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EXISTENCE RESULTS FOR SYSTEMS WITH NONLINEAR COUPLED NONLOCAL INITIAL CONDITIONS

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Abstract. The purpose of the present paper is to study the existence of solutions to initial value problems for nonlinear first order differential systems subject to nonlinear nonlocal initial conditions of functional type. The approach uses vector-valued metrics and matrices convergent to zero. Two existence results are given by means of Schauder and Leray-Schauder fixed point principles and the existence and uniqueness of the solution is obtained via a fixed point theorem due to Perov. Two examples are given to illustrate the theory.

Keywords: nonlinear differential system; nonlocal boundary condition; nonlinear boundary condition; fixed point; vector-valued norm; matrix convergent to zero

MSC 2010: 34A34, 34A12, 34B10, 47H10

1. Introduction

Nonlocal problems for different classes of differential equations and systems are intensively studied in the literature by a variety of methods (see for example [2], [4], [5], [6], [10]–[17], [21], [23], [24], [28], [30], [33], [35]–[38], [41], [42], [46], [48]–[54] and the references therein). For problems with nonlinear boundary conditions we refer the reader to [3], [18]–[20], [22], [25]–[27], [29], [31], [32], [34], [44] and the references therein.

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In the recent papers [8], [9], [39], [40], a new method based on vector-valued norms and matrices convergent to zero was used for the treatment of first order differential systems under nonlocal conditions expressed by *linear* functionals. The aim of this paper is to extend the use of that technique to nonlocal conditions given by *nonlinear* functionals.

We shall consider the problem

$$(1.1)\qquad \begin{cases} x'(t) = f_1(t, x(t), y(t)), \\ y'(t) = f_2(t, x(t), y(t)), \quad \text{a.e. on } [0,1], \\ x(0) = \alpha[x, y], \\ y(0) = \beta[x, y]. \end{cases}$$

Here, $f_1, f_2\colon\ [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ are L^1-Carathéodory functions, $\alpha, \beta\colon\ (C[0,1])^2 \to \mathbb{R}$ are nonlinear continuous functionals, and the solution (x, y) is sought in $W^{1,1}(0,1;\mathbb{R}^2)$. The technique we use differs from that in [9], [39], [40] by the necessity of working with nonlinear operators on the product space $C[0,1] \times \mathbb{R}$. This way, the nonlinear functionals expressing the nonlocal conditions become part of the nonlinear operators associated to the problem. More exactly, we rewrite the problem (1.1) as a system of the form

$$x_a = \left(a + \int_0^t f_1(s, x(s), y(s))\,\mathrm{d}s,\ \alpha[x, y]\right),$$

$$y_b = \left(b + \int_0^t f_2(s, x(s), y(s))\,\mathrm{d}s,\ \beta[x, y]\right),$$

where by x_a, y_b we mean the pairs $(x, a), (y, b) \in C[0,1] \times \mathbb{R}$. This, in turn, can be viewed as a fixed point problem in $(C[0,1] \times \mathbb{R})^2$ for the completely continuous operator

$$T = (T_1, T_2)\colon\ (C[0,1] \times \mathbb{R})^2 \to (C[0,1] \times \mathbb{R})^2,$$

where T_1 and T_2 are given by

$$T_1[x_a, y_b] = \left(a + \int_0^t f_1(s, x(s), y(s))\,\mathrm{d}s,\ \alpha[x, y]\right),$$

$$T_2[x_a, y_b] = \left(b + \int_0^t f_2(s, x(s), y(s))\,\mathrm{d}s,\ \beta[x, y]\right).$$

In what follows, we introduce some notations, definitions and basic results which are used throughout this paper. Three different fixed point principles are used in order to prove the existence of solutions for the problem (1.1), namely the fixed point principles of Perov, Schauder and Leray-Schauder (see [45], [46]). The technique that makes use of the vector-valued metrics and matrices convergent to zero has an essential role in all three cases. Therefore, we recall the fundamental results that are used in the next sections (see [1], [43], [46]).

