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The grouping of policy values

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β étant une constante arbitraire réelle. Par suite, en posant

$$2c = m,\ |\Re^{\frac{1}{2}}| = \sqrt{x^2 + y^2} = \varrho = \varphi(x, y),\ \text{arc tg}\ \frac{y}{x} = w,\ z = x + iy =$$
$$= \varrho(\cos w + i \sin w),$$

on trouve

$$\log F(z) = 2c \log |\Re^{\frac{1}{2}}| + 2ciw + \alpha + i\beta = m\ (\log \varrho + iw) - \alpha + i\beta,$$

d'où il suit

$$F(z) = e^{\log F(z)} = (e^{\log \varrho + iw})^m \,.\, e^{\alpha + i\beta} = A\ [\varrho(\cos w + i \sin w)]^m = Az^m.$$

A étant défini par l'équation $A = e^{\alpha + i\beta}$.

Donc la fonction cherchée $F(z)$ doit être exprimée par l'équation

$$F(z) = A\, z^m, \tag{11}$$

où A désigne une constante complexe et m une constante réelle. Réciproquement, la formule (11) donne une solution du problème proposé, la constante complexe A et la constante réelle m étant tout à fait arbitraires. En effet, pour $A = 0$ ou $m = 0$ le module de la fonction $A\, z^m$ reste constant et, par suite, ne croît pas et ne decroît pas, quand le module de z croît. Pour $A \neq 0$, $m > 0$ le module de cette fonction croît et pour $A \neq 0$, $m < 0$ il décroît toujours quand le module de z croît.

5. Dans les sciences empiriques ainsi que dans la statistique on se laisse guider par le principe suivant: si une quantité croît toujours ou décroît toujours, quand une autre quantité croît, l'une de ces deaux quantités dépend de l'autre. On voit bien que, certaines conditions indiquées au texte du théorème énoncé plus haut étant remplies, ce principe devient une vérité mathématique.

The grouping of policy values.

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I.

The main difficulty in the drawing up of the balance of an Insurance Company is in the ascertaining of the values of the individual policies. Since the number of cases is usually very large, we have to make our calculations as little cumbersome as possible. The actuary has to solve the problem of „grouping" policy values.

Disregarding merely approximative methods (e. g. Lidstone's Z method), we can proceed by way of grouping policies of the same kind and of the same attained age into one whole, for which we determine unique policy value. In every policy we take account only of certain auxiliary constants for the whole duration of the policy, which it is sufficient to determine at the time of entry. As far as I know, **Zillmer** has shown this for the first time in his book „Die mathematischen Rechnungen bei Lebens- und Rentenversicherung". (1st ed. 1861, 2nd ed. 1887), where he discusses the method of calculation for single-life assurances with whole-life or temporary payments of premiums, for temporary assurances payable at moment of death, for endowment assurances and for whole-life assurances. Mr. P. Smolenski discussed this question in a paper: Sul calculo delle riserve col metodo dei valori ausiliari in the new Italian review Giornale dell'Istituto Italiano degli attuari. He shows there that even with regard to certain more complicated cases it is possible to use successfully the method of grouping policy values according to the age attained by assistance of certain auxiliary constants.

I shall show that it is possible to use this method of grouping for any kind of assurance whatever. The general equation of equilibrium is

$$a_x(a_0 a_1 \dots a_\omega) + A_x(c_0 c_1 \dots c_\omega) = a_x(p_0 p_1 \dots p_\omega).\text{*)}$$

From the known relation

$$A_x(c_0, c_1, c_2 \dots c_k \dots) = a_x(c_0, c_1 - c_0, \dots c_k - c_{k-1} \dots) - d a_x(c_0 c_1 \dots c_k \dots)$$

follows that we can transform the equation of equilibrium so that only the expressions $a_x(a_0, a_1 \dots)$ will occur in it.

The policy value after ν years is

$${}_\nu V_x = a_{x+\nu}(a_\nu\, a_{\nu+1} \dots) + A_x(c_\nu\, c_{\nu+1} \dots) - a_x(p_\nu\, p_{\nu+1} \dots)$$

where the expression on the right side can again be transformed so that only members of the formula $a_{x+\nu}(a'_\nu \dots)$ will occur in it.

We have to prove that the expression

$$a_{x+\nu}(a_\nu \dots a_\omega)$$

can be alwyas expressed with the assistance of values dependent only on the age attained $z = x + \nu$ and with the assistance of certain constants independent of ν, which remain the same throughout the duration of the assurance.

