$, we have $P \cong \mathbb{Z}_p$. It follows that $|G_v^*| \mid (p-1)^2$, and hence $PG_v^*/G_v^* \cong \mathbb{Z}_p$ is also a Sylow p-subgroup of G_v/G_v^*. By [2], Corollary 3.5 B, every transitive permutation group of prime degree p is either 2-transitive or solvable and has a normal Sylow p-subgroup. Clearly, G_v/G_v^* is solvable because G_v is solvable. Thus, PG_v^*/G_v^* is regular and normal in G_v/G_v^*, which implies that PG_v^* is normal in G_v. Since any regular abelian permutation group is self-centralizing (see [14], Proposition 4.4), $PG_v^*/G_v^* \cong \mathbb{Z}_p$ is self-centralizing in G_v/G_v^*. Thus, by N/C-Theorem (see [7], Chapter I, Theorem 4.5), we have $(G_v/G_v^*)/(PG_v^*/G_v^*) \lesssim \mathrm{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1}$, and hence $G_v^{N(v)} = G_v/G_v^* \lesssim \mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$, where $\mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$ is the Frobenius group of order $p(p-1)$. It follows that $G_{uv}^{N(v)} = G_{uv}/G_v^* \lesssim \mathbb{Z}_{p-1}$ and $G_{uv}^{N(u)} = G_{uv}/G_u^* \lesssim \mathbb{Z}_{p-1}$. Let $|G_{uv}/G_v^*| = m$. Then $G_{uv}/G_v^* \cong G_{uv}/G_u^* \cong \mathbb{Z}_m$ and $G_v/G_v^* \cong \mathbb{Z}_p \rtimes \mathbb{Z}_m$, where $\mathbb{Z}_p \rtimes \mathbb{Z}_m$ is a subgroup of the Frobenius group $\mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$.

Recall that $G_{uv}^* = 1$. Thus, $G_u^* G_v^* = G_u^* \times G_v^*$. Since the kernel of G_v^* acting on $N(u)$ is $G_u^* \cap G_v^* = G_{uv}^* = 1$, we have that G_v^* is faithful on $N(u)$. It follows that $G_v^* \cong G_v^*/(G_u^* \cap G_v^*) \cong G_v^* G_u^*/G_u^* \leqslant G_{uv}/G_u^* \cong \mathbb{Z}_m \leqslant \mathbb{Z}_{p-1}$. Thus, $|G_v^*| \mid p-1$ and G_v^* is a subgroup of the cyclic group $\mathbb{Z}_m$. Let $|G_v^*| = n$. Then $G_v^* \cong \mathbb{Z}_n$, $n \mid m$ and $|G_{uv}| = mn$.

Since $|G_v^*| \mid (p-1)$, by the Sylow Theorem, P is the unique normal Sylow p-subgroup of PG_v^*, forcing that P is characteristic in PG_v^*. It follows from the normality of PG_v^* in G_v that P is normal in G_v. Note that $P \cong \mathbb{Z}_p$, $|G_v^*| \mid p-1$ and $|G_{uv}| \mid (p-1)^2$. Thus, we can easily deduce that $P \cap G_v^* = 1$ and $P \cap G_{uv} = 1$. Since P and G_v^* are normal in G_v, $PG_v^* = P \times G_v^*$ and $G_v = P \rtimes G_{uv}$.

Both G_{uv}/G_u^* and G_{uv}/G_v^* are cyclic groups of order m, and there is a natural homomorphism of G_{uv} into $G_{uv}/G_u^* \times G_{uv}/G_v^*$ with kernel $G_u^* \cap G_v^*$. As noted above, $G_u^* \cap G_v^* = 1$ and so this homomorphism is an embedding of G_{uv} into an abelian group. Therefore G_{uv} is an abelian group of order dividing m^2.

Let $G_v^* = \langle a \rangle$ and $G_{uv}/G_v^* = \langle bG_v^* \rangle$. Then $G_{uv} = \langle a, b \rangle = \langle a \rangle \langle b \rangle$ because G_{uv} is abelian. Since $G_v^* \cong \mathbb{Z}_n$ and $G_{uv}/G_v^* \cong \mathbb{Z}_m$, the order $o(a) = n$ and $o(b) \geqslant m$. On the other hand, $\mathbb{Z}_n \cong G_u^* \cong G_u^* G_v^*/G_v^* \leqslant G_{uv}/G_v^* \cong \mathbb{Z}_m$ implies that $b^{m/n} G_v^* \in G_u^* G_v^*/G_v^*$, that is, $b^{m/n} \in G_u^* G_v^* = G_u^* \times G_v^* \cong \mathbb{Z}_n^2$. It follows that $(b^{m/n})^n = b^m = 1$ and $o(b) \leqslant m$. Thus, $o(b) = m$. Note that $|G_{uv}| = mn$. We have $G_{uv} = \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_m$. Recall that $PG_v^* = P \times G_v^*$ and $G_v = P \rtimes G_{uv}$. Thus, $G_v \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_m) \times \mathbb{Z}_n$, where $\mathbb{Z}_p \rtimes \mathbb{Z}_m$ is a subgroup of the Frobenius group $\mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$ and $n \mid m$.