Let X be a nonempty set.

Definition 1.1. By a *vector-valued metric* on X we mean a mapping $d\colon X \times X \to \mathbb{R}^n_+$ such that

(i) $d(u,v) \geqslant 0$ for all $u, v \in X$ and if $d(u,v) = 0$ then $u = v$;
(ii) $d(u,v) = d(v,u)$ for all $u, v \in X$;
(iii) $d(u,v) \leqslant d(u,w) + d(w,v)$ for all $u, v, w \in X$.

Here, if $x, y \in \mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, by $x \leqslant y$ we mean $x_i \leqslant y_i$ for $i = 1, 2, \ldots, n$. We call the pair (X, d) a *generalized metric space.* For such a space convergence and completeness are similar to those in usual metric spaces.

Definition 1.2. A square matrix M with nonnegative elements is said to be *convergent to zero* if

$$M^k \to 0 \quad \text{as } k \to \infty.$$

The property of being convergent to zero is equivalent to each of the following conditions from the characterization lemma below (see [7], pages 9, 10, [45], [46], [47], pages 12, 88):

Lemma 1.1. *Let M be a square matrix of nonnegative numbers. The following statements are equivalent:*

(i) *M is a matrix convergent to zero;*
(ii) *$I - M$ is nonsingular and $(I - M)^{-1} = I + M + M^2 + \ldots$ (where I stands for the unit matrix of the same order as M);*
(iii) *the eigenvalues of M are located inside the unit disc of the complex plane;*
(iv) *$I - M$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.*

Note that, according to the equivalence of the statements (i) and (iv), a matrix M is convergent to zero if and only if the matrix $I - M$ is *inverse-positive.* Also, the equivalence of (i) and (iii) shows that a matrix M is convergent to zero if and only if $\varrho(M) < 1$, where $\varrho(M)$ is the spectral radius of M.

The following lemma is a consequence of the previous characterizations.

Lemma 1.2. *Let A be a matrix that is convergent to zero. Then for each matrix B of the same order whose elements are nonnegative and sufficiently small, the matrix $A+B$ is also convergent to zero.*

Definition 1.3. Let (X,d) be a generalized metric space. An operator $T\colon X\to X$ is said to be *contractive* (with respect to the vector-valued metric d on X) if there exists a convergent to zero (Lipschitz) matrix M such that

$$d(T(u),T(v))\leqslant Md(u,v)\quad\text{for all } u,v\in X.$$

Theorem 1.1 (Perov). *Let (X,d) be a complete generalized metric space and $T\colon X\to X$ a contractive operator with Lipschitz matrix M. Then T has a unique fixed point u^* and for each $u_0\in X$ we have*

$$d(T^k(u_0),u^*)\leqslant M^k(I-M)^{-1}d(u_0,T(u_0))\quad\textit{for all } k\in\mathbb{N}.$$

Theorem 1.2 (Schauder). *Let X be a Banach space, $D\subset X$ a nonempty closed bounded convex set and $T\colon D\to D$ a completely continuous operator (i.e., T is continuous and $T(D)$ is relatively compact). Then T has at least one fixed point.*

Theorem 1.3 (Leray-Schauder). *Let $(X,|\cdot|_X)$ be a Banach space, $R>0$ and $T\colon \bar{b}_X(0;R)\to X$ a completely continuous operator. If $|u|_X<R$ for every solution u of the equation $u=\lambda T(u)$ and any $\lambda\in(0,1)$, then T has at least one fixed point.*

In this paper, by $|x|_C$, where $x\in C[0,1]$, we mean

$$|x|_C=\max_{t\in[0,1]}|x(t)|.$$

Also, the notation $|x|_{L^1}$ will stand for the L^1-norm in $L^1(0,1)$.