The values

$$a_0, a_1, \dots a_k \dots a_\omega$$

can be always expressed by the polynomial $P_n(k)$ of the n-th degree so that

$$a_k = P_n(k) = \alpha_0 + \alpha_1 k + \dots + \alpha_n k^n$$

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*) I am using the general equation and designation introduced by Professor Schoenbaum, the applicability of which is shown very clearly in this case.

Since there are $\omega + 1$ of these values, n is at the most equal to ω. If the values a_k form an arithmetical series of a certain degree, n denotes this degree.

Then

$$a_{x+\nu}(a) = \sum_{k=\nu}^{\omega} a_k \frac{D_{x+k}}{D_{x+\nu}} = \sum_{k=\nu}^{\omega} (\alpha_0 + \alpha_1 k + \ldots + \alpha_n k^n) \frac{D_{x+k}}{D_{x+\nu}} =$$

$$= \alpha_0 \sum_{k=\nu}^{\omega} \frac{D_{x+n}}{D_{x+\nu}} + \alpha_1 \sum_{k=\nu}^{\omega} k \frac{D_{x+k}}{D_{x+\nu}} + \ldots + \alpha_n \sum_{k=\nu}^{\omega} k_n \frac{D_{x+k}}{D_{x+\nu}}$$

but

$$k^r = (x + k - x)^r = \sum_{i=0}^{r} (-1)^i \binom{r}{i} (x+k)^{r-i} x^i$$

and hence

$$\sum_{k=\nu}^{\omega} k^r \frac{D_{x+k}}{D_{x+\nu}} = \sum_{k=\nu}^{\omega} \sum_{i=0}^{r} (-1)^i \binom{r}{i} (x+k)^{r-i} x^i \frac{D_{x+k}}{D_{x+\nu}}.$$

The expression

$$\sum_{k=\nu}^{\omega} (x+k)^{r-i} \frac{D_{x+k}}{D_{x+\nu}}$$

now really depends only on the age attained $z = x + k$. If we introduce the designation

$$\overset{<(r-i)}{a_{x+\nu}}$$

for it, then

$$\sum_{h=\nu}^{\omega} k^r \frac{D_{x+k}}{D_{x+\nu}} = \sum_{i=0}^{r} (-1)^i \binom{r}{i} x^i \overset{<(r-i)}{a_z},$$

hence follows

$$a_{x+\nu}(a) = \overset{<(0)}{a_z} \sum_{i=0}^{n} (-1)^i \alpha_i x^i + \overset{<(1)}{a_z} \sum_{i=1}^{n} (-1)^{i-1} \alpha_i \binom{i}{1} x^{i-1} + \ldots$$

$$\ldots + \overset{<(k)}{a_z} \sum_{i=k}^{n} (-1)^{i-k} \alpha_i \binom{i}{k} x^{i-k} + \ldots + \alpha_n \overset{<(n)}{a_z}.$$

The reuired expression has been found; the auxiliary constants are for the whole duration of the policy

$$C_k = \sum_{i=k}^{n} (-1)^{i-k} \alpha_i \binom{i}{k} x^{i-k}$$

where C_k are numbers dependent on the values α_i and where x is the age of entry.

II.

The newly introduced values $\overset{<(h)}{a}_x$ have one disadvantage i. e. they are too large for larger n. The proof given in the first paragraph has, therefore, rather a theoretical value. For a practical application of the method of grouping it is, however, possible to change the procedure so that it leads to results easily managable numerically and consequently of great importance for the practive of the actuary. Let us suppose, that in the fundamental equation of equilibrium there are except the constants no other expressions but ${}_nE_x$, ${}_{m|n}a_x$, ${}_{m,n}\overset{<}{a}_x$, ${}_{m|n}A_x$, ${}_{m,n}\overset{<}{A}_x$, where m n are arbitrary numbers, equal to zero respectively. As, however, in practice only such cases occur the great importance of these methods for practical application is demonstrated.

We can confine ourselves to cases, where the expressions ${}_nE_x$, ${}_{m,n}\overset{<}{a}_x$, ${}_{m|n}a_x$ occur in the equation of equilibrium. From the relation mentioned above

$$A_x(c_1, c_2, \ldots) = a_x(c_1, c_2 - c_1, \ldots) - da_x(c_1, c_2, \ldots)$$

follows, that ${}_{m|n}A_x$ and ${}_{m|n}\overset{<}{A}_x$ can be expressed with the assistance of the first three values and that, therefore, it is sufficient to confine ourselves to these cases.

1. If the expression ${}_nE_x$ occurs in the equation of equilibrium, we have to distinguish two possibilities in determining the value of the policy.

a) $k < n$.