Let $m < p-1$. Then G_{uv} cannot act on $N(v) \setminus \{u\}$ transitively, and hence G_v is 1-transitive on $N(v)$. It follows that G is 1-transitive on X, that is, $s = 1$. Let $m = p-1$ and $n < p-1$. Then $G_{uv}^{N(v)} \cong \mathbb{Z}_{p-1}$ and G_v is 2-transitive on $N(v)$. However, $G_v^* \cong \mathbb{Z}_n$ is not transitive on $N(u) \setminus \{v\}$ because $n < p-1$. Thus, in this case $s = 2$. Let $m = n = p-1$. Then $G_v^{N(v)} \cong F_{p(p-1)}$ is 2-transitive on $N(v)$ and $G_v^* \cong \mathbb{Z}_{p-1}$ is transitive on $N(u) \setminus \{v\}$, which implies that $s = 3$. This completes the proof. □

Note that $D_{10} \cong F_{10}$ and $D_{20} \cong F_{10} \times \mathbb{Z}_2$. Then [15], Theorem 4.1, is a consequence of Theorem 2.1. The following corollary gives the structure of solvable vertex stabilizer of (G,s)-transitive graph with valency seven, which can be derived easily from Theorem 2.1.

Corollary 2.2. *Let X be a connected (G,s)-transitive graph of valency seven with $G \leqslant \mathrm{Aut}(X)$ and $s \geqslant 1$. Suppose that G_v is solvable. Then one of the following holds:*

(1) *$s = 1$, and $G_v \cong \mathbb{Z}_7$, F_{14}, F_{21}, $F_{14} \times \mathbb{Z}_2$ or $F_{21} \times \mathbb{Z}_3$;*
(2) *$s = 2$, and $G_v \cong F_{42}$, $F_{42} \times \mathbb{Z}_2$, or $F_{42} \times \mathbb{Z}_3$;*
(3) *$s = 3$, and $G_v \cong F_{42} \times \mathbb{Z}_6$.*

3. Realization

Let X be a connected (G,s)-transitive graph of prime valency $p \geqslant 5$ for $G \leqslant \mathrm{Aut}(X)$ and let $v \in V(X)$. Take two positive integers m and n such that $m \mid p-1$ and $n \mid m$. In this section, we show that each type of $G_v \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_m) \times \mathbb{Z}_n$ in Theorem 2.1 can be realized with G as a group of automorphisms of the complete bipartite graph $K_{p,p}$.

Clearly, $\mathrm{Aut}(K_{p,p}) = \mathrm{S}_p \mathrm{wr} \mathrm{S}_2$. Then $\mathrm{Aut}(K_{p,p})$ contains an arc-transitive subgroup $A = F_{p(p-1)} \,\mathrm{wr}\, \mathrm{S}_2 = ((\mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}) \times (\mathbb{Z}_p \rtimes \mathbb{Z}_{p-1})) \rtimes \mathrm{S}_2 = (((\langle a_1 \rangle \rtimes \langle b_1 \rangle) \times (\langle a_2 \rangle \rtimes \langle b_2 \rangle)) \rtimes \langle c \rangle$, where $o(a_1) = o(a_2) = p$, $o(b_1) = o(b_2) = p-1$, $o(c) = 2$, $a_1^c = a_2$, $a_2^c = a_1$, $b_1^c = b_2$ and $b_2^c = b_1$. Furthermore, A has a normal subgroup $N = \langle a_1, a_2 \rangle \cong \mathbb{Z}_p^2$.

Let $\{u,v\} \in E(K_{p,p})$. Without loss of generality, we may assume that c interchanges u and v, $A_v = (\langle a_1 \rangle \rtimes \langle b_1 \rangle) \times \langle b_2 \rangle$ and $A_u = (\langle a_2 \rangle \rtimes \langle b_2 \rangle) \times \langle b_1 \rangle$. Set $H = \langle (b_1 b_2)^{(p-1)/m}, b_2^{(p-1)/n}, c \rangle$. Note that $n \mid m$. Since $((b_1 b_2)^{(p-1)/m})^{m/n} b_2^{-(p-1)/n} = b_1^{(p-1)/n}$, we have $b_1^{(p-1)/n} \in H$. It follows that $H = \langle (b_1 b_2)^{(p-1)/m}, b_1^{(p-1)/n}, c \rangle$. Since c interchanges b_1 and b_2, we infer that c normalizes $\langle (b_1 b_2)^{(p-1)/m}, b_1^{(p-1)/n} \rangle \cong \mathbb{Z}_m \times \mathbb{Z}_n$. Thus, $|H| = 2mn$.

Let $G = NH$. Then $G \leqslant A$ because N is normal in A. Since $|H| = 2mn$, we have $G = N \rtimes H$. It follows that $|G| = 2mnp^2$. Clearly, G is arc-transitive because $\langle a_1, a_2, c\rangle \leqslant G$. Thus, $|G_v| = mnp$. Since $A_v = (\langle a_1\rangle \rtimes \langle b_1\rangle) \times \langle b_2\rangle$, we have $\langle a_1, (b_1b_2)^{(p-1)/m}, b_2^{(p-1)/n}\rangle \leqslant G_v$. Since b_1 normalizes $\langle a_1\rangle$ and b_2 centralizes a_1, we can easily deduce that $\langle a_1, (b_1b_2)^{(p-1)/m}, b_2^{(p-1)/n}\rangle = (\langle a_1\rangle \rtimes \langle (b_1b_2)^{(p-1)/m}\rangle) \times \langle b_2^{(p-1)/n}\rangle \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_m) \times \mathbb{Z}_n$. It follows that $G_v \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_m) \times \mathbb{Z}_n$ because $|G_v| = mnp$.

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