2. Existence and uniqueness of the solution

In this section we show that the existence of solutions to the problem (1.1) follows from Perov's fixed point theorem in case that the nonlinearities f_1, f_2 and the functionals α, β satisfy Lipschitz conditions of the type:

$$\begin{cases}|f_1(t,x,y)-f_1(t,\bar{x},\bar{y})|\leqslant a_1|x-\bar{x}|+b_1|y-\bar{y}|\\ |f_2(t,x,y)-f_2(t,\bar{x},\bar{y})|\leqslant a_2|x-\bar{x}|+b_2|y-\bar{y}|,\end{cases}\tag{2.1}$$

for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}$ and a.e. $t \in [0,1]$, and

$$\begin{cases} |\alpha[x,y] - \alpha[\bar{x},\bar{y}]| \leqslant A_1|x-\bar{x}|_C + B_1|y-\bar{y}|_C \\ |\beta[x,y] - \beta[\bar{x},\bar{y}]| \leqslant A_2|x-\bar{x}|_C + B_2|y-\bar{y}|_C, \end{cases} \tag{2.2}$$

for all $x, y, \bar{x}, \bar{y} \in C[0,1]$.

For a given number $\theta > 0$, denote

$$\begin{aligned} m_{11}(\theta) &= \max\Big\{\frac{1}{\theta}, a_1 + \theta A_1\Big\}, \quad m_{12}(\theta) = b_1 + \theta B_1, \\ m_{21}(\theta) &= a_2 + \theta A_2, \quad m_{22}(\theta) = \max\Big\{\frac{1}{\theta}, b_2 + \theta B_2\Big\}. \end{aligned}$$

Theorem 2.1. *Assume that* f_1, f_2 *satisfy the Lipschitz conditions* (2.1) *and* α, β *satisfy the conditions* (2.2). *In addition assume that for some* $\theta > 0$, *the matrix*

$$M_\theta = \begin{bmatrix} m_{11}(\theta) & m_{12}(\theta) \\ m_{21}(\theta) & m_{22}(\theta) \end{bmatrix} \tag{2.3}$$

is convergent to zero. Then the problem (1.1) *has a unique solution.*

Proof. We shall apply Perov's fixed point theorem in $(C[0,1] \times \mathbb{R})^2$ endowed with the vector-valued norm $\|\cdot\|_{(C[0,1]\times\mathbb{R})^2}$,

$$\|u\|_{(C[0,1]\times\mathbb{R})^2} = \begin{bmatrix} |x_a| \\ |y_b| \end{bmatrix},$$

for $u = (x_a, y_b)$. Here

$$|x_a| = |(x,a)| = |x|_C + \theta|a|,$$

which represents a norm on $C[0,1] \times \mathbb{R}$.

We have to prove that T is contractive with respect to the convergent to zero matrix M_θ, more exactly that

$$\|T(u) - T(\bar{u})\|_{(C[0,1]\times\mathbb{R})^2} \leqslant M_\theta \|u - \bar{u}\|_{(C[0,1]\times\mathbb{R})^2},$$

for all $u = (x_a, y_b)$, $\bar{u} = (\bar{x}_{\bar{a}}, \bar{y}_{\bar{b}}) \in (C[0,1] \times \mathbb{R})^2$.