The expression ${}_{n-k}E_{x+k} = \dfrac{D_{x+n}}{D_{x+k}} = \dfrac{1}{D_z} \cdot D_{x+n}$ occurs in the value, which is always multiplied by a certain invariable constant; it is sufficient to introduce an auxiliary number $\alpha = D_{x+n} \cdot \text{const.}$ which obviously is invariable for the whole duration of the policy. In the total value we have to multiply $\Sigma\alpha$ by the expression $1/D_z$, whereby we arrive at the corresponding members of the total value.

b) $k \geqq n$.

In the expression of the policy value, no member occurs, as it has dropped out from further calculation. We have, therefore, to be careful to draw attention to this fact in arranging the balance cards without injuring the clearness of the valuation.

2. If in the equation of equilibrium the expression ${}_{m|n}a_x$ occurs, then in the policy value ${}_kV_x$ we have to distinguish three cases:

a) $k \leqq m$ $\qquad {}_{m-k|n}a_{x+k} = \dfrac{N_{x+m} - N_{x+m+n}}{D_z},$

The auxiliary numbers are

$$\alpha = N_{x+m} \,.\, \text{const.} \qquad \beta = N_{x+m+n} \,.\, \text{const.}$$

b)

$$m < k < n + m \quad {}_{n+m}\;{}_{k}a_{x+k} = \frac{N_z - N_{x+m+n}}{D_z}.$$

Here only the second auxiliary number is accounted for, the first auxiliary number drops out and is replaced by the expression N_z . const., which depends only on the age attained and is therefore, invariable for all values of the age z, where k fulfills the condition mentioned above. We have, therefore, to substitute $\Sigma\alpha$ of the values in case of a) by a member $N_z \,.\, \Sigma$ const. of the values in case of b) in order to arrive at a complete corresponding member.

c) $k \geqq m + n$.

From the expression nothing has to be accounted for in the value. By a skilful arranging of the balance card for such kinds of policies we can atrive at a complete mechanisation of the choice of arbitrary numbers because the first auxiliary number α has not to be considered after a certain time and is replaced by an other expression (as we have mentioned under b)).

3. In case of life-annuities with payments linearly inoreasing it is sufficient to examine only annuties with iimited increase, since if with the increase also the payments are stopped, it is sufficient to substract from the first value the corresponding multip'e of the referred life-annuity.

In the equation of equilibrium, therefore, he member

$${}_{m}\,{}_{n}\overset{<}{a}_{x} = \frac{1}{D_x}\,(S_{x+m} - S_{x+m+n})$$

occurs. There are three possible cases of the policy value:

a) $k \leqq m$

$${}_{m-k}\,{}_{n}\overset{<}{a}_{x+k} = \frac{1}{D_{x+k}}\,(S_{x+m} - S_{x+m+n})$$

the auxiliary numbers are

$$\alpha = S_{x+m}\,\text{const.} \qquad \beta = S_{x+m+n} \,.\, \text{const.}$$

b) $m < k < m + n$.

The member

$$\frac{1}{D_{x+k}} \,.\, (\overline{k - m}\, N_{x+k} + S_{x+k} - S_{x+m+n}) =$$

$$= \frac{1}{D_z}\,(zN_z + S_z - (m + x)\,N_z - S_{x+m+n})$$

occurs in the value. Here the auxiliary number β has to be considered as under a), but the auxiliary number α drops out here and is replaced by the expression $(z \cdot N_z + S_z) \cdot \text{const.}$, since the first factor depends only on the age attained and the expression $\overline{m + x} N_z \cdot \text{const.}$ where $m + x \cdot \text{const.} = \gamma$ is the further constant auxiliary number for the whole duration of the policy.

c) $k \geqq m + n$.

In the policy value is the member

$$n \frac{N_{x+k}}{D_{x+k}} = n a_z$$

so that from the numbers α, β, γ none is considered, but on the other hand, we need a new number $\delta = n \cdot \text{const.}$

Again as in the preced ng case a convenient arrangement of the balance cards mechanises completely the use of this method of auxiliary numbers.

The proof required has been carried out. For practical purposes it is, of course, not always advisable to substitute the expressions for whole-life insurance by expressions for single-life annuities, since the calculations would become needlessly cumbersome thereby. Further we have to add that in general the numbers m and n are in a close connection with the duration of the policy so that auxiliary numbers are simpler than we had to deduce them for general use.

I shall quote as an example a whole-life insurance, the premium of which is paid annually in advance; the first r premiums are paid in full, the further decrease annually by $100\,\alpha'\%$.

