Mathematica Bohemica

Andrzej Myśliński; Konrad Koniarski
Hybrid level set phase field method for topology optimization of contact problems

Mathematica Bohemica, Vol. 140 (2015), No. 4, 419–435

Persistent URL: http://dml.cz/dmlcz/144460

HYBRID LEVEL SET PHASE FIELD METHOD FOR TOPOLOGY OPTIMIZATION OF CONTACT PROBLEMS

Andrzej Myśliński, Konrad Koniarski, Warszawa

(Received September 30, 2013)

Abstract. The paper deals with the analysis and the numerical solution of the topology optimization of system governed by variational inequalities using the combined level set and phase field rather than the standard level set approach. The standard level set method allows to evolve a given sharp interface but is not able to generate holes unless the topological derivative is used. The phase field method indicates the position of the interface in a blurry way but is flexible in the holes generation. In the paper a two-phase topology optimization problem is formulated in terms of the modified level set function and regularized using the Cahn-Hilliard based interfacial energy term rather than the standard perimeter term. The derivative formulae of the cost functional with respect to the level set function is calculated. Modified reaction-diffusion equation updating the level set function is derived. The necessary optimality condition for this optimization problem is formulated. The finite element and finite difference methods are used to discretize the state and adjoint systems. Numerical examples are provided and discussed.

Keywords: topology optimization; unilateral problem; level set approach; phase field method

MSC 2010: 35J86, 49K20, 49Q10, 49Q12, 74N20, 74P10

1. Introduction

Shape and/or topology optimization problems of systems governed by PDEs arise in many applications. Examples include various branches of industry, biology or image processing [1], [2], [6], [8], [23]. The present paper is concerned with the topology optimization problem for an elastic body in unilateral contact with a rigid foundation. The contact phenomenon with Tresca friction is governed by the second order elliptic variational inequality [11], [21]. The structural optimization problem consists in finding the material distribution in a given design domain occupied by the body such that the normal contact stress along the boundary of the body is minimized.

Shape and/or topology optimization problems have been studied in literature from the analytical as well as numerical point of view. Topology optimization problems are usually ill-posed and require regularization [5], [7], [15], [16], [18], [19], [21], [25]. Existence results for this class of optimization problems may be found in [5]. The material derivative [21] or topology derivative methods [20] are employed to calculate the derivatives of the cost functional with respect to the shape boundary variations or to the insertion or removal of a void (hole) from the material of the body, respectively, and to formulate a necessary optimality condition.

Many successful numerical methods have been proposed to solve shape or topology optimization problems. For the review of these methods see [8], [23]. Especially, Simple Isotropic Material Penalization method, Evolutionary Structural Optimization approach [8] or topology derivative method [20] are the main methods used to solve topology optimization problems. Recently the use of the level set methods [17] and the phase field methods [4] has been proposed to solve the topology optimization problems [3]–[5], [7], [13], [14], [16], [18], [22]–[25]. In numerical algorithms of structural optimization the level set method is employed for capturing the evolution of the domain boundary on a fixed mesh and finding an optimal domain [1], [2]. It is based on the implicit representation of the boundaries of the optimized structure. It introduces a continuous auxiliary function over the whole global domain and embeds the optimized domain interface as the zero level set of this higher dimensional function. Phase field models in the form of Cahn-Hilliard or Allen-Hilliard equations [4] have been first introduced in metallurgy to describe phase separation in binary alloy systems. Next, these approaches have been used to provide mathematical models in different areas, including crack propagation, image processing, tumor growth. The basic concept of the phase field model is the representation of two fluid or material phases by two minima of a double-well potential with a smooth transition region representing the interface. The evolution equations for the smooth fields corresponding to the phase field variable are obtained using a variational approach associated with searching minimum of the corresponding free energy or entropy. The topology optimization problem in multiphase setting can be transformed further into a phase field problem where the optimal topology is characterized as the steady state of the phase transition.

The standard level set method is capable of precisely locating the position of the subdomains interfaces but cannot generate voids. It requires the use of the topology derivative method to indicate the voids areas. Moreover, this method requires reinitialization of the level set function. On the other hand, the phase field method is capable of generating voids but cannot precisely determine the position of the interfaces. The phase field method has many similarities with the level set approach and can be viewed as a physically motivated level set method. In the paper, taking into account the similarity of these two approaches, we merge them into one hybrid level set phase field method. Using the suitable features of both approaches the hybrid method is capable of generating voids and sharply locating the position of subdomains interfaces. This method does not require the topological derivative to indicate the void area.

The paper is concerned with the analysis and numerical solution of the topology optimization of systems governed by the elliptic variational inequalities modeling the elastic unilateral contact problem with Tresca friction using the combined level set and phase field rather than the standard level set approach. The aim of the optimization problem is to find the distribution of the material of the body in unilateral contact with the rigid foundation that minimizes the normal contact stress. Two-phase topology optimization problem is formulated in terms of the modified level set function. This optimization problem is regularized using the Cahn-Hilliard interface energy term rather than the perimeter term. Derivatives formulae of the cost functional with respect to the modified level set function are calculated. Interface evolution is governed by the modified gradient flow equation of reaction-diffusion type. The necessary optimality condition for this optimization problem is formulated. The numerical implementation issues are described. Numerical examples are provided and discussed.