Indeed, we have

$$
\begin{aligned}
(2.4)\qquad & |T_1[x_a, y_b] - T_1[\bar{x}_{\bar{a}}, \overline{y}_{\bar{b}}]| \\
& \leqslant \left| \int_0^t |f_1(s, x(s), y(s)) - f_1(s, \bar{x}(s), \overline{y}(s))| \,\mathrm{d}s \right|_C + |a - \bar{a}| \\
& \quad + \theta |\alpha[x, y] - \alpha[\bar{x}, \overline{y}]| \\
& \leqslant \left| a_1 \int_0^t |x(s) - \bar{x}(s)| \,\mathrm{d}s + b_1 \int_0^t |y(s) - \overline{y}(s)| \right|_C + \theta A_1 |x - \bar{x}|_C \\
& \quad + \theta B_1 |y - \overline{y}|_C + |a - \bar{a}| \\
& \leqslant (a_1 + \theta A_1)|x - \bar{x}|_C + (b_1 + \theta B_1)|y - \overline{y}|_C + \frac{1}{\theta} \cdot \theta |a - \bar{a}| \\
& \leqslant \max\left\{\frac{1}{\theta}, a_1 + \theta A_1\right\} |x_a - \bar{x}_{\bar{a}}| + (b_1 + \theta B_1)|y_b - \overline{y}_{\bar{b}}| \\
& = m_{11}(\theta)|x_a - \bar{x}_{\bar{a}}| + m_{12}(\theta)|y_b - \overline{y}_{\bar{b}}|.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
(2.5)\qquad & |T_2[x_a, y_b] - T_2[\bar{x}_{\bar{a}}, \overline{y}_{\bar{b}}]| \\
& \leqslant (a_2 + \theta A_2)|x_a - \bar{x}_{\bar{a}}| + \max\left\{\frac{1}{\theta}, b_2 + \theta B_2\right\} |y_b - \overline{y}_{\bar{b}}| \\
& = m_{21}(\theta)|x_a - \bar{x}_{\bar{a}}| + m_{22}(\theta)|y_b - \overline{y}_{\bar{b}}|.
\end{aligned}
$$

Now, both inequalities (2.4), (2.5) can be put together and be rewritten equivalently as

$$
\begin{bmatrix} |T_1[x_a, y_b] - T_1[\bar{x}_{\bar{a}}, \overline{y}_{\bar{b}}]| \\ |T_2[x_a, y_b] - T_2[\bar{x}_{\bar{a}}, \overline{y}_{\bar{b}}]| \end{bmatrix} \leqslant M_\theta \begin{bmatrix} |x_a - \bar{x}_{\bar{a}}| \\ |y_b - \overline{y}_{\bar{b}}| \end{bmatrix}
$$

or using the vector-valued norm

$$
\|T(u) - T(\bar{u})\|_{(C[0,1]\times\mathbb{R})^2} \leqslant M_\theta \|u - \bar{u}\|_{(C[0,1]\times\mathbb{R})^2},
$$

where M_θ is given by (2.3) and assumed to be convergent to zero. The result follows now from Perov's fixed point theorem. □

3. EXISTENCE OF AT LEAST ONE SOLUTION

In the beginning of this section, we give an application of Schauder's fixed point theorem. More precisely, we show that the existence of solutions to the problem (1.1) follows from Schauder's fixed point theorem in case that f_1, f_2 satisfy a relaxed growth condition of the type:

$$\begin{cases} |f_1(t,x,y)| \leqslant a_1|x| + b_1|y| + c_1(t), \\ |f_2(t,x,y)| \leqslant a_2|x| + b_2|y| + c_2(t), \end{cases} \tag{3.1}$$

for all $x, y \in \mathbb{R}$ and a.e. $t \in [0,1]$, where $c_1, c_2 \in L^1(0,1;\mathbb{R}_+)$. In addition, we assume that

$$\begin{cases} |\alpha[x,y]| \leqslant A_1|x|_C + B_1|y|_C + C_1, \\ |\beta[x,y]| \leqslant A_2|x|_C + B_2|y|_C + C_2, \end{cases} \tag{3.2}$$

for all $x, y \in C[0,1]$.