The equation of equilibrium is

$$p(a_{x\overline{n}|} - \alpha' \cdot {}_{r|n-r}\overset{<}{a}_x) = A_{\overline{x+n}|}.$$

In determining the value of the policy we have to distinguish

1. $k \leqq r$

$${}_kV_x = A_{x+k,\, \overline{n-k}|} - p(a_{x+k,\, \overline{n-k}|} - \alpha' \cdot {}_{r-k|n-r}\overset{<}{a}_{x+k}) =$$

$$= 1 - (d + p) \frac{N_z - N_{x+n}}{D_z} + \alpha' p \frac{S_{x+r} - S_{x+n} - (n - r) N_{x+n}}{D_z}.$$

If the sum insured is S, the auxiliary numbers

$$S,\ S(d + p) = \alpha,\ S(d + p) N_{x+n} = \beta.\ S\alpha' p S_{x+r} = \gamma$$

$$S\alpha' p(S_{x+n} + \overline{n - r}\, N_{x+n}) = \delta$$

occur.

2. $k > r$

$$
{}_kV_x = A_{x+\overline{k, n-k}} - p\,[a_{x+k,\,n-\overset{<}{k}}\;(1 - \alpha'\,k - r) - \alpha' \cdot {}_{n-k}\,a_{x-k}] =
$$

$$
= 1 - (d + p)\frac{N_z - N_{x+n}}{D_z} + \alpha' p\,(k - r)\frac{N_z}{D_z} +
$$

$$
+ \alpha' p\frac{S_z - S_{x+n} - (n - r)\,N_{x+n}}{D_z} = 1 - (d + p)\frac{N_z - N_{x+n}}{D_z} +
$$

$$
+ \alpha' p z a_z - \alpha' p\,(x + r)\,a_z + \alpha' p\frac{S_z}{D_z} - \alpha' p\frac{S_{x+n} + (n - r)\,N_{x+n}}{D_z}.
$$

Besides the former auxiliary numbers we have to introduce two more such numbers

$$
\alpha' p S = \gamma'. \quad \alpha' p S(x + r) = \varepsilon.
$$

The total value of all policies, where the age of the person insured is the same is given by the expression

$$
V_z = \Sigma S - a_z \Sigma \alpha + \frac{1}{D_z}\Sigma \beta + z a_z \Sigma \gamma' - a_z \Sigma \varepsilon +
$$

$$
+ \frac{1}{D_z}(\Sigma \gamma + S_z \Sigma \gamma') - \frac{1}{D_z}\Sigma \delta
$$

where we ought to consider that we have to leave out the members with γ' and ε for values with a duration smaller than r.

III.

Similarly, we can easily define the method of grouping values of those policies which contain further expressions in their equation of equilibrium. We can group the values in the assurance à terme fixe or in the assurance of a certain annuity payable after the death of the person insured until the end of n years from the conclusion of the policy (so-called familly insurance) in an analogous way.

In the assurance à terme fixe, there follows from the equation of equilibrium:

$$
p \cdot a_{\overline{x, n}} = v^n
$$

the policy value

$$
{}_kV_x = v^{n-k} - p \cdot a_{\overline{x+k, n-k}|}.
$$

But

$$
v^{n-x} = v^{n+x}.\, r^{x+k} = v^{n+x}.\, r^z.
$$

If the sum insured be S, it is sufficient to introduce auxiliary values

$$
S v^{n+x} = \alpha, \quad pS = \beta, \quad pSN_{x+n} = \gamma,
$$

whence the total value for all policies with the age attained x is

$$
V_z = r^z . \Sigma \alpha - a_z \Sigma \beta + \frac{1}{D_z} . \Sigma \gamma.
$$

In the familly assurance the equation of equilibrium is

$$p a_{\overline{x,n}} = a_{\overline{n+1}} - a_{x,n+1}$$

and hence the policy value

$$_kV_x = a_{\overline{n-k+1}} - a_{x+k,\overline{n-k+1}} - p \cdot a_{x+k,\overline{n-k}} =$$

$$= \frac{v}{i}(1 - v^{n-k+1}) - (1 + p)\, a_z + \frac{1}{D_z}(N_{x+n+1} + pN_{x+n}).$$

Let us again denote

$$\frac{v}{i}\, S = \alpha, \quad \frac{S}{i}\, v^{n+x+2} = \beta, \quad (1 + p)\, S = \gamma, \quad S(N_{x+n+1} + pN_{x+n}) = \delta,$$

where the total value V_z is given by the expression

$$V_z = \Sigma\alpha - v^z\, \Sigma\beta - a_z\Sigma\gamma + \frac{1}{D_z}\, \Sigma\delta.$$

IV.