2. Problem formulation

Consider deformations of an elastic body occupying a two-dimensional domain Ω with a smooth boundary Γ (see Figure 1). Assume $\Omega \subset D$ where D

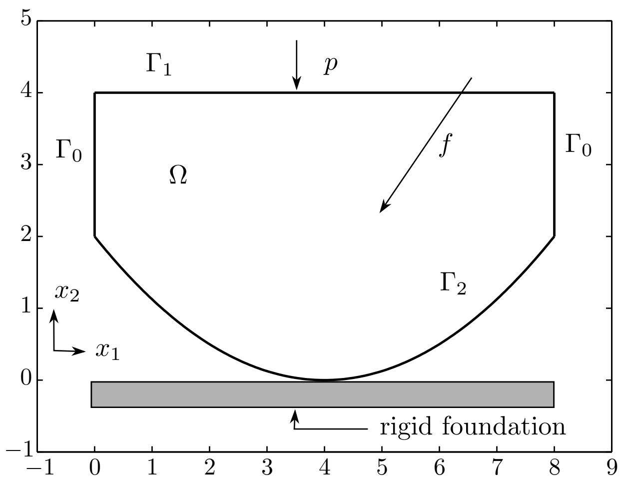

Figure 1. Initial domain Ω.

is a bounded smooth hold-all subset of $\mathbb{R}^2$. The body is subject to body forces $f(x) = (f_1(x), f_2(x))$, $x \in \Omega$. Moreover, surface tractions $p(x) = (p_1(x), p_2(x))$, $x \in \Gamma$, are applied to a portion Γ_1 of the boundary Γ. We assume that the body is clamped along the portion Γ_0 of the boundary Γ, and that the contact conditions are prescribed on the portion Γ_2, where $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j$, $i, j = 0, 1, 2$, $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_2$.

Let $\varrho = \varrho(x) \colon \Omega \to \mathbb{R}$ denote the material density function at any generic point x in the design domain Ω. It is a phase field variable taking values close to 1 in the presence of material, while $\varrho = 0$ corresponds to regions of the domain Ω where the material is absent, i.e. there is a void. The phase field approach assumes that between the material and the void there is a diffusive interfacial layer of a thickness proportional to a small length scale parameter $\varepsilon > 0$. At this interface the phase field variable ϱ rapidly but smoothly changes its value [3], [4], [19]. We require that $0 \leqslant \varrho \leqslant 1$. The ϱ values outside this range do not seem to correspond to admissible material distributions. The elastic tensor $\mathcal{A}$ of the material body is assumed to be a function depending on the density function ϱ:

$$\mathcal{A} = g(\varrho)\mathcal{A}_0, \quad \mathcal{A}_0 = \{a_{ijkl}\}_{i,j,k,l=1}^{2} \tag{2.1}$$

and $g(\varrho)$ is a suitably chosen function [5], [20]. We denote by $u = (u_1, u_2)$, $u = u(x)$, $x \in \Omega$, the displacement of the body and by $\sigma(x) = \{\sigma_{ij}(u(x))\}$, $i, j = 1, 2$, the stress field in the body. We consider elastic bodies obeying Hooke's law, i.e., for $x \in \Omega$ and $i, j, k, l = 1, 2$

$$\sigma_{ij}(u(x)) = g(\varrho) a_{ijkl}(x) e_{kl}(u(x)). \tag{2.2}$$

We use here and throughout the paper the summation convention over repeated indices [11]. The strain $e_{kl}(u(x))$, $k, l = 1, 2$, is defined by:

$$e_{kl}(u(x)) = \frac{1}{2}(u_{k,l}(x) + u_{l,k}(x)), \tag{2.3}$$

where $u_{k,l}(x) = \partial u_k(x)/\partial x_l$. The stress field σ satisfies the system of equations in the domain Ω [11]

$$-\sigma_{ij}(x)_{,j} = f_i(x), \quad x \in \Omega,\ i, j = 1, 2, \tag{2.4}$$

where $\sigma_{ij}(x)_{,j} = \partial \sigma_{ij}(x)/\partial x_j$, $i, j = 1, 2$. The following boundary conditions are imposed on the boundary $\partial\Omega$:

$$u_i(x) = 0 \quad \text{on } \Gamma_0,\ i = 1, 2, \tag{2.5}$$

$$\sigma_{ij}(x) n_j = p_i \quad \text{on } \Gamma_1,\ i, j = 1, 2, \tag{2.6}$$

$$u_N \leqslant 0, \quad \sigma_N \leqslant 0, \quad u_N \sigma_N = 0 \quad \text{on } \Gamma_2, \tag{2.7}$$

$$|\sigma_T| \leqslant 1, \quad u_T \sigma_T + |u_T| = 0 \quad \text{on } \Gamma_2, \tag{2.8}$$

where $n = (n_1, n_2)$ is the unit outward versor to the boundary Γ. Here $u_N = u_i n_i$ and $\sigma_N = \sigma_{ij} n_i n_j$, $i, j = 1, 2$, represent the normal components of displacement u and stress σ, respectively. The tangential components of displacement u and stress σ are given by $(u_T)_i = u_i - u_N n_i$ and $(\sigma_T)_i = \sigma_{ij} n_j - \sigma_N n_i$, $i, j = 1, 2$, respectively. $|u_T|$ denotes the Euclidean norm in $\mathbb{R}^2$ of the tangent vector u_T. The results concerning the existence of unique solutions to (2.4)–(2.8) can be found in [11], [21].

2.1. Variational formulation of contact problem. Let us formulate the contact problem (2.4)–(2.8) in the variational form. Denote by $V_{\rm sp}$ and K the space and the set of kinematically admissible displacements:

$$V_{\rm sp} = \{z \in [H^1(\Omega)]^2 \colon\ z_i = 0 \text{ on } \Gamma_0,\ i = 1, 2\}, \tag{2.9}$$

$$K = \{z \in V_{\rm sp} \colon\ z_N \leqslant 0 \text{ on } \Gamma_2\}, \tag{2.10}$$

where $H^1(\Omega)$ denotes the Sobolev space of square integrable functions and their first derivatives [11], [21] and $[H^1(\Omega)]^2 = H^1(\Omega) \times H^1(\Omega)$. Denote also by Λ the set

$$\Lambda = \{\zeta \in L^2(\Gamma_2) \colon\ |\zeta| \leqslant 1\}.$$

The variational formulation of problem (2.4)–(2.8) has the form: *find a pair* $(u, \lambda) \in K \times \Lambda$ *satisfying*

$$\begin{aligned} \int_\Omega g(\varrho) a_{ijkl} e_{ij}(u) e_{kl}(\varphi - u)\,\mathrm{d}x &- \int_\Omega f_i(\varphi_i - u_i)\,\mathrm{d}x \\ - \int_{\Gamma_1} p_i(\varphi_i - u_i)\,\mathrm{d}s &+ \int_{\Gamma_2} \lambda(\varphi_T - u_T)\,\mathrm{d}s \geqslant 0, \quad \varphi \in K, \end{aligned} \tag{2.11}$$

$$\int_{\Gamma_2} (\zeta - \lambda) u_T\,\mathrm{d}s \leqslant 0, \quad \zeta \in \Lambda, \tag{2.12}$$