Theorem 3.1. *If the conditions* (3.1), (3.2) *hold and the matrix* M_θ *defined in* (2.3) *is convergent to zero for some* $\theta > 0$, *then the problem* (1.1) *has at least one solution.*

Proof. In order to apply Schauder's fixed point theorem, we look for a nonempty, bounded, closed and convex subset B of $(C[0,1]\times\mathbb{R})^2$ so that $T(B) \subset B$. Let x_a, y_b be any elements of $C[0,1]\times\mathbb{R}$. Then, using the same norm on $C[0,1]\times\mathbb{R}$ as in the proof of the previous theorem, we obtain

$$\begin{aligned} |T_1[x_a,y_b]| &= \left| a + \int_0^t f_1(s,x(s),y(s))\,\mathrm{d}s|_C + \theta|\alpha[x,y] \right| \\ &\leqslant |a| + \left| \int_0^t (a_1|x(s)| + b_1|y(s)| + c_1(s))\,\mathrm{d}s \right|_C + \theta A_1|x|_C + \theta B_1|y|_C + \theta C_1 \\ &\leqslant a_1|x|_C + b_1|y|_C + |c_1|_{L^1} + \theta A_1|x|_C + \theta B_1|y|_C + \theta C_1 + |a| \\ &= (a_1+\theta A_1)|x|_C + (b_1+\theta B_1)|y|_C + \frac{1}{\theta}\cdot\theta|a| + |c_1|_{L^1} + \theta C_1 \\ &\leqslant \max\left\{\frac{1}{\theta}, a_1+\theta A_1\right\}|x_a| + (b_1+\theta B_1)|y_b| + c_0 \\ &= m_{11}(\theta)|x_a| + m_{12}(\theta)|y_b| + c_0, \end{aligned} \tag{3.3}$$

where $c_0 := |c_1|_{L^1} + \theta C_1$. Similarly

$$\begin{aligned} |T_2[x_a,y_b]| &\leqslant (a_2+\theta A_2)|x_a| + \max\left\{\frac{1}{\theta}, b_2+\theta B_2\right\}|y_b| + C_0 \\ &= m_{21}(\theta)|x_a| + m_{22}(\theta)|y_b| + C_0, \end{aligned} \tag{3.4}$$

where $C_0 := |c_2|_{L^1} + \theta C_2$. Now, from (3.3), (3.4) we have

$$\begin{bmatrix} |T_1[x_a, y_b]| \\ |T_2[x_a, y_b]| \end{bmatrix} \leqslant M_\theta \begin{bmatrix} |x_a| \\ |y_b| \end{bmatrix} + \begin{bmatrix} c_0 \\ C_0 \end{bmatrix},$$

where M_θ is given by (2.3) and is assumed to be convergent to zero. Next we look for two positive numbers R_1, R_2 such that if $|x_a| \leqslant R_1$ and $|y_b| \leqslant R_2$, then $|T_1[x_a, y_b]| \leqslant R_1$, $|T_2[x_a, y_b]| \leqslant R_2$. To this end it is sufficient that

$$M_\theta \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} + \begin{bmatrix} c_0 \\ C_0 \end{bmatrix} \leqslant \begin{bmatrix} R_1 \\ R_2 \end{bmatrix},$$

whence

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \geqslant (I - M_\theta)^{-1} \begin{bmatrix} c_0 \\ C_0 \end{bmatrix}.$$

Notice that $I - M_\theta$ is invertible and its inverse $(I - M_\theta)^{-1}$ has nonnegative elements since M_θ is convergent to zero. Thus, if $B = B_1 \times B_2$, where

$$B_1 = \{x_a \in C[0,1] \times \mathbb{R} \colon\ |x_a| \leqslant R_1\} \quad \text{and} \quad B_2 = \{y_b \in C[0,1] \times \mathbb{R} \colon\ |y_b| \leqslant R_2\},$$

then $T(B) \subset B$. Also, the operator T is completely continuous since f_1, f_2 have been assumed to be L^1-Carathéodory. Thus Schauder's fixed point theorem can be applied. ☐