The method under consideration has the great advantage that we can use it also in order to group balance values, which are mainly important in the balance of an Insurance Company. Already Zillmer has shown this in the book mentioned above.

In the balances there usually occur values of the form $_{k+t}V_x$, where k is an integer and t is a fraction. Usually these values are determined by linear interpolation: i. e. suposing that the premium p is payable annually in advance, we may put

$$_{k+t}V_x = (1 - t)\, (_kV_k + p) + t\, _{k+1}V_x.$$

Usually it is assumed for the balance, that the new entries are spread equally over the whole year*) so that we can put $t = \frac{1}{2}$ whence

$$_{k+\frac{1}{2}}V_z = \tfrac{1}{2}\, (_kV_z + p + {}_{k+1}V_z).$$

If we take e. g. an assurance à terme fixe, there follows from it after a short transformation

$$_{k+\frac{1}{2}}V_x = \left(1 + \frac{i}{2}\right) v^{n-k} - p\, \frac{a_z + a_{z+1} - 1}{2} + \tfrac{1}{2}\left(\frac{1}{D_z} + \frac{1}{D_{z+1}}\right) pN_{x+n},$$

where z is the number of full years finished up to the day of the balance.

If we have a larger number of such values in an age attained z for the sums S and if we put, as before, the auxiliary numbers

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*) Even then, if this supposition should not be justified in a concrete case, we can use the method described, since the supposition $t = \frac{1}{2}$ is not actually essentiel for it.

$$\left(1+\frac{i}{2}\right)v^{n+x}S=\alpha,\quad pS=\beta,\quad SpN_{x+n}=\gamma.$$

then the total balance value is given by the relation:

$$V_z = v^z\Sigma\alpha - \frac{a_z + a_{z+1} - 1}{2}\Sigma\beta + \tfrac{1}{2}\left(\frac{1}{D_z}+\frac{1}{D_{z+1}}\right)\Sigma\gamma.$$

Further it is obvious, that we can group not only net values, but also values simplified by Zillmer's method or other one. This is obvious, since in the equation of equilibrium of these values no other expressions occur than these, of which we have proved that they can be easily expressed by assistance of fixed factors and that of factors dependent only on the age attained.

Both these circumstances increase, as we have mentioned before, the value of this rather elementary method for practical use. The calculations themselves are very simple and especially if we organize and mechanize the method appropriately, we can determine the required values with comparatively little effort.

LITERATURA.

Journal of the Institute of Actuaries; vol. LXI, Nro. 301.

Harry Freeman; **Notes on a short method of valuation of pension funds.** V článku ukazuje se, jak je možno značně zjednodušiti početní práci při bilancích různých fondů, pro které nutno konstruovati z vlastních zkušeností nový aktivitní řád l_x. Známe-li hodnoty p_x pro věky postupující po 5 letech, je možno použíti jedné ze tří aproximativních formulí:

$$l_{x+5} = p_x{}^{2,5}\, p_{x+5}{}^{2,5}\, l_x$$
$$l_{x+5} = p_x{}^3\, p_{x+2}{}^2\, l_x$$
$$l_{x+5} = [\tfrac{1}{2}(p_x + p_{x+5})]^2\, l_x.$$

Nejlépe obyčejně vyhovuje formule druhá. Pomocí diferenčních schemat je možno snadno odvoditi základní čísla jak pro hodnotu placeného pojistného, tak pro hodnotu různých dávek. Autor dále ukazuje na praktických příkladech, jak velmi dobré výsledky tato metoda dává v porovnání s výsledky získanými plným nekráceným výpočtem pomocí obvyklých formulí. Nutno toliko v každém případě přihlédnouti bedlivě k tomu, zda průběh p_x je dosti rovnoměrný. V konečných stářích je vždy doporučitelno napočísti hodnoty pro jednotlivé věky; v nízkých letech pouze tehdy, uplatňuje-li se silněji vliv výstupů, což vede k značným odchylkám v průběhu p_x.

H. J. Tappenden: **A valuation of non-participing policies without classification.** Premiová reserva pojistky mění se v důsledku konstantního úročení a proměnného risika; je na snadě uvažovati ji za výsledek spoření při poměrné úrokové míře. Pro praktické výpočty pak naopak je výhodné pokládati premiovou reservu za výsledek spoření různých částí s různými — ale konstantními — úrokovými měrami. Tedy pro pojistky za běžnou premii je možno klásti