$i, j, k, l = 1, 2$. The function λ is interpreted as a Lagrange multiplier corresponding to the term $|u_T|$ in the equality constraint in (2.8) [11]. This function is equal to the tangent stress along the boundary Γ_2, i.e., $\lambda = \sigma_{T|\Gamma_2}$. The function λ belongs to the space $H^{-1/2}(\Gamma_2)$, i.e., the space of traces on the boundary Γ_2 of functions from the space $H^1(\Omega)$. Here, following [11], the function λ is assumed to be more regular, i.e. $\lambda \in L^2(\Gamma_2)$. The results concerning the existence of solutions to system (2.11)–(2.12) under the assumptions introduced can be found, among others, in [11], [21]:

Theorem 2.1. *There exists a unique solution* $(u,\lambda) \in K \times \Lambda$ *to system* (2.11)–(2.12).

2.2. Topology optimization problem. Before formulating a structural optimization problem for (2.11)–(2.12) let us introduce the set $U_{\rm ad}$ of admissible domains. Denote by $\mathrm{Vol}(\Omega)$ the volume of the domain Ω equal to

$$\mathrm{Vol}(\Omega) = \int_\Omega \varrho(x)\,\mathrm{d}x. \tag{2.13}$$

Domain Ω is assumed to satisfy the volume constraint of the form

$$\mathrm{Vol}(\Omega) - \mathrm{Vol}^{\rm giv} \leqslant 0, \tag{2.14}$$

where the constant $\mathrm{Vol}^{\rm giv} = C_0 > 0$ is given. In the case when the shape optimization for problem (2.11)–(2.12) is considered the domain Ω is assumed to satisfy volume constraint (2.14) as equality. In the case of the topology optimization $\mathrm{Vol}^{\rm giv}$ is assumed to be the initial domain volume and (2.14) is satisfied in the form $\mathrm{Vol}(\Omega) = r_{\rm fr}\,\mathrm{Vol}^{\rm giv}$ with $r_{\rm fr} \in (0,1)$, see [20]. The set $U_{\rm ad}$ has the form

$$\begin{aligned} U_{\rm ad} = \{\Omega\colon\ E \subset \Omega \subset D \subset \mathbb{R}^2\colon\ &\Omega \text{ is Lipschitz continuous,} \\ &\Omega \text{ satisfies condition (2.14)}\}, \end{aligned} \tag{2.15}$$

where $E \subset \mathbb{R}^2$ is a given domain such that Ω as well as all of its perturbations satisfy $E \subset \Omega$. The constant $C_1 > 0$ is assumed to exist. The set $U_{\rm ad}$ is assumed to be nonempty. In order to define a cost functional we shall also need the set $M^{\rm st}$ of auxiliary functions

$$M^{\rm st} = \{\eta = (\eta_1,\eta_2) \in [H^1(D)]^2\colon\ \eta_i \leqslant 0 \text{ on } D,\ i=1,2,\ \|\eta\|_{[H^1(D)]^2} \leqslant 1\}, \tag{2.16}$$

where the norm is $\|\eta\|_{[H^1(D)]^2} = \Big(\sum\limits_{i=1}^{2} \|\eta_i\|^2_{H^1(D)}\Big)^{1/2}$. Recall from [13], [15], [16] the cost functional approximating the normal contact stress on the contact boundary

$$J_\eta(u(\Omega)) = \int_{\Gamma_2} \sigma_N(u)\eta_N(x)\,\mathrm{d}s, \tag{2.17}$$

depending on the given auxiliary bounded function $\eta(x) \in M^{\rm st}$. The functions σ_N and η_N are the normal components of the stress field σ corresponding to a solution u satisfying system (2.11)–(2.12) and the auxiliary function η, respectively.

Consider the following structural optimization problem: *for a given function* $\eta \in M^{\mathrm{st}}$, *find a domain* $\Omega^\star \in U_{\mathrm{ad}}$ *such that*

$$J_\eta(u(\Omega^\star)) = \min_{\Omega \in U_{\mathrm{ad}}} J_\eta(u(\Omega)). \tag{2.18}$$

Adding to (2.15) a perimeter constraint $P_D(\Omega) \leqslant C_1$, where $P_D(\Omega) = \int_\Gamma \,\mathrm{d}x$ is a perimeter of a domain Ω in D [16], [21] and $C_1 > 0$ is a given constant the existence of an optimal domain $\Omega^\star \in U_{\mathrm{ad}}$ to the problem (2.18) is ensured (see [5], [21]).

Theorem 2.2. *Assume the number of connected components of the complement set* Ω^c *of domain* Ω *with respect to* $D \subset \mathbb{R}^2$ *is bounded. There exists a solution to* $\widehat{\Omega} \subset U_{\mathrm{ad}}$ *to the problem* (2.18).

Proof. The class of admissible domains is endowed with the complementary Hausdorff topology that guarantees the class itself to be compact. The existence of an optimal domain $\Omega^\star \in U_{\mathrm{ad}}$ to the topology optimization problem (2.18) follows from Šverák's theorem and arguments provided in [5]. □

The optimization problem (2.18) has been analysed and numerically solved using either the classical level set approach [16] or the phase field approach [13]. Let us recall the main features of these approaches.