In what follows, we give an application of the Leray-Schauder principle and we assume that the nonlinearities f_1, f_2 and also the functionals α, β satisfy more general growth conditions, namely:

$$\begin{cases} |f_1(t,x,y)| \leqslant \omega_1(t,|x|,|y|), \\ |f_2(t,x,y)| \leqslant \omega_2(t,|x|,|y|), \end{cases} \tag{3.5}$$

for all $x, y \in \mathbb{R}$ and a.e. $t \in [0,1]$, and

$$\begin{cases} |\alpha[x,y]| \leqslant \omega_3(|x|_C, |y|_C), \\ |\beta[x,y]| \leqslant \omega_4(|x|_C, |y|_C), \end{cases} \tag{3.6}$$

for all $x, y \in C[0,1]$. Here ω_1, ω_2 are L^1-Carathéodory functions on $[0,1] \times \mathbb{R}_+^2$, nondecreasing in their second and third arguments, and ω_3, ω_4 are continuous functions on $\mathbb{R}_+^2$, nondecreasing in both variables.

Theorem 3.2. *Assume that the conditions* (3.5), (3.6) *hold. In addition assume that there exists* $R_0 = (R_1^0, R_2^0) \in (0,\infty)^2$ *such that for* $\varrho = (\varrho_1, \varrho_2) \in (0,\infty)^2$

$$\begin{cases} \int_0^1 \omega_1(s,\varrho_1,\varrho_2)\,\mathrm{d}s + \omega_3(\varrho_1,\varrho_2) \geqslant \varrho_1 \\ \int_0^1 \omega_2(s,\varrho_1,\varrho_2)\,\mathrm{d}s + \omega_4(\varrho_1,\varrho_2) \geqslant \varrho_2 \end{cases} \quad \text{implies } \varrho \leqslant R_0. \tag{3.7}$$

Then the problem (1.1) *has at least one solution.*

Proof. The result follows from the Leray-Schauder fixed point theorem once we have proved the boundedness of the set of all solutions of the equation $u = \lambda T(u)$, for $\lambda \in (0,1)$. Let $u = (x_a, y_b)$ be such a solution. Then $x_a = \lambda T_1(x_a, y_b)$ and $y_b = \lambda T_2(x_a, y_b)$, or equivalently

$$\begin{cases} (x,a) = \lambda\Bigl(a + \int_0^t f_1(s,x(s),y(s))\,\mathrm{d}s,\ \alpha[x,y]\Bigr), \\ (y,b) = \lambda\Bigl(b + \int_0^t f_2(s,x(s),y(s))\,\mathrm{d}s,\ \beta[x,y]\Bigr). \end{cases}$$

First, we obtain that

$$\begin{aligned} |x(t)| &= \lambda\left|a + \int_0^t f_1(s,x(s),y(s))\,\mathrm{d}s\right| \leqslant |a| + \int_0^t |f_1(s,x(s),y(s))|\,\mathrm{d}s \\ &\leqslant |a| + \int_0^1 \omega_1(s,|x(s)|,|y(s)|)\,\mathrm{d}s \leqslant |a| + \int_0^1 \omega_1(s,\varrho_1,\varrho_2)\,\mathrm{d}s \end{aligned} \tag{3.8}$$

where $\varrho_1 = |x|_C$, $\varrho_2 = |y|_C$. Also

$$|a| = |\lambda\alpha[x,y]| \leqslant \omega_3(\varrho_1,\varrho_2). \tag{3.9}$$

Similarly, we have

$$|y(t)| \leqslant |b| + \int_0^1 \omega_2(s,\varrho_1,\varrho_2)\,\mathrm{d}s \tag{3.10}$$

and

$$|b| \leqslant \omega_4(\varrho_1,\varrho_2). \tag{3.11}$$

Then from (3.8)–(3.11), we deduce

$$\begin{cases} \varrho_1 \leqslant \int_0^1 \omega_1(s,\varrho_1,\varrho_2)\,\mathrm{d}s + \omega_3(\varrho_1,\varrho_2), \\ \varrho_2 \leqslant \int_0^1 \omega_2(s,\varrho_1,\varrho_2)\,\mathrm{d}s + \omega_4(\varrho_1,\varrho_2). \end{cases}$$

This by (3.7) guarantees that

$$\varrho \leqslant R_0. \tag{3.12}$$

It follows that

$$|a| \leqslant \omega_3(R_0) =: R_1^1, \quad |b| \leqslant \omega_4(R_0) =: R_2^1. \tag{3.13}$$

Finally (3.12) and (3.13) show that the solutions $u = (x_a, y_b)$ are a priori bounded independently of λ. Also, the operator T is completely continuous since ω_1, ω_2 have been assumed to be L^1-Carathéodory.