2.3. Level set based topology optimization. Let $t \in [0, t_0)$, $t_0 > 0$ given, denote the artificial time variable, $V = V(t,x) \in C^2(0, t_0; C^2(D; \mathbb{R}^2))$ a velocity field and I the identity operator. Consider the evolution of a domain Ω under a velocity field V in time t. Under a suitable regular mapping $T(t, V)$ we have [24]

$$\Omega_t = T(t, V)(\Omega) = (I + tV)(\Omega), \quad t > 0.$$

By Ω_t^- (Ω_t^+) we denote the interior (exterior) of the domain Ω_t. The domain Ω_t and its boundary $\partial\Omega_t$ are defined by a function $\phi = \phi(t,x)\colon\ [0,t) \times \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$\begin{aligned} \phi(t,x) = 0 \quad \text{if } x \in \partial\Omega_t, \quad \phi(t,x) < 0 \quad \text{if } x \in \Omega_t^-, \\ \phi(t,x) > 0 \quad \text{if } x \in \Omega_t^+. \end{aligned} \tag{2.19}$$

Function ϕ satisfying (2.19) is called the level set function [17]. The gradient of this function is defined as $\nabla\phi = \big(\partial\phi/\partial x_1, \partial\phi/\partial x_2\big)$, the local unit outward normal n to the boundary $\partial_t\Omega$ is equal to $n = \nabla\phi/|\nabla\phi|$ and the mean curvature is $\kappa = \nabla \cdot n$. In the level set approach the Heaviside function $H(\phi)$ and the Dirac function $\delta(\phi)$

are used to transform integrals on the domain Ω onto domain D, see [17]. These functions are defined in [17] as

$$H(\phi) = 1 \quad \text{if } \phi \geqslant 0, \quad H(\phi) = 0 \quad \text{if } \phi < 0, \quad \delta(x) = H'(\phi)|\nabla\phi(x)|. \tag{2.20}$$

Since in one spatial dimension the Dirac function $\delta(\phi) = H'(\phi)$ is identically zero everywhere except at $\phi = 0$ it allows to rewrite the function $\delta(x)$ using the one-dimensional Dirac function $\delta(\phi)$, i.e.

$$\delta(x) = \delta(\phi(x))|\nabla\phi(x)|, \quad x \in D. \tag{2.21}$$

Using (2.20) and (2.21) we can write

$$\int_\Omega f(x)\,\mathrm{d}x = \int_D f(x)H(\phi)\,\mathrm{d}x \quad \text{and} \quad \int_{\partial\Omega} f(x)\,\mathrm{d}s = \int_D f(x)\delta(\phi)|\nabla\phi|\,\mathrm{d}s. \tag{2.22}$$

Assume that the velocity field $V = V(t,x)$ is known for every point x lying on the boundary $\partial\Omega_t$, i.e., such that $\phi(t,x) = 0$. Therefore the equation governing the evolution of the interface $\partial\Omega_t$ in $[0,t_0] \times D$, known as the Hamilton-Jacobi equation, has the form [1], [17]

$$\frac{\partial\phi(t,x)}{\partial t} + V(t,x) \cdot \nabla_x\phi(t,x) = 0, \quad \phi(0,x) = \phi_0(x), \tag{2.23}$$

where $\phi_0(x)$ is a given signed distance function of the set Ω_t. The velocity field V in (2.23) is chosen as the shape derivative of the cost functional (2.17) with respect to the boundary variations of the domain. Topological derivative of this cost functional is used to indicate the areas of voids or weak material inside domain Ω. The shape and topology derivatives of the cost functional (2.17) are provided and a necessary optimality condition is shown in [16]. For other applications of the standard level set approach to analyse and solve numerically structural optimization problems see [1], [7], [23].

2.4. Phase field based topology optimization. The phase field approach is based on the assumption that the material occupying the domain Ω consists of two phases, i.e. strong and weak materials, see [1]. The weak material distribution corresponds to voids. The concentration of the phases is described by the material density function $0 \leqslant \varrho \leqslant 1$. While in the domain Ω the concentration of one of the phases is ϱ the other phase is obtained as $1-\varrho$. The function ϱ is used to describe the phase transition, see [19]. To indicate the evolution of the material density function ϱ let us assume this function depends not only on $x \in \Omega$ but also on the artificial time variable $t \in [0, t_0)$, $t_0 > 0$ given, i.e. $\varrho = \varrho(t, x)$. Let us introduce the regularized cost functional $J(\varrho, u)$ in the form

$$J(\varrho, u) = J_\eta(u) + E(\varrho), \tag{2.24}$$

where the functional $J_\eta(u)$ is given by (2.17) with $u = u(\varrho)$ and the Ginzburg-Landau free energy term $E(\varrho)$ and the total free energy function $\psi(\varrho)$ are given by [4], [5], [23], [24]

$$E(\varrho) = \int_\Omega \psi(\varrho)\,\mathrm{d}\Omega, \quad \psi(\varrho) = \frac{\gamma\varepsilon}{2}|\nabla\varrho|^2 + \frac{\gamma}{\varepsilon}\psi_B(\varrho), \tag{2.25}$$

where $\gamma > 0$ is a constant, $\varepsilon > 0$ is a parameter related to the interfacial energy density and $\psi_B(\varrho)$ is a double-well potential which characterizes the two phases, see [4], [5]. Usually it is taken as an even-order polynomial of the form, see [12],

$$\psi_B(\varrho) = \varrho^2(1 - \varrho^2). \tag{2.26}$$

The first term in the total free energy function $\psi(\varrho)$ is called the interface energy. It represents [4], [10], [25] a measure of the perimeter of the interfaces between the phases and in this sense it is the relaxed version of the global perimeter constraint. The term (2.26) is called the bulk energy. It is a non-convex smooth function attaining minimum in the pure phases $\varrho = 0$ and $\varrho = 1$. The values assumed by $\psi_B(\varrho)$ for intermediate values of ϱ are larger than for pure phases and are not preferred in the optimization process. Parameter ε measures the width of the transition zone. The structural optimization problem (2.18) in terms of the function ϱ takes the form: *find* $\varrho^\star \in U_{\mathrm{ad}}^\varrho$ *such that*

$$J(\varrho^\star, u^\star) = \min_{\varrho \in U_{\mathrm{ad}}^\varrho} J(\varrho, u), \tag{2.27}$$

where $u^\star = u(\varrho^\star)$ denotes a solution to the state system (2.11)–(2.12) in domain D rather than Ω depending on $\varrho^\star$. The set $U_{\mathrm{ad}}^\varrho = \{\varrho\colon \mathrm{Vol}(\Omega) = \mathrm{Vol}^{\mathrm{giv}}\}$ denotes the set of admissible material density functions.

The definition of the phase transition model is based on the concept of the flow of the gradient $\partial L/\partial\varrho$ of the Lagrangian L of the optimization problem (2.27) with respect to ϱ in the norm of a suitable chosen Hilbert space H:

$$\frac{\partial\varrho}{\partial t}(t, x) = -\frac{\partial L}{\partial\varrho}(\varrho) \quad \text{in } \Omega, \quad t \in [0, T), \quad \varrho(0, x) = \varrho_0(x), \quad x \in D, \tag{2.28}$$

where $\varrho_0(x)$ is a given function. Selecting the space H, see [4], as the subspace of a space $[H^1(\Omega)]'$ dual to $H^1(\Omega)$ (2.28) leads to a necessary optimality condition in the form of the modified Cahn-Hilliard equation. For details see [13].

3. Hybrid approach to topology optimization

The level set and the phase field approaches are well-known due to their topological flexibility [3], [6], [9], [19], [25]. Both the approaches are very flexible and allow a wide range of extensions for model-based matching, registration and segmentation, optical flow with discontinuities, fluid flow. In these methodologies the process of splitting a curve into several curves is a smooth one. However, these two approaches differs significantly in the representation of the discontinuity set. The level set method allows to represent trace and evolve a given sharp interface. This fits very well to the framework of the calculus of shape derivatives in which the current interface is given precisely. On the other hand, the phase field function is able to indicate the position of a interface in a blurry way only determined by the order of a grid size. The classical level set framework is restricted to closed curves and thus it does not allow to represent crack tips or to generate a hole using a single level set function. Topological derivative is used to generate holes in the framework of the level set method [20]. On the other hand, the phase field method appears to be more flexible and practicable for these applications. The phase field representation is global by definition and respects the features of the topology in the entire domain occupied by a structure without requiring any initialization.

Taking into account the similarity of these two approaches, hybrid interface tracking methods are combining within one approach elements of the level set and the phase field approaches. Using the suitable features of both approaches the hybrid method is able to generate voids and to sharply locate the position of subdomains interfaces. This method does not require the topological derivative to indicate the void area. The hybrid method developed in the paper is based on the definition of a modified level set function indicating subdomains of the different phases similarly to the material density function in the phase field method and regularizes the original topology optimization problem using the terms of Ginzburg-Landau free energy term. Therefore the regularized topology optimization problem is formulated in terms of the modified level set function only.

Structural optimization problems with a level set function and different phase field like gradient flow equations are considered in [7], [19], [22], [25]. The relation between phase field and sharp interface tracking models in optimal control problems is considered in [3]. Using the method of the matched asymptotic expansions it is shown that for the compliance topology optimization problem in linear elasticity the sharp interface limit of the necessary optimality condition for the phase field model when the interface width parameter is passing to zero coincides with the necessary optimality condition for this optimization problem obtained by the shape calculus [1].

3.1. Hybrid formulation of the topology optimization problem. Consider a slightly modified level set function ϕ as compared to the standard one (2.19),

$$0 < \phi(x) \leqslant 1 \quad \text{for } x \in \Omega \setminus \partial\Omega, \quad \phi(x) = 0 \quad \text{for } x \in \partial\Omega, \qquad -1 \leqslant \phi(x) < 0 \quad \text{for } x \in D \setminus \Omega. \tag{3.1}$$

Note that the level set function (3.1) is close to the phase field function ϱ governing the evolution of phases in the phase field method or to the so-called binary level set method [19]. This function is bounded and takes values close to $+1$ or -1 in regions sufficiently distant from the interfaces. Consider the regularized cost functional (2.17)

$$J_R(\phi) = J_\eta(u(\phi)) + E_R(\phi), \quad E_R(\phi) = \frac{1}{2}\tau \int_D |\nabla\phi|^2 \, \mathrm{d}\Omega, \tag{3.2}$$

where $\tau > 0$ is a regularization parameter. The structural optimization problem (2.18) takes the form: *find* $\phi \in U_{\mathrm{ad}}^\phi$ *such that:*

$$\min_{\phi \in U_{\mathrm{ad}}^\phi} J_R(\phi), \tag{3.3}$$

where the admissible set U_{ad}^ϕ (2.15) in terms of ϕ has the form:

$$U_{\mathrm{ad}}^\phi = \left\{ \phi \in H^1(D) \colon \ \mathrm{Vol}(\phi) = \int_D H(\phi)\, \mathrm{d}x - \mathrm{Vol}^{\mathrm{giv}} \leqslant 0 \right\}. \tag{3.4}$$

A pair $(u, \lambda) \in K \times \Lambda$ solves the state system (2.11)–(2.12) in the domain D rather than Ω:

$$\int_D H(\phi) a_{ijkl} e_{ij}(u) e_{kl}(\varphi - u)\, \mathrm{d}x - \int_D H(\phi) f_i(\varphi_i - u_i)\, \mathrm{d}x - \int_{\Gamma_1} p_i(\varphi_i - u_i)\, \mathrm{d}s + \int_{\Gamma_2} \lambda(\varphi_T - u_T)\, \mathrm{d}s \geqslant 0, \quad \varphi \in K, \tag{3.5}$$

$$\int_{\Gamma_2} (\zeta - \lambda) u_T \, \mathrm{d}s \leqslant 0, \quad \zeta \in \Lambda. \tag{3.6}$$

The existence of a unique solution to (3.5)–(3.6) follows from Theorem 2.1.