Thus Leray-Schauder's fixed point theorem can be applied. $\square$

4. EXAMPLES

In what follows, we give two examples that illustrate our theory.

Example 4.1. Consider the nonlocal problem

$$\begin{cases} x' = \dfrac{1}{4}\sin x + ay + g(t) \equiv f_1(t,x,y), \\ y' = \cos\Big(ax + \dfrac{1}{4}y\Big) + h(t) \equiv f_2(t,x,y), \\ x(0) = \dfrac{1}{8}\sin\Big(x\Big(\dfrac{1}{4}\Big) + y\Big(\dfrac{1}{4}\Big)\Big), \\ y(0) = \dfrac{1}{8}\cos\Big(x\Big(\dfrac{1}{4}\Big) + y\Big(\dfrac{1}{4}\Big)\Big), \end{cases} \tag{4.1}$$

where $t \in [0,1]$, $a \in \mathbb{R}$ and $g, h \in L^1(0,1)$. We have $a_1 = 1/4$, $b_1 = |a|$, $a_2 = |a|$, $b_2 = 1/4$ and $A_1 = B_1 = A_2 = B_2 = 1/8$. Consider $\theta = 2$. Hence

$$M_\theta = \begin{bmatrix} \frac{1}{2} & |a| + \frac{1}{4} \\ |a| + \frac{1}{4} & \frac{1}{2} \end{bmatrix}. \tag{4.2}$$

Since the eigenvalues of M_θ are $\lambda_1 = -|a| + 1/4$, $\lambda_2 = |a| + 3/4$, the matrix (4.2) is convergent to zero if $|\lambda_1| < 1$ and $|\lambda_2| < 1$. It is also known that a matrix of this type is convergent to zero if $|a| + 1/4 + 1/2 < 1$ (see [45]). Therefore, if $|a| < 1/4$, the matrix (4.2) is convergent to zero and by Theorem 2.1 the problem (4.1) has a unique solution.

Example 4.2. Consider the nonlocal problem

$$
\tag{4.3}
\begin{cases}
x' = \frac{1}{4} x \sin\left(\frac{y}{x}\right) + ay \sin\left(\frac{x}{y}\right) + g(t) \equiv f_1(t,x,y),\\
y' = ax \sin\left(\frac{y}{x}\right) + \frac{1}{4} y \sin\left(\frac{x}{y}\right) + h(t) \equiv f_2(t,x,y),\\
x(0) = \frac{1}{8} \sin\left(x\left(\frac{1}{4}\right) + y\left(\frac{1}{4}\right)\right),\\
y(0) = \frac{1}{8} \cos\left(x\left(\frac{1}{4}\right) + y\left(\frac{1}{4}\right)\right),
\end{cases}
$$

where $t \in [0,1]$, $a \in \mathbb{R}$ and $g, h \in L^1(0,1)$. Since

$$
\begin{aligned}
|f_1(t,x,y)| &\leqslant \frac{1}{4}|x| + |a||y| + |g(t)|,\\
|f_2(t,x,y)| &\leqslant |a||x| + \frac{1}{4}|y| + |h(t)|,
\end{aligned}
$$

we are under the assumptions from the first part of Section 3. Also, the matrix M_θ is that from Example 1 if we consider $\theta = 2$. Therefore, according to Theorem 3.1, if that matrix is convergent to zero, then the problem (4.3) has at least one solution. Note that the functions $f_1(t,x,y)$, $f_2(t,x,y)$ from this example do not satisfy Lipschitz conditions in x, y and consequently Theorem 2.1 does not apply.

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