4. Necessary optimality condition

So let us formulate the necessary optimality condition for problem (3.3)–(3.6). In order to do it we introduce the Lagrangian $L(\phi,\tilde{\lambda})\colon\ H^1(D)\times\mathbb{R}\to\mathbb{R}$ by

$$\begin{aligned}(4.1)\quad L(\phi,\tilde{\lambda}) &= L(\phi,u_\varepsilon,\lambda_\varepsilon,p^a,q^a,\tilde{\lambda})\\ &= J_R(\phi)+\int_D H(\phi)a_{ijkl}e_{ij}(u_\varepsilon)e_{kl}(p^a)\,\mathrm{d}x-\int_D H(\phi)f_i(p_i^a)\,\mathrm{d}x\\ &\quad-\int_{\Gamma_1}p_ip_i^a\,\mathrm{d}s+\int_{\Gamma_2}\lambda_\varepsilon(p_T^a)\,\mathrm{d}s+\int_{\Gamma_2}q^au_{\varepsilon T}\,\mathrm{d}s+\tilde{\lambda}c(\phi)+\frac{1}{2\mu}c^2(\phi),\end{aligned}$$

where $\tilde{\lambda}\in\mathbb{R}$, $c(\phi)=[\mathrm{Vol}(\phi)]$, $\mu>0$ is a given real. By $(p^a,q^a)\in K_1\times\Lambda_1$ we denote an adjoint state satisfying the system

$$(4.2)\qquad \int_D H(\phi)a_{ijkl}e_{ij}(\eta+p^a)e_{kl}(\varphi)\,\mathrm{d}x+\int_{\Gamma_2}q^a\varphi_T\,\mathrm{d}s=0,\quad \varphi\in K_1,$$

$$(4.3)\qquad \int_{\Gamma_2}\zeta(p_T^a+\eta_T)\,\mathrm{d}s=0,\quad \zeta\in\Lambda_1.$$

The sets K_1 and Λ_1 are given by

$$(4.4)\qquad K_1=\{\xi\in V_{\mathrm{sp}}\colon\ \xi_N=0\text{ on }A^{\mathrm{st}}\},$$

$$(4.5)\qquad \Lambda_1=\{\zeta\in\Lambda\colon\ \zeta(x)=0\text{ on }B_1\cup B_2\cup B_1^+\cup B_2^+\},$$

while the coincidence set is $A^{\mathrm{st}}=\{x\in\Gamma_2\colon\ u_N+v=0\}$. Moreover $B_1=\{x\in\Gamma_2\colon\ \lambda(x)=-1\}$, $B_2=\{x\in\Gamma_2\colon\ \lambda(x)=+1\}$, $\widetilde{B}_i=\{x\in B_i\colon\ u_N(x)+v=0\}$, $i=1,2$, $B_i^+=B_i\setminus\widetilde{B}_i$, $i=1,2$. Using (4.2)–(4.5) as well as the results on differentiability of variational inequalities [21] we obtain [16] the derivative of the Lagrangian L with respect to ϕ:

$$\begin{aligned}(4.6)\qquad \int_D\frac{\partial L}{\partial\phi}(\phi,\tilde{\lambda})\zeta\,\mathrm{d}x &= \int_D\big[H(\phi)(a_{ijkl}e_{ij}(u_\varepsilon)e_{kl}(p^a+\eta)\\ &\quad-f(p^a+\eta))+\tau\Delta\phi\big]\zeta\,\mathrm{d}x+\int_D\Big(\tilde{\lambda}+\frac{1}{\mu}c(\phi)\Big)\zeta\,\mathrm{d}x,\quad \zeta\in H^1(D).\end{aligned}$$

The necessary optimality condition for problem (3.3)–(3.6) follows by standard arguments [11], [21]:

Theorem 4.1. *If* $(\widehat{\phi}, \tilde{\lambda}^\star) \in U^\phi_{\rm ad} \times \mathbb{R}$ *is an optimal solution to problem* (3.3)–(3.6) *then*

$$L(\widehat{\phi}, \tilde{\lambda}) \leqslant L(\widehat{\phi}, \tilde{\lambda}^\star) \leqslant L(\phi, \tilde{\lambda}^\star), \quad (\phi, \tilde{\lambda}) \in U^\phi_{\rm ad} \times \mathbb{R},\ \tilde{\lambda} \geqslant 0. \tag{4.7}$$

Condition (4.7) implies [11], [21] that for all $\phi \in U^\phi_{\rm ad}$ and $\tilde{\lambda} \in \mathbb{R}$, $\tilde{\lambda} \geqslant 0$ we have

$$\frac{\partial L(\widehat{\phi}, \tilde{\lambda})}{\partial \phi} \geqslant 0 \quad \text{and} \quad \frac{\partial L(\phi, \tilde{\lambda}^\star)}{\partial \tilde{\lambda}} \leqslant 0. \tag{4.8}$$

5. Implementation issues

Uzawa type algorithm is employed to solve numerically the optimization problem (3.3). First as in (2.19) we assume that due to the evolution of the subdomains the function ϕ is also time dependent. The minimization of the Lagrangian $L(\phi, \tilde{\lambda})$ with respect to ϕ is realized by solving the time dependent PDE [17]

$$\begin{gathered} \frac{\partial \phi(t,x)}{\partial t} = \nabla_\phi L(\phi, \tilde{\lambda}) \quad \text{in } (0,\infty) \times D, \\ \phi(0,x) = \phi_0(x) \quad \text{in } D, \\ \nabla \phi \cdot n = 0 \quad \text{on } \partial D \end{gathered} \tag{5.1}$$

to reach the steady state $\partial\phi/\partial t = 0$. It implies the gradient $\nabla_\phi L(\phi, \tilde{\lambda})$ given by (4.6) equals zero. $\phi_0(x)$ is a given function. The explicit Euler scheme [2] is used to solve numerically the equation (5.1), i.e.,

$$\phi^{n+1} = \phi^n + \Delta t^n \frac{\partial L(\phi^n, \tilde{\lambda}^n)}{\partial \phi}, \tag{5.2}$$

where $\phi^n = \phi(x, t^n)$, Δt^n denotes the nth time step and $\partial L(\phi^n, \tilde{\lambda}^n)/\partial\phi$ is given by (4.6). To satisfy CFL stability condition the stepsize Δt^n is assumed to satisfy, see [17],

$$\Delta t^n = \frac{\alpha h}{\max\limits_{x \in D} \left| \partial L(\phi^n(x), \tilde{\lambda}^n)/\partial\phi \right|}, \tag{5.3}$$

where α is a suitable given number and h is the uniform mesh size. The updating scheme for the Lagrange multiplier $\tilde{\lambda}$ is as follows:

$$\tilde{\lambda}^{n+1} = \tilde{\lambda}^n + \frac{1}{\mu^n} \operatorname{Vol}(\phi), \tag{5.4}$$

with the penalty parameter $\mu^{n+1} \in (0, \mu^n)$, $\mu^0 > 0$ given.

5.1. Numerical example. The discretized topology optimization problem (3.3)–(3.6) is solved numerically. As an example a body occupying the 2D domain

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \colon\ 0 \leqslant x_1 \leqslant 8 \wedge 0 < v(x_1) \leqslant x_2 \leqslant 4\} \tag{5.5}$$

is considered. The boundary Γ of the domain Ω is divided into three pieces

$$\begin{aligned} \Gamma_0 &= \{(x_1, x_2) \in \mathbb{R}^2 \colon\ x_1 = 0, 8 \wedge 0 < v(x_1) \leqslant x_2 \leqslant 4\}, \\ \Gamma_1 &= \{(x_1, x_2) \in \mathbb{R}^2 \colon\ 0 \leqslant x_1 \leqslant 8 \wedge x_2 = 4\}, \\ \Gamma_2 &= \{(x_1, x_2) \in \mathbb{R}^2 \colon\ 0 \leqslant x_1 \leqslant 8 \wedge v(x_1) = x_2\}. \end{aligned} \tag{5.6}$$

The domain Ω and the boundary Γ_2 depend on the function v. The initial position of the boundary Γ_2 is given as in Figure 1. The computations are carried out for the elastic body characterized by Poisson's ratio $\nu = 0.29$, the Young modulus $E = 2.1 \cdot 10^{11}\,\mathrm{N/m}^2$ for the strong phase and the Young modulus $E' = 10^{-4} \cdot E$ for the weak phase. The body is loaded by boundary traction $p_1 = 0$, $p_2 = -5.6 \cdot 10^6\,\mathrm{N}$ along Γ_1, body forces $f_i = 0$, $i = 1, 2$. An auxiliary function η is selected as piecewise constant (or linear) on D and is approximated by a piecewise constant (or bilinear) functions. The computational domain $D = [0, 8] \times [0, 4]$ is selected. The domain D is discretized with a fixed rectangular mesh of 80×40.

Figure 2 presents the optimal domain obtained by solving the topology optimization problem (3.3) in the computational domain D using a Uzawa type algorithm and employing the optimality condition (4.7). Weak material areas where the material density function has low values appear in the central part of the body and near the fixed edges. While for the initial domain the normal contact stress is concentrated

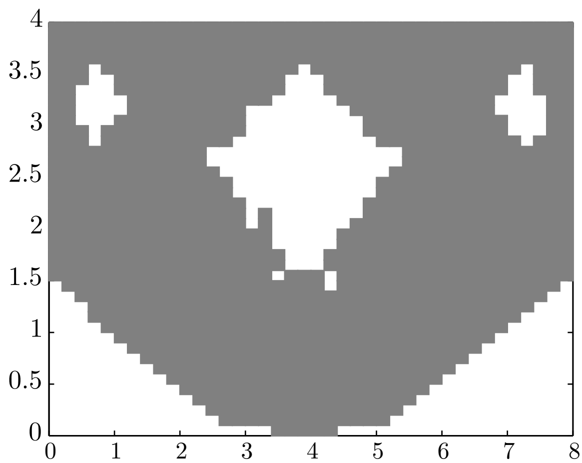

Figure 2. Optimal domain $\Omega^\star$.

in the middle of the contact zone and takes high values the optimal normal contact stress obtained is almost constant and uniformly distributed along the contact boundary. Moreover, its maximal value has been significantly reduced comparing to the initial one (see Figure 3). The convergence history and the decrease of the cost functional value during the computational process is shown in Figure 4. In the beginning of the computations the cost functional value is rapidly decreasing and later it is relatively slowly approaching the minimal value.

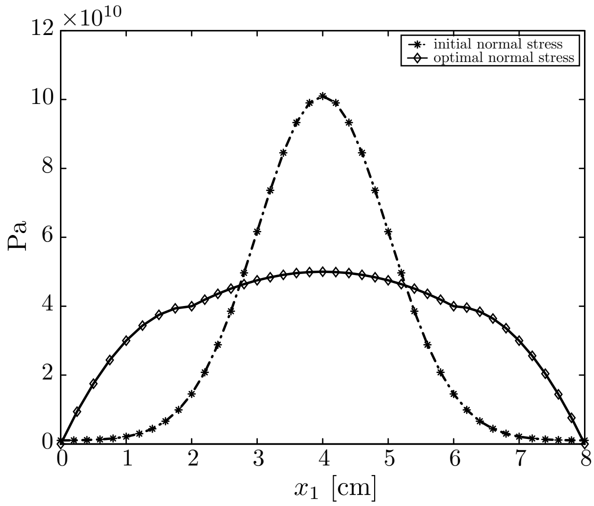

Figure 3. Initial and optimal normal contact stress.

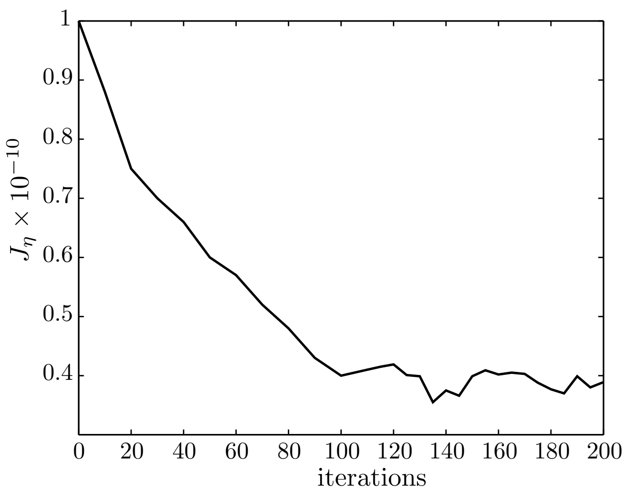

Figure 4. The decrease of the cost functional value during computational process.

6. Concluding remarks

The topology optimization problem for the elastic contact problem with the prescribed friction is analysed and solved numerically in the paper using the level set approach combined with the phase field approach. The friction term complicates both the form of the gradients of the cost functional as well as of the numerical process. The obtained numerical results seems to be in accordance with physical reasoning. They indicate that the proposed method allows for significant improvements of the structure from one iteration to the next and is more efficient than the algorithms based on the standard level set approach. Comparing to the standard level set approach the proposed approach does not require to solve Hamilton-Jacobi equation and to perform the reinitialization process of the signed distance function. Moreover, the proposed method has also hole nucleation capabilities as the topological derivative based methods.

References

[1] *G. Allaire, F. Jouve, A.-M. Toader*: Structural optimization using sensitivity analysis and a level-set method. J. Comput. Phys. *194* (2004), 363–393.

[2] *G. Aubert, P. Kornprobst*: Mathematical Problems in Image Processing. Partial Differential Equations and the Calculus of Variations. Applied Mathematical Sciences 147, Springer, New York, 2006.

[3] *L. Blank, H. Garcke, M. Hassan Farshbaf-Shaker, V. Styles*: Relating phase field and sharp interface approaches to structural topology optimization. ESAIM Control Optim. Calc. Var. *20* (2014), 1025–1058.

[4] *L. Blank, H. Garcke, L. Sarbu, T. Srisupattarawanit, V. Styles, A. Voigt*: Phase-field approaches to structural topology optimization. Constrained Optimization and Optimal Control for Partial Differential Equations (G. Leugering et al., eds.). International Series of Numerical Mathematics 160, Birkhäuser, Basel, 2012, pp. 245–256.

[5] *B. Bourdin, A. Chambolle*: The phase-field method in optimal design. IUTAM Symposium on Topological Design Optimization of Structures, Machines and Materials. Status and Perspectives. Proc. of the Conf. Rungstedgaard, Denmark, 2005 (M. P. Bendsøe et al., eds.). Springer, Dordrecht, 2006.

[6] *T. F. Chan, L. A. Vese*: Active contours without edges. IEEE Trans. Image Process. *10* (2001), 266–277.

[7] *J. S. Choi, T. Yamada, K. Izui, S. Nishiwaki, J. Yoo*: Topology optimization using a reaction-diffusion equation. Comput. Methods Appl. Mech. Eng. *200* (2011), 2407–2420.

[8] *J. D. Deaton, R. V. Grandhi*: A survey of structural and multidisciplinary continuum topology optimization: post 2000. Struct. Multidiscip. Optim. *49* (2014), 1–38.

[9] *M. Droske, W. Ring, M. Rumpf*: Mumford-Shah based registration: a comparison of a level set and a phase field approach. Comput. Vis. Sci. *12* (2009), 101–114.

[10] *A. L. Gain, G. H. Paulino*: Phase-field based topology optimization with polygonal elements: a finite volume approach for the evolution equation. Struct. Multidiscip. Optim. *46* (2012), 327–342.

[11] *J. Haslinger, R. A. E. Mäkinen*: Introduction to Shape Optimization. Theory, Approximation, and Computation. Advances in Design and Control 7, SIAM, Philadelphia, 2003.

[12] *M. Kronbichler, G. Kreiss*: A hybrid level-set-phase-field method for two-phase flow with contact lines. Technical Report 2011-026, University of Uppsala, Department of Information Technology, 2011.

[13] *A. Myśliński*: Phase field approach to topology optimization of contact problems. Proc. of the 10th World Congress on Structural and Multidisciplinary Optimization (R. Haftka, ed.). ISSMO, 2013, Paper 5434, 9 pages.

[14] *A. Myśliński*: Shape and topology optimization of elastic contact problems using piecewise constant level set method. Proc. of the 11th International Conf. on Computational Structural Technology (B. H. V. Topping, ed.). Civil-Comp Press, Stirlingshire, 2012, Paper 233, 12 pages.

[15] *A. Myśliński*: Radial basis function level set method for structural optimization. Control Cybern. *39* (2010), 627–645.

[16] *A. Myśliński*: Level set method for shape and topology optimization of contact problems. Eng. Anal. Bound. Elem. *32* (2008), 986–994; System Modeling and Optimization 2009 (A. Korytowski et al., eds.). IFIP Adv. Inf. Commun. Technol. 312, Elsevier, Oxford (2009), pp. 397–410.

[17] *S. Osher, R. Fedkiw*: Level Set Methods and Dynamic Implicit Surfaces. Applied Mathematical Sciences 153, Springer, New York, 2003.

[18] *P. Penzler, M. Rumpf, B. Wirth*: A phase-field model for compliance shape optimization in nonlinear elasticity. ESAIM, Control Optim. Calc. Var. *18* (2012), 229–258.

[19] *M. Scherer, R. Denzer, P. Steinmann*: A fictitious energy approach for shape optimization. Int. J. Numer. Methods Eng. *82* (2010), 269–302.

[20] *J. Sokołowski, A. Żochowski*: On topological derivative in shape optimization. Optimal Shape Design and Modelling (T. Lewiński et al., eds.). Academic Printing House EXIT, Warsaw, Poland, 2004, pp. 55–143.

[21] *J. Sokołowski, J.-P. Zolésio*: Introduction to Shape Optimization. Shape Sensitivity Analysis. Springer Series in Computational Mathematics 16, Springer, Berlin, 1992.

[22] *A. Takezawa, S. Nishiwaki, M. Kitamura*: Shape and topology optimization based on the phase field method and sensitivity analysis. J. Comput. Phys. *229* (2010), 2697–2718.

[23] *N. P. van Dijk, K. Maute, M. Langelaar, F. van Keulen*: Level-set methods for structural topology optimization: a review. Struct. Multidiscip. Optim. *48* (2013), 437–472.

[24] *M. Wallin, M. Ristinmaa, H. Askfelt*: Optimal topologies derived from a phase-field method. Struct. Multidiscip. Optim. *45* (2012), 171–183.

[25] *T. Yamada, K. Izui, S. Nishiwaki, A. Takezawa*: A topology optimization method based on the level set method incorporating a fictitious interface energy. Comput. Methods Appl. Mech. Eng. *199* (2010), 2876–2891.

Authors' address: *Andrzej Myśliński*, *Konrad Koniarski*, Systems Research Institute, Polish Academy of Sciences, ul. Newelska 6, 01-447 Warsaw, Poland, e-mail: `myslinsk@ibspan.waw.pl`, `konrad.koniarski@gmail.